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GENERALIZED LANDSBERG MANIFOLDS OF SCALAR CURVATURE

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ABSTRACT. We prove that every generalized Landsberg manifold of scalar curvature R is a Riemannian manifold of constant curvature, provided that $R \neq 0$.

Introduction

A Finsler manifold (space) is called a Landsberg manifold (space) if the Berwald connection **BFC**$= \left(G_i^k, G_i{}^k{}_j, 0\right)$ coincides with the Rund connection **RFC**$= \left(G_i^k, F_i{}^k{}_j, 0\right)$ (cf. Antonelli et al. [1], p. 98). In the literature there exists a remarkable research work developed on the geometry of Landsberg manifolds (see Bao-Chern-Shen [2], Matsuomoto [5], Numata [7]). The following result is the starting point of our paper.

THEOREM A (Numata [7]). *Let* $\mathbb{F}^m (m \geq 3)$ *be a* Landsberg *manifold of scalar curvature* R. *Then* $\mathbb{F}^m$ *is a* Riemannian *manifold of constant curvature, provided* $R \neq 0$.

With the above definition of Landsberg manifolds in mind, we may introduce a new class of Finsler manifolds, as follows. We say that $\mathbb{F}^m$ is a generalized Landsberg manifold if the h-curvature Finsler tensor fields of the Berwald and Rund connections coincide. Apart from Landsberg manifolds, generalized Landsberg manifolds include h-flat Finsler manifolds with respect to the Cartan connection. The main purpose of the present paper is to extend the above Numata's result to generalized Landsberg manifolds (see Theorem 1). To reach this goal, we first prove that C-reducible generalized Landsberg manifolds are

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Landsberg manifolds (see Theorem 2). Then we show that generalized Landsberg manifolds of scalar curvature are C-reducible (see Theorem 3). Although our objective is to prove the result of Numata [7] in the more general setting of generalized Landsberg manifolds, but due to the clear differences in the two approaches, our techniques in this paper are totally different from those in [7].

Throughout the paper we make use of Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range. We also use the notation $\mathcal{A}_{(ij)}$ to denote the interchange of indices i, j and subtraction. For instance,

$$\mathcal{A}_{(ij)}\{T_i{}^r{}_k S_{rj}\} = T_i{}^r{}_k S_{rj} - T_j{}^r{}_k S_{ri}.$$

All manifolds and mappings are considered to be of class of differentiability at least C^5. Finally, we should mention that all covariant derivatives in the paper are considered with respect to the Cartan connection. The h- and v-covariant derivatives of a Finsler tensor field are denoted by "|" and "||" respectively.

1. Preliminaries

Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold, where M is a real m-dimensional smooth manifold and F is the fundamental function of $\mathbb{F}^m$. Consider $TM^\circ = TM \backslash \{\theta(M)\}$, where TM is the tangent bundle of M and θ is the zero section of TM. Denote by VTM° the vertical vector bundle over TM°, that is, $VTM^\circ = Ker\pi_*$, where π_* is the tangent mapping of the canonical projection $\pi : TM^\circ \to M$. The local coordinates on M and TM° are denoted by (x^i) and (x^i, y^i), $i \in \{1, \cdots, m\}$ respectively.

Throughout the paper, we assume that the Finsler metric $g = \big(g_{ij}(x, y)\big)$, where we set

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

is positive definite. Thus g may be thought of as a Riemannian metric on VTM° (see Bejancu [3], p. 20).

The main tool for studying Finsler geometry is the canonical non-linear connection $GTM^\circ = (G_i^k)$, where

$$G_i^k = \frac{\partial G^k}{\partial y^i} \; ; \; G^k = \frac{1}{4} g^{kh} \left(\frac{\partial^2 F^2}{\partial y^h \partial x^t} y^t - \frac{\partial F^2}{\partial x^h} \right).$$

Then the tangent bundle of TM° has the decomposition

$$T\,TM^\circ = GTM^\circ \oplus VTM^\circ.$$

Thus a field of frames on TM° adapted to the above decomposition is $\{\delta/\delta x^i,\ \partial/\partial y^i\}$ where we set

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j},$$

and $\{\partial/\partial y^i\}$ is a basis for the module of local sections of VTM°. The three classical Finsler connections: the Berwald, Cartan and Rund connections are completely determined by a pair (GTM°, ∇), where ∇ is a linear connection on VTM°. In order to present the local coefficients of these Finsler connections we set

$$\nabla_{\delta/\delta x^j} \frac{\partial}{\partial y^i} = F_i{}^k{}_j \frac{\partial}{\partial y^k} \quad \text{and} \quad \nabla_{\partial/\partial y^j} \frac{\partial}{\partial y^i} = C_i{}^k{}_j \frac{\partial}{\partial y^k}.$$

Then the local coefficients of the Cartan connection **CFC**, the Berwald connection **BFC** and the Rund connection **RFC** are given by the triplets $(G_i^k,\ F_i{}^k{}_j,\ C_i{}^k{}_j)$, , $(G_i^k, G_i{}^k{}_j, 0)$ and $(G_i^k, F_i{}^k{}_j, 0)$ respectively, where we put (see Antonelli et al. [1], pp. 80, 86, 97)

$$F_i{}^k{}_j = \frac{1}{2} g^{kh} \left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right),$$

$$C_i{}^k{}_j = \frac{1}{2} g^{kh} \frac{\partial g_{hi}}{\partial y^j} \; ; \; G_i{}^k{}_j = \frac{\partial G_i^k}{\partial y^j}.$$

Moreover, we have

$$G_i{}^k{}_j = F_i{}^k{}_j + C_i{}^k{}_{j|0}, \tag{1.1}$$

where we set $C_{i\ j|0}^{\ k} = C_{i\ j|h}^{\ k} y^h$.

Next, we denote by $R_{i\ jh}^{\ k}$, $H_{i\ jh}^{\ k}$ and $K_{i\ jh}^{\ k}$ the h-curvature Finsler tensor fields of **CFC, BFC** and **RFC**. Also, we denote by $R^k_{\ ij}$ the curvature Finsler tensor field of the canonical non-linear connection $GTM^\circ = (G_i^k)$, that is, we have

$$R^k_{\ ij} = \frac{\delta G_i^k}{\delta x^j} - \frac{\delta G_j^k}{\delta x^i}.$$

We now recall some well known identities (see sections 17 and 18 in Matsumoto [6]).

$$(1.2)\ R_{i\ kh}^{\ t} = K_{i\ kh}^{\ t} + C_{i\ r}^{\ t} R^r_{\ kh},$$

$$(1.3)\ R_{i\ kh}^{\ t} = H_{i\ kh}^{\ t} + C_{i\ r}^{\ t} R^r_{\ kh} + \mathcal{A}_{(kh)} \left\{ C_{k\ r|0}^{\ t} C_{i\ h|0}^{\ r} + C_{i\ h|0|k}^{\ t} \right\},$$

$$(1.4)\ K_{i\ kh}^{\ t} = H_{i\ kh}^{\ t} + \mathcal{A}_{(kh)} \left\{ C_{k\ r|0}^{\ t} C_{i\ h|0}^{\ r} + C_{i\ h|0|k}^{\ t} \right\},$$

$$(1.5)\ R_{i\ kh}^{\ t} = R^t_{\ kh\|i} + \mathcal{A}_{(kh)} \left\{ R^t_{\ kr} C_{i\ h}^{\ r} + C^t_{hr|0} C_{i\ h|0}^{\ r} + C^t_{ih|0|k} \right\}$$

$$(1.6) \qquad (a)\ R_{ijkh} + R_{jikh} = 0\ ;\ (b)\ R^t_{\ jh} = y^i R_{i\ jh}^{\ t}.$$

By (1.1) we deduce that $\mathbb{F}^m$ is a Landsberg manifold if and only if $C_{i\ j|0}^{\ k} = 0$. Also, contracting (1.4) by g_{it}, we deduce that $K_{ijkh} = H_{ijkh}$, if and only if ,

$$(1.7) \qquad (a)\ \mathcal{A}_{kh} \left\{ C_{jkr|0} C^{\ r}_{ih|0} \right\} = 0 \text{ and } (b)\ \mathcal{A}_{kh} \left\{ C_{ijh|0|k} \right\} = 0.$$

Hence $\mathbb{F}^m$ is a generalized Landsberg manifold, if and only if, both relations in (1.7) are satisfied. When $R_{ijkh} = 0$, we say that $\mathbb{F}^m$ is an h-flat Finsler manifold with respect to **CFC**.

In this case from (1.6)b, it follows $R^t_{\ jh} = 0$. Hence from (1.5), if follows that both conditions in (1.7) are satisfied. Thus any h-flat Finsler manifold with respect to **CFC** is a generalized Landsberg manifold.

Further, we recall that on TM there exist the Liouville vector field $L = y^i \partial/\partial y^i$ and the Hilbert form $\eta = \eta_i dx^i$. These two Finsler tensor fields are related by the following

$$(1.8) \qquad (a)\ \eta_i = \frac{1}{F} g_{ij} y^j = \frac{\partial F}{\partial y^i} \text{ and } (b)\ \eta_i y^i = F.$$

The local components of the angular metric $h = (h_{ij})$ are given by

$$(1.9) \qquad h_{ij} = g_{ij} - \eta_i \eta_j.$$

Then by direct calculations we obtain

$$(1.10) \qquad (a)\ h_{ij|k} = 0 \text{ and } (b)\ \frac{\partial h_{ij}}{\partial y^k} = 2g_{ijk} - \frac{1}{F}\left(h_{ik}\eta_j + h_{jk}\eta_i\right).$$

The following interesting class of Finsler manifolds has been introduced by Matsumoto [4]. A Finsler manifold $\mathbb{F}^m$ is said to be C-reducible if the Cartan tensor field satisfies

$$(1.11) \qquad C_{ijk} = \frac{1}{m+1}\left(h_{ij}C_k + h_{ik}C_j + h_{jk}C_i\right),$$

where $C_i = C_{ijk} g^{jk}$.

Finally, we say that $\mathbb{F}^m$ is of scalar curvature $R(x, y)$ if the Finsler tensor field R_{ijk} satisfies (Antonelli [1], p.106)

$$(1.12) \qquad R_{ijk} y^j = F^2 R h_{ik}.$$

Then it can be proved (see Antonelli et al [1], p. 106) that we have

$$(1.13) \qquad R_{ijk} = h_{ik}R_j - h_{ij}R_k \ ; \ R_j = \frac{F^2}{3}\frac{\partial R}{\partial y^j} + FR\eta_j.$$

2. The Main Result

It is the purpose of the present section to give a generalization of Numata's theorem. More precisely, we prove the following theorem.

THEOREM 1. *Let $\mathbb{F}^m (m \geq 3)$ be a generalized* Landsberg *manifold of scalar curvature R. Then $\mathbb{F}^m$ is a* Riemannian *manifold of constant curvature, provided $R \neq 0$.*

For the proof of this theorem we need the following two theorems.

THEOREM 2. *Let $\mathbb{F}^m (m \geq 3)$ be a generalized* Landsberg *manifold. If $\mathbb{F}^m$ is C-reducible, then it is a* Landsberg *manifold.*

Proof. Using (1.11) and (1.10 a) we deduce that

$$C_{ijk|0} = \frac{1}{m+1}\left(h_{ij}C_{k|0} + h_{ik}C_{j|0} + h_{jk}C_{i|0}\right). \tag{2.1}$$

On the other hand, by using (1.8) and (1.9) we infer that

$$h_{ij} = h_{ir}h_{js}g^{rs}, \tag{2.2}$$

and

$$C_{i|0} = h_{ir}C_{s|0}g^{rs}. \tag{2.3}$$

Then, by some lengthy calculations using $(2.1) - (2.3), (1.7a)$ becomes

$$A_{(jk)}\left\{h_{ij}h_{\ell k}g^{rs}C_{r|0}C_{s|0} + h_{\ell k}C_{i|0}C_{j|0} + h_{ij}C_{\ell|0}C_{k|0}\right\} = 0. \tag{2.4}$$

Next, using again (1.8) and (1.9) we derive

$$h_{ij}g^{ij} = m - 1. \tag{2.5}$$

Finally, contracting (2.4) by $g^{ij}g^{tk}$ and using (2.2), (2.3) and (2.5) we obtain

$$(m+1)(m-2)g^{rs}C_{r|0}C_{s|0} = 0.$$

As $m \geq 3$ and the Finsler metric g is positive definite, we deduce that $C_{i|0} = 0$, for any $i \in \{1, \cdots, m\}$. Hence by (2.1) we obtain $C_{ijk|0} = 0$, that is, $\mathbb{F}^m$ is a Landsberg manifold. □

THEOREM 3. *Let $\mathbb{F}^m$ be a generalized* Landsberg *manifold of non zero scalar curvature R. Then $\mathbb{F}^m$ is a C-reducible Finsler manifold.*

Proof. By using (1.7) and (1.5) we deduce that the h-curvature tensor of the Cartan connection on $\mathbb{F}^m$ is given by

$$R_{ijk\ell} = \frac{\partial R_{jk\ell}}{\partial y^i} - g_i{}^r{}_j R_{rk\ell}.$$

Thus (1.6 a) becomes

$$2C_i{}^r{}_j R_{rk\ell} = \frac{\partial R_{jk\ell}}{\partial y^i} + \frac{\partial R_{ik\ell}}{\partial y^j}. \tag{2.6}$$

Contracting (2.6) by y^k we obtain

$$2C_i{}^r{}_j R_{rk\ell} y^k = \frac{\partial (R_{jk\ell} y^k)}{\partial y^i} + \frac{\partial (R_{ik\ell} y^k)}{\partial y^j} - R_{ji\ell} - R_{ij\ell}. \tag{2.7}$$

By using (1.12), (1.13), (1.10b), (1.8) and (1.9) in (2.7) after some lengthy calculations we infer that

$$C_{ij\ell} = -\frac{1}{3R}\left(h_{ij}\frac{\partial R}{\partial y^\ell} + h_{i\ell}\frac{\partial R}{\partial y^j} + h_{j\ell}\frac{\partial R}{\partial y^i}\right). \tag{2.8}$$

As our (2.8) coincides with (12) in Numata [7], we conclude that $\mathbb{F}^m$ is C-reducible. □

Now we prove Theorem 1. Suppose $\mathbb{F}^m (m \geq 3)$ is a generalized Landsberg manifold of non zero scalar curvature R. Then by Theorems 2 and 3 it follows that $\mathbb{F}^m$ is a Landsberg manifold. Thus we apply Theorem A and obtain the assertion of Theorem 1.

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