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BLOCK INCOMPLETE FACTORIZATION PRECONDITIONERS FOR A SYMMETRIC H-MATRIX

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ABSTRACT. We propose new parallelizable block incomplete factorization preconditioners for a symmetric block-tridiagonal H-matrix. Theoretical properties of these block preconditioners are compared with those of block incomplete factorization preconditioners for the corresponding comparison matrix. Numerical results of the preconditioned CG(PCG) method using these block preconditioners are compared with those of PCG method using a standard incomplete factorization preconditioner to see the effectiveness of the block incomplete factorization preconditioners.

1. Introduction

The discretization of partial differential equations in 2D or 3D, by finite difference or finite element approximation, leads often to large sparse block-tridiagonal linear systems. In this paper, we consider the linear system of equations

$$Ax = b, \quad x, b \in \mathbb{R}^n \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse symmetric block-tridiagonal H-matrix blocked in the form

$$A = \begin{pmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ C_1^T & B_2 & C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_{m-2}^T & B_{m-1} & C_{m-1} \\ 0 & \cdots & 0 & C_{m-1}^T & B_m \end{pmatrix}. \tag{2}$$

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It is assumed that the diagonal blocks B_i of A are symmetric matrices with the same order. Since A is a large sparse matrix, direct solvers become prohibitively expensive because of the large amount of work and storage required. As an alternative, the conjugate gradient (CG) iterative method [3] is widely used for a symmetric M-matrix A which guarantees the positive-definiteness of A. Given an initial guess x_0, CG algorithm compute iteratively new approximations x_k to the true solution $x^* = A^{-1}b$. The iterate x_k is accepted as a solution if the residual $r_k = b - Ax_k$ satisfies $\|r_k\|/\|b\| \leq$ (Tolerance). In general, the convergence is not guaranteed or may be extremely slow. Hence, the original problem (1) must be transformed into a more tractable form. To do so, we consider an easily invertible matrix K called the *preconditioning matrix* or *preconditioner* and apply the iterative solvers either to the left preconditioned linear system $K^{-1}Ax = K^{-1}b$ or to the right preconditioned linear system $AK^{-1}y = b$, where $y = Kx$. The preconditioner K should be chosen so that $K^{-1}A$ or AK^{-1} is a good approximation to the identity matrix.

The purpose of this paper is to propose new parallelizable block incomplete factorization preconditioners for a symmetric block-tridiagonal H-matrix which extend the ideas for an M-matrix introduced by Yun [7]. The block incomplete factorization preconditioners for H-matrices to be proposed in this paper are obtained by performing the standard incomplete factorization on each matrix block independently, so that they have no block recurrence which requires sparse approximate inverses for pivot blocks and thus they can be computed in parallel based on matrix blocks.

In section 2, we review some basic properties of the incomplete factorization for H-matrices. In section 3, we propose new parallelizable block incomplete factorization preconditioners for a symmetric block-tridiagonal H-matrix, and their theoretical properties are compared with those of block incomplete factorization preconditioners for its comparison matrix. In section 4, we describe how to construct the effective block preconditioners for a special type of matrix which arises from five-point discretization of the second-order partial differential equation. In section 5, we present numerical results of the PCG with the block incomplete factorization preconditioners developed in this paper, and their results are compared with those of the PCG with a standard incomplete factorization preconditioner. Lastly, some conclusions are drawn.

2. H-matrices and incomplete factorization

A vector x is nonnegative (positive), denoted $x \geq 0$ ($x > 0$), if all its entries are nonnegative (positive). Similarly, a matrix B is said to be nonnegative, denoted $B \geq 0$, if all its entries are nonnegative. We compare two matrices $A \geq B$, when $A - B \geq 0$, and two vectors $x \geq y$ ($x > y$) when $x - y \geq 0$ ($x - y > 0$). Given a matrix $A = (a_{ij})$, we define the matrix $|A| = (|a_{ij}|)$. It follows that $|A| \geq 0$ and that $|AB| \leq |A||B|$ for any two matrices A and B of compatible size. For any two matrices A and B of the same size, the *Hadamard matrix product* $A \odot B$ is defined by $a_{ij}b_{ij}$, where a_{ij} and b_{ij} are the entries of A and B respectively. A matrix $A = (a_{ij})$ is an *M-matrix* if $a_{ij} \leq 0$ for all $i \neq j$ and $A^{-1} \geq 0$. The *comparison matrix* $\langle A \rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} := \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

A matrix A is an H-matrix if $\langle A \rangle$ is an M-matrix. H-matrices have been studied by many authors in connection to iterative solutions of linear systems; see Beauwens [1], Frommer and Mayer [2]. Notice that M-matrices and strictly or irreducibly diagonally dominant matrices are contained in the class of all H-matrices. Actually, an H-matrix $A = (a_{ij})$ may be equivalently characterized by being *generalized strictly diagonally dominant*, i.e.,

$$|a_{ii}|u_i > \sum_{j \neq i} |a_{ij}|u_j, \qquad i = 1, 2, \ldots, n$$

for some vector $u = (u_1, u_2, \ldots, u_n)^T > 0$. The *spectral radius* $\rho(A)$ of a matrix A is $\rho(A) = \max\{|\lambda| \colon \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the *spectrum* of A, that is, the set of eigenvalues of A. It was shown in [6] that for $n \times n$ real matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$. A representation $A = K - N$ is called a *splitting* of A when K is nonsingular. A splitting $A = K - N$ is a *convergent splitting* of A if $\rho(K^{-1}N) < 1$. It is well-known that if $A = K - N$ is a convergent splitting, then any stationary iterative method of the form

$$Kx_{k+1} = Nx_k + b, \qquad k \geq 0$$

converges to the exact solution of $Ax = b$ for every choice of x_0 [6].

LEMMA 2.1. *Let A be an H-matrix, let B be a matrix of order n, and let $C = A \odot B$. If $0 \le b_{ij} \le 1$ for $i \ne j$ and $b_{ii} \ge 1$ for $i = 1, 2, \ldots, n$, then C is also an H-matrix.*

Proof. Since A is an H-matrix, there exists a positive vector x such that $\langle A \rangle x > 0$. Thus, for all $i = 1, 2, \ldots, n$

$$(\langle C \rangle x)_i = |a_{ii} b_{ii}| x_i - \sum_{j \ne i} |a_{ij} b_{ij}| x_j \ge |a_{ii}| x_i - \sum_{j \ne i} |a_{ij}| x_j > 0.$$

Hence, C is an H-matrix. □

LEMMA 2.2. *Let $A = [A_{ij}]$ be a symmetric H-matrix that is partitioned into block matrix form. Then*

(a) *The block diagonal part of A is a symmetric H-matrix. In particular, each block A_{ii} is a symmetric H-matrix.*
(b) *The block lower and upper triangular parts of A are H-matrices.*

Proof. To prove part (a), let $B = [B_{ij}]$ be partitioned consistently with the partitioning of A and let the entries B_{ii} be unity and the entries B_{ij}, $i \ne j$, be zero. If we compute $A \odot B$, then

$$A \odot B = \text{block diagonal part of } A.$$

Thus, from Lemma 2.1, the block diagonal part of A is a symmetric H-matrix. In addition, we can easily show that each block A_{ii} is a symmetric H-matrix. Part (b) is proved similarly. □

A general algorithm for building incomplete LU factorization can be derived by performing Gaussian elimination and dropping some of elements in predetermined off-diagonal positions. Let P_n denote the set of all pairs of indices of off-diagonal matrix entries, that is,

$$P_n = \{(i, j) \mid i \ne j,\ 1 \le i \le n,\ 1 \le j \le n\}.$$

Then, it was shown in [4] that incomplete LU factorizations of M-matrices exist for any zero pattern set $P \subset P_n$. The following theorem which is a little variant of Theorem 2.5 in [5] states the existence of incomplete factorizations of symmetric H-matrices.

THEOREM 2.3. *Let A be a symmetric H-matrix. Then, for every symmetric zero pattern set $P \subset P_n$ (i.e., $(i, j) \in P$ implies $(j, i) \in P$), there exist an upper triangular matrix $U = (u_{ij})$, a diagonal matrix D*

whose i-th diagonal element is u_{ii}^{-1}, and a symmetric matrix $R = (r_{ij})$, with $u_{ij} = 0$ if $(i,j) \in P$ and $r_{ij} = 0$ if $(i,j) \notin P$, such that $A = U^T DU - R$. Moreover, U is an H-matrix.

In Theorem 2.3, $A = U^T DU - R$ is called an *incomplete factorization* of A corresponding to a zero pattern set P.

THEOREM 2.4 ([5]). *Let A be an $n \times n$ H-matrix. Let $A = LU - N$ and $\langle A \rangle = \tilde{L}\tilde{U} - \tilde{N}$ be the incomplete LU factorizations of A and $\langle A \rangle$ corresponding to a zero pattern set $P \subset P_n$ respectively. Then each of the following holds:*

(a) $\tilde{U} \le \langle U \rangle$.
(b) $|L^{-1}| \le \tilde{L}^{-1}$.
(c) $|U^{-1}| \le \tilde{U}^{-1}$.
(d) $|N| \le \tilde{N}$.
(e) $|(LU)^{-1}N| \le (\tilde{L}\tilde{U})^{-1}\tilde{N}$.

For symmetric H-matrices, the following theorem similar to Theorem 2.4 can be easily obtained using Theorems 2.3 and 2.4.

THEOREM 2.5. *Let A be a symmetric H-matrix. Let $A = U^T DU - R$ and $\langle A \rangle = \tilde{U}^T \tilde{D}\tilde{U} - \tilde{R}$ be the incomplete factorizations of A and $\langle A \rangle$ corresponding to a symmetric zero pattern set $P \subset P_n$ respectively. Then each of the following holds:*

(a) $\tilde{U} \le \langle U \rangle$.
(b) $|U^{-1}| \le \tilde{U}^{-1}$.
(c) $|(U^T D)^{-1}| \le (\tilde{U}^T \tilde{D})^{-1}$.
(d) $|R| \le \tilde{R}$.
(e) $|D| \le \tilde{D}$.

LEMMA 2.6. *Let A be a symmetric H-matrix. Let $A = U^T DU - R$ and $\langle A \rangle = \tilde{U}^T \tilde{D}\tilde{U} - \tilde{R}$ be the incomplete factorizations of A and $\langle A \rangle$ corresponding to a symmetric zero pattern set $P \subset P_n$ respectively. Then each of the following holds:*

(a) $|DU| \le |\tilde{D}\tilde{U}|$, $\quad |I - DU| \le I - \tilde{D}\tilde{U}$.
(b) $|U^T D| \le |\tilde{U}^T \tilde{D}|$, $\quad |I - U^T D| \le I - \tilde{U}^T \tilde{D}$.

Proof. For the proof of part (a), let $U = (u_{ij})$ and $\tilde{U} = (\tilde{u}_{ij})$. Then, from Theorem 2.5(a) $0 < \tilde{u}_{ii} \le |u_{ii}|$ and $\tilde{u}_{ij} \le -|u_{ij}| \le 0$ for $i \ne j$. Thus, for $i < j$

$$\left|\frac{u_{ij}}{u_{ii}}\right| = |u_{ij}|\frac{1}{|u_{ii}|} \leq |u_{ij}|\frac{1}{\tilde{u}_{ii}} \leq -\tilde{u}_{ij}\frac{1}{\tilde{u}_{ii}} = -\frac{\tilde{u}_{ij}}{\tilde{u}_{ii}} = \left|\frac{\tilde{u}_{ij}}{\tilde{u}_{ii}}\right|.$$

It follows that $|DU| \leq |\tilde{D}\tilde{U}|$. Since DU is a unit upper triangular matrix and $\tilde{D}\tilde{U}$ is a unit upper triangular M-matrix, $|I - DU| = |DU| - I$ and $I - \tilde{D}\tilde{U} = |\tilde{D}\tilde{U}| - I$. Using these properties,

$$|I - DU| = |DU| - I \leq |\tilde{D}\tilde{U}| - I = I - \tilde{D}\tilde{U}.$$

Hence, part (a) is proved. Notice that $B \leq C$ implies $B^T \leq C^T$ for any two matrices B and C of compatible size. Thus, part (b) follows from part (a) by taking transposes. □

3. Block incomplete factorizations

We first consider block incomplete factorization preconditioners for a symmetric block-tridiagonal H-matrix of the simplest form

$$A = \begin{pmatrix} B_1 & C_1 \\ C_1^T & B_2 \end{pmatrix}. \tag{3}$$

It is assumed that the diagonal blocks B_i of A are $\ell \times \ell$ square matrices. Since A is a symmetric H-matrix, it follows from Lemma 2.2 that B_1 and B_2 are symmetric H-matrices. From the incomplete factorization process, we can find an upper triangular matrix U_i, a symmetric matrix R_i, and a diagonal matrix D_i such that $B_i = U_i^T D_i U_i - R_i$ is the incomplete factorization of B_i for each $i = 1, 2$, see Theorem 2.3. If $A = K - N$ is a splitting of A and K is a matrix which is easily invertible, then K can be used as a preconditioner for nonstationary iterative methods. The effectiveness of the preconditioner K depends on how well K approximates A.

THEOREM 3.1. *Let A be a symmetric H-matrix of the form* (3). *For each $i = 1, 2$, let $B_i = U_i^T D_i U_i - R_i$ and $\langle B_i \rangle = \tilde{U}_i^T \tilde{D}_i \tilde{U}_i - \tilde{R}_i$ be the incomplete factorizations of B_i and $\langle B_i \rangle$ corresponding to a symmetric* *zero pattern set $P \subset P_\ell$ respectively. Let*

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad U_\alpha = \begin{pmatrix} U_1 & C_1 \\ 0 & U_2 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} U_1 & (U_1^T D_1)^{-1} C_1 \\ 0 & U_2 \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{U}_2 \end{pmatrix}, \quad \tilde{U}_\alpha = \begin{pmatrix} \tilde{U}_1 & -|C_1| \\ 0 & \tilde{U}_2 \end{pmatrix}, \quad \tilde{U}_\beta = \begin{pmatrix} \tilde{U}_1 & -(\tilde{U}_1^T \tilde{D}_1)^{-1} |C_1| \\ 0 & \tilde{U}_2 \end{pmatrix},$$

and

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} \tilde{D}_1 & 0 \\ 0 & \tilde{D}_2 \end{pmatrix}.$$

If we let

$$\begin{array}{lll} M = U^T D U, & M_\alpha = U_\alpha^T D U_\alpha, & M_\beta = U_\beta^T D U_\beta, \\ R = M - A, & R_\alpha = M_\alpha - A, & R_\beta = M_\beta - A, \\ \tilde{M} = \tilde{U}^T \tilde{D} \tilde{U}, & \tilde{M}_\alpha = \tilde{U}_\alpha^T \tilde{D} \tilde{U}_\alpha, & \tilde{M}_\beta = \tilde{U}_\beta^T \tilde{D} \tilde{U}_\beta, \\ \tilde{R} = \tilde{M} - \langle A \rangle, & \tilde{R}_\alpha = \tilde{M}_\alpha - \langle A \rangle, & \tilde{R}_\beta = \tilde{M}_\beta - \langle A \rangle, \end{array}$$

then each of the following holds:

(a) $|U^{-1}| \le \tilde{U}^{-1}, \ |U_\alpha^{-1}| \le \tilde{U}_\alpha^{-1}, \ |U_\beta^{-1}| \le \tilde{U}_\beta^{-1}$,
(b) $|(U^T D)^{-1}| \le (\tilde{U}^T \tilde{D})^{-1}, \ |(U_\alpha^T D)^{-1}| \le (\tilde{U}_\alpha^T \tilde{D})^{-1}, \ |(U_\beta^T D)^{-1}| \le (\tilde{U}_\beta^T \tilde{D})^{-1}$,
(c) $|R| \le \tilde{R}, \ |R_\alpha| \le \tilde{R}_\alpha, \ |R_\beta| \le \tilde{R}_\beta$,
(d) $\rho(M^{-1}R) \le \rho(\tilde{M}^{-1}\tilde{R}), \ \rho(M_\alpha^{-1}R_\alpha) \le \rho(\tilde{M}_\alpha^{-1}\tilde{R}_\alpha), \ \rho(M_\beta^{-1}R_\beta) \le \rho(\tilde{M}_\beta^{-1}\tilde{R}_\beta)$,
(e) $\rho(\tilde{M}_\beta^{-1}\tilde{R}_\beta) \le \rho(\tilde{M}_\alpha^{-1}\tilde{R}_\alpha) \le \rho(\tilde{M}^{-1}\tilde{R}) < 1$.

Proof. For the proof of parts (a) and (b), we will show only that $|U_\beta^{-1}| \le \tilde{U}_\beta^{-1}$ and $|(U_\beta^T D)^{-1}| \le (\tilde{U}_\beta^T \tilde{D})^{-1}$ since other properties can be proved similarly. By simple calculations, one obtains

$$|U_\beta^{-1}| = \begin{pmatrix} |U_1^{-1}| & |-U_1^{-1}(U_1^T D_1)^{-1} C_1 U_2^{-1}| \\ 0 & |U_2^{-1}| \end{pmatrix},$$

$$\tilde{U}_\beta^{-1} = \begin{pmatrix} \tilde{U}_1^{-1} & \tilde{U}_1^{-1}(\tilde{U}_1^T \tilde{D}_1)^{-1} |C_1| \tilde{U}_2^{-1} \\ 0 & \tilde{U}_2^{-1} \end{pmatrix},$$

$$|(U_\beta^T D)^{-1}| = \begin{pmatrix} |(U_1^T D_1)^{-1}| & 0 \\ |-(U_2^T D_2)^{-1} C_1^T U_1^{-1} (U_1^T D_1)^{-1}| & |(U_2^T D_2)^{-1}| \end{pmatrix},$$

$$(\tilde{U}_\beta^T \tilde{D})^{-1} = \begin{pmatrix} (\tilde{U}_1^T \tilde{D}_1)^{-1} & 0 \\ (\tilde{U}_2^T \tilde{D}_2)^{-1} |C_1|^T \tilde{U}_1^{-1} (\tilde{U}_1^T \tilde{D}_1)^{-1} & (\tilde{U}_2^T \tilde{D}_2)^{-1} \end{pmatrix}.$$

From Theorem 2.5(b) and (c), $|U_i^{-1}| \le \tilde{U}_i^{-1}$ and $|(U_i^T D_i)^{-1}| \le (\tilde{U}_i^T \tilde{D}_i)^{-1}$ for $i = 1, 2$. Hence, $|U_\beta^{-1}| \le \tilde{U}_\beta^{-1}$ and $|(U_\beta^T D)^{-1}| \le (\tilde{U}_\beta^T \tilde{D})^{-1}$ hold. For the proof of part (c), we will show only that $|R_\alpha| \le \tilde{R}_\alpha$. If we compute $|R_\alpha|$ and $\tilde{R}_\alpha$, then

$$|R_\alpha| = \begin{pmatrix} |R_1| & |(U_1^T D_1 - I) C_1| \\ |C_1^T (D_1 U_1 - I)| & |C_1^T D_1 C_1 + R_2| \end{pmatrix},$$

$$\tilde{R}_\alpha = \begin{pmatrix} \tilde{R}_1 & (I - \tilde{U}_1^T \tilde{D}_1)|C_1| \\ |C_1^T|(I - \tilde{D}_1 \tilde{U}_1) & |C_1^T| \tilde{D}_1 |C_1| + \tilde{R}_2 \end{pmatrix}.$$

From Lemma 2.6, $|I - U_1^T D_1| \le I - \tilde{U}_1^T \tilde{D}_1$ and $|I - D_1 U_1| \le I - \tilde{D}_1 \tilde{U}_1$. From Theorem 2.5(d) and (e), $|R_i| \le \tilde{R}_i$ for $i = 1, 2$ and $|D_1| \le \tilde{D}_1$. Thus, $|R_\alpha| \le \tilde{R}_\alpha$ holds. For the proof of part (d), we will show only that $\rho(M_\beta^{-1} R_\beta) \le \rho(\tilde{M}_\beta^{-1} \tilde{R}_\beta)$. Using parts (a), (b) and (c),

$$|M_\beta^{-1} R_\beta| \le |U_\beta^{-1}| |(U_\beta^T D)^{-1}| |R_\beta| \le \tilde{U}_\beta^{-1} (\tilde{U}_\beta^T \tilde{D})^{-1} \tilde{R}_\beta = \tilde{M}_\beta^{-1} \tilde{R}_\beta.$$

Hence, $|M_\beta^{-1} R_\beta| \le \tilde{M_\beta}^{-1} \tilde{R}_\beta$ implies $\rho(M_\beta^{-1} R_\beta) \le \rho(\tilde{M}_\beta^{-1} \tilde{R}_\beta)$. Since $\langle A \rangle$ and $\langle B_i \rangle$ are M-matrices, part (e) was proved in [7]. □

The following Examples 3.2 and 3.3 show that Theorem 3.1(e) does not hold for a symmetric H-matrix A. In other words, it is not true that

$$\rho(M_\beta^{-1} R_\beta) \le \rho(M_\alpha^{-1} R_\alpha) \le \rho(M^{-1} R).$$

EXAMPLE 3.2. Consider a symmetric 2×2 block matrix A of the form

$$A = \begin{pmatrix} B_1 & C_1 \\ C_1^T & B_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & -3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix},$$

where B_1, B_2, and C_1 are 2×2 square matrices. Since $\langle A \rangle$ is an M-matrix, A is an H-matrix. Let $B_1 = U_1^T D_1 U_1$ and $B_2 = U_2^T D_2 U_2$ be factorizations of B_1 and B_2, respectively. Then one obtains

$$U_\alpha = \begin{pmatrix} U_1 & C_1 \\ 0 & U_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} U_1 & (U_1^T D_1)^{-1} C_1 \\ 0 & U_2 \end{pmatrix}.$$

Since $M_\alpha = U_\alpha^T D U_\alpha$, $M_\beta = U_\beta^T D U_\beta$, $R_\alpha = M_\alpha - A$, and $R_\beta = M_\beta - A$, by simple calculations

$$M_\alpha^{-1} R_\alpha = \begin{pmatrix} 0 & 1/20 & 7/80 & -19/320 \\ 0 & -1/20 & 3/80 & 9/320 \\ 0 & -1/10 & -3/10 & 3/20 \\ 0 & 1/30 & -1/15 & 3/40 \end{pmatrix}, \quad M_\beta^{-1} R_\beta = \begin{pmatrix} 0 & 0 & 27/160 & -11/128 \\ 0 & 0 & -5/32 & 9/128 \\ 0 & 0 & -7/20 & 3/16 \\ 0 & 0 & -1/20 & 1/16 \end{pmatrix}.$$

If we compute $\rho(M_\alpha^{-1} R_\alpha)$ and $\rho(M_\beta^{-1} R_\beta)$ using the MATLAB software,

$$\rho(M_\alpha^{-1} R_\alpha) \doteqdot 0.2418, \qquad \rho(M_\beta^{-1} R_\beta) \doteqdot 0.3259.$$

Hence, $\rho(M_\alpha^{-1} R_\alpha) < \rho(M_\beta^{-1} R_\beta)$.

EXAMPLE 3.3. Consider a symmetric 2×2 block matrix A of the form

$$A = \begin{pmatrix} B_1 & C_1 \\ C_1^T & B_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & -4 & 0 & 1 \\ 2 & 0 & -3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix},$$

where B_1, B_2, and C_1 are 2×2 square matrices. Since $\langle A \rangle$ is an M-matrix, A is an H-matrix. Let $B_1 = U_1^T D_1 U_1$ and $B_2 = U_2^T D_2 U_2$ be factorizations of B_1 and B_2, respectively. Then one obtains

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad U_\alpha = \begin{pmatrix} U_1 & C_1 \\ 0 & U_2 \end{pmatrix}.$$

Since $M = U^T D U$, $M_\alpha = U_\alpha^T D U_\alpha$, $R = M - A$, and $R_\alpha = M_\alpha - A$, by simple calculations

$$M^{-1} R = \begin{pmatrix} 0 & 0 & -2/5 & -1/10 \\ 0 & 0 & -1/5 & 1/5 \\ 3/5 & -1/10 & 0 & 0 \\ -1/5 & -3/10 & 0 & 0 \end{pmatrix}, \quad M_\alpha^{-1} R_\alpha = \begin{pmatrix} 0 & 7/50 & 28/125 & 2/125 \\ 0 & 1/50 & -21/125 & -3/250 \\ 0 & -3/10 & -7/25 & -1/50 \\ 0 & 1/10 & 4/25 & -3/50 \end{pmatrix}.$$

If we compute $\rho(M^{-1}R)$ and $\rho(M_\alpha^{-1}R_\alpha)$ using the MATLAB software,

$$\rho(M^{-1}R) \doteqdot 0.3761, \qquad \rho(M_\alpha^{-1}R_\alpha) \doteqdot 0.3864.$$

Hence, $\rho(M^{-1}R) < \rho(M_\alpha^{-1}R_\alpha)$.

Next, we consider block incomplete factorization preconditioners for a symmetric block-tridiagonal H-matrix of the general form (2). Generalization of Theorem 3.1 to an H-matrix of the form (2) is not difficult, so that the following theorem is described without proof. For simplicity of exposition, let $(Q)_{ij}$ denote the (i,j)-block component of the block matrix Q.

THEOREM 3.4. *Let A be a symmetric block-tridiagonal H-matrix of the form (2). For each $i = 1, 2, \ldots, m$, let $B_i = U_i^T D_i U_i - R_i$ and $\langle B_i \rangle = \tilde{U}_i^T \tilde{D}_i \tilde{U}_i - \tilde{R}_i$ be the incomplete factorizations of B_i and $\langle B_i \rangle$, respectively. Let*

$$D = \text{blockdiag}(D_1, D_2, \ldots, D_m), \quad U = \text{blockdiag}(U_1, U_2, \ldots, U_m),$$

$$U_\alpha = [(U_\alpha)_{ij}], \ (U_\alpha)_{ij} = \begin{cases} U_i & \text{if } i = j \\ C_i & \text{if } j = i+1 \text{ and } 1 \le i \le m-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$U_\beta = [(U_\beta)_{ij}], \ (U_\beta)_{ij} = \begin{cases} U_i & \text{if } i = j \\ (U_i^T D_i)^{-1} C_i & \text{if } j = i+1 \text{ and } 1 \le i \le m-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{D} = \text{blockdiag}(\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_m), \quad \tilde{U} = \text{blockdiag}(\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_m),$$

$$\tilde{U}_\alpha = [(\tilde{U}_\alpha)_{ij}], \ (\tilde{U}_\alpha)_{ij} = \begin{cases} \tilde{U}_i & \text{if } i = j \\ -|C_i| & \text{if } j = i+1 \text{ and } 1 \le i \le m-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{U}_\beta = [(\tilde{U}_\beta)_{ij}], \ (\tilde{U}_\beta)_{ij} = \begin{cases} \tilde{U}_i & \text{if } i = j \\ -(\tilde{U}_i^T \tilde{D}_i)^{-1} |C_i| & \text{if } j = i+1 \text{ and } 1 \le i \le m-1 \\ 0 & \text{otherwise.} \end{cases}$$

If we let

$$M = U^T D U, \qquad M_\alpha = U_\alpha^T D U_\alpha, \qquad M_\beta = U_\beta^T D U_\beta,$$
$$R = M - A, \qquad R_\alpha = M_\alpha - A, \qquad R_\beta = M_\beta - A,$$

$$\tilde{M} = \tilde{U}^T \tilde{D} \tilde{U}, \qquad \tilde{M}_\alpha = \tilde{U}_\alpha^T \tilde{D} \tilde{U}_\alpha, \qquad \tilde{M}_\beta = \tilde{U}_\beta^T \tilde{D} \tilde{U}_\beta,$$
$$\tilde{R} = \tilde{M} - \langle A \rangle, \qquad \tilde{R}_\alpha = \tilde{M}_\alpha - \langle A \rangle, \qquad \tilde{R}_\beta = \tilde{M}_\beta - \langle A \rangle,$$

then each of the following holds:

(a) $|U^{-1}| \le \tilde{U}^{-1}$, $|U_\alpha^{-1}| \le \tilde{U}_\alpha^{-1}$, $|U_\beta^{-1}| \le \tilde{U}_\beta^{-1}$,
(b) $|(U^T D)^{-1}| \le (\tilde{U}^T \tilde{D})^{-1}$, $|(U_\alpha^T D)^{-1}| \le (\tilde{U}_\alpha^T \tilde{D})^{-1}$, $|(U_\beta^T D)^{-1}| \le (\tilde{U}_\beta^T \tilde{D})^{-1}$,
(c) $|R| \le \tilde{R}$, $|R_\alpha| \le \tilde{R}_\alpha$, $|R_\beta| \le \tilde{R}_\beta$,
(d) $\rho(M^{-1}R) \le \rho(\tilde{M}^{-1}\tilde{R})$, $\rho(M_\alpha^{-1}R_\alpha) \le \rho(\tilde{M}_\alpha^{-1}\tilde{R}_\alpha)$, $\rho(M_\beta^{-1}R_\beta) \le \rho(\tilde{M}_\beta^{-1}\tilde{R}_\beta)$,
(e) $\rho(\tilde{M}_\beta^{-1}\tilde{R}_\beta) \le \rho(\tilde{M}_\alpha^{-1}\tilde{R}_\alpha) \le \rho(\tilde{M}^{-1}\tilde{R}) < 1$.

Since U_i's can be computed independently of one another, three types of the block incomplete factorization preconditioners M, M_α, and M_β presented in Theorem 3.4 can be computed *in parallel*. This inherent parallelism is a big advantage of three types of the block incomplete factorization preconditioners. The PCG method is used to test the effectiveness of the block preconditioners in Theorem 3.4, so the PCG algorithm with a preconditioner K is described below. Here, A and K are assumed to be a symmetric (positive or negative) definite matrix.

ALGORITHM: PCG (PRECONDITIONED CG)

Choose x_0 and compute $r_0 = b - Ax_0$

Solve $K\omega_0 = r_0$ and set $p_0 = \omega_0$

For $i = 0, 1, 2, \ldots$

$\alpha_i = (r_i, \omega_i)/(p_i, Ap_i)$

$x_{i+1} = x_i + \alpha_i p_i$

$r_{i+1} = r_i - \alpha_i A p_i$

If $\|r_{i+1}\|_2/\|b\|_2 <$ (Tolerance), stop

Solve $K\omega_{i+1} = r_{i+1}$

$\beta_i = (r_{i+1}, \omega_{i+1})/(r_i, \omega_i)$

$p_{i+1} = \omega_{i+1} + \beta_i p_i$

4. Construction of Block incomplete factorization preconditioners

The construction of three types of the block incomplete factorization preconditioners presented in Theorem 3.4 will be considered in this section for a special type of H-matrix A whose structure is of the form (2) with B_i's tridiagonal matrices and C_i's diagonal matrices. This type of matrix A arises from five-point discretization of the following elliptic second-order PDE (partial differential equation)

$$(4) \qquad (a(x,y)u_x(x,y))_x + (b(x,y)u_y(x,y))_y - c(x,y)u(x,y) = f(x,y)$$

with $a(x,y) > 0$, $b(x,y) > 0$, $c(x,y) \geq 0$, and $(x,y) \in \Omega$, where Ω is a square region, and with suitable boundary conditions on $\partial\Omega$ which denotes the boundary of Ω.

For simplicity, the block incomplete factorization preconditioners described in Theorem 3.4 were constructed based on the incomplete factorizations of 1×1 block submatrices B_i. These ideas can be generalized to the block incomplete factorization preconditioners based on the incomplete factorizations of $k \times k$ block submatrices, which are from now on called *k-block incomplete factorization preconditioners*.

We just describe how to construct 2-block incomplete factorization preconditioners for 4×4 block-tridiagonal matrix A of the form (2) since these ideas can be easily extended to the construction of general k-block incomplete factorization preconditioners for $m \times m$ block-tridiagonal matrix of the form (2). Let ℓ denote the order of submatrices B_i and C_i. First, A is partitioned into

$$A = \begin{pmatrix} \mathcal{B}_1 & \mathcal{C}_1 \\ \mathcal{C}_1^T & \mathcal{B}_2 \end{pmatrix}$$

where $\mathcal{B}_1 = \begin{pmatrix} B_1 & C_1 \\ C_1^T & B_2 \end{pmatrix}$, $\mathcal{B}_2 = \begin{pmatrix} B_3 & C_3 \\ C_3^T & B_4 \end{pmatrix}$, and $\mathcal{C}_1 = \begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}$. Since A is assumed to be a symmetric H-matrix, from Lemma 2.2 $\mathcal{B}_i$'s are also symmetric H-matrices. It follows from Theorem 2.3 that the incomplete factorization of $\mathcal{B}_i$ exists. For each $i = 1, 2$, let $\mathcal{B}_i = \mathcal{U}_{ij}^T \mathcal{D}_{ij} \mathcal{U}_{ij} - \mathcal{R}_{ij}$ be the incomplete factorization of $\mathcal{B}_i$, where $0 \leq j \leq \ell - 1$, and the nonzero structures of $\mathcal{U}_{ij}$'s for $\ell = 7$ are illustrated in Fig. 1.

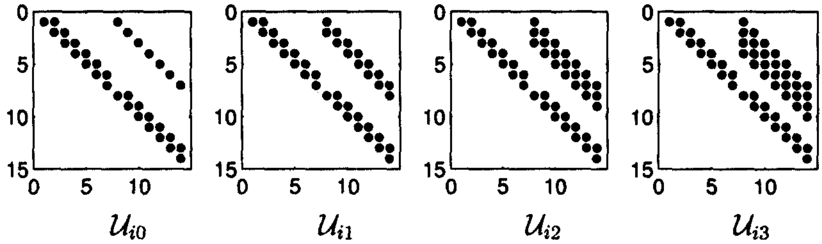

Fig. 1. Nonzero structures of $\mathcal{U}_{ij}$'s

If we let for each $0 \le j \le \ell - 1$

$$\mathcal{D}_j^2 = \begin{pmatrix} \mathcal{D}_{1j} & 0 \\ 0 & \mathcal{D}_{2j} \end{pmatrix}, \quad \mathcal{U}_j^2 = \begin{pmatrix} \mathcal{U}_{1j} & 0 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

$$(\mathcal{U}_\alpha)_j^2 = \begin{pmatrix} \mathcal{U}_{1j} & \mathcal{C}_1 \\ 0 & \mathcal{U}_{2j} \end{pmatrix}, \quad (\mathcal{U}_\beta)_j^2 = \begin{pmatrix} \mathcal{U}_{1j} & (\mathcal{U}_{1j}^T \mathcal{D}_{1j})^{-1} \mathcal{C}_1 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

then $M_j^2 = (U_j^2)^T \mathcal{D}_j^2 U_j^2$, $(M_\alpha)_j^2 = ((\mathcal{U}_\alpha)_j^2)^T \mathcal{D}_j^2 (\mathcal{U}_\alpha)_j^2$, and $(M_\beta)_j^2 = ((\mathcal{U}_\beta)_j^2)^T \mathcal{D}_j^2 (\mathcal{U}_\beta)_j^2$ are 2-block incomplete factorization preconditioners of types M, M_α, and M_β respectively, where the superscript 2 is used to represent 2-block preconditioners. From Fig. 1, the nonzero structures of $\mathcal{U}_j^2$, $(\mathcal{U}_\alpha)_j^2$, and $(\mathcal{U}_\beta)_j^2$ for $\ell = 7$ are illustrated in Fig. 2 to 4.

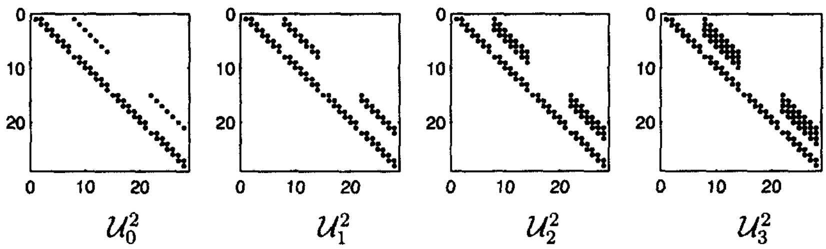

Fig. 2. Nonzero structures of $\mathcal{U}_j^2$'s

From Fig. 4, it can be seen that $(M_\beta)_j^2$ has much more fill-ins than other block preconditioners even if j is small. In the similar way as was done for 2-block preconditioners, k-block preconditioners M_j^k, $(M_\alpha)_j^k$, and $(M_\beta)_j^k$ can be easily constructed. Notice that $(M_\alpha)_j^1 = (M_\alpha)_0^1$ and

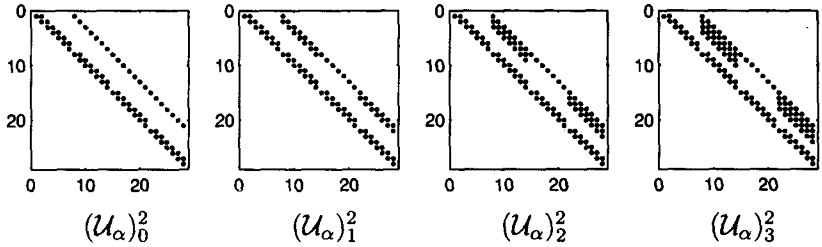

Fig. 3. Nonzero structures of $(\mathcal{U}_\alpha)_j^2$'s

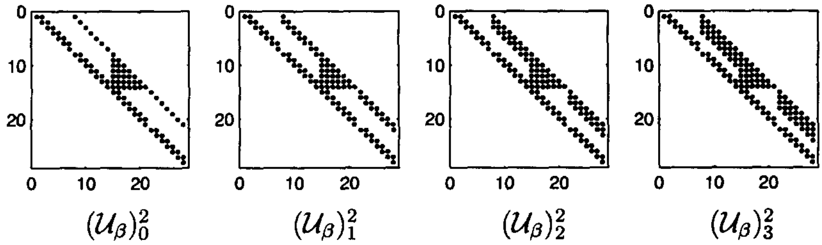

Fig. 4. Nonzero structures of $(\mathcal{U}_\beta)_j^2$'s

$(M_\beta)_j^1 = (M_\beta)_0^1$ for all $j = 0, 1, \ldots, \ell-1$ since B_i's are tridiagonal matrices and thus the complete factorizations of B_i's have no fill-in elements.

5. Numerical results

In this section, we provide numerical results of the PCG method using the k-block incomplete factorization preconditioners proposed in this paper for solving $Ax = b$ with the special type of matrix A described in section 4. For each type of block preconditioner, numerical experiments are carried out for $0 \leq j \leq 2$ and various values of k. To evaluate the effectiveness of the k-block incomplete factorization preconditioners, we also provide numerical results of the PCG method using the standard incomplete factorization preconditioner which is called $IC(0)$ *preconditioner*. In all cases, the PCG was started with $x_0 = 0$ and it was stopped when $\|r_i\|_2/\|b\|_2 < 10^{-8}$, where $\|\cdot\|_2$ refers to L_2-norm. All numerical experiments have been carried out on the Cray C90 supercomputer using

64-bit arithmetic. In Tables 1 and 2, *ITER* refers to the number of iterations satisfying the stopping criterion mentioned above, *P-time* refers to the CPU time to compute block preconditioners, and *I-time* refers to the CPU time for the PCG with these preconditioners. All CPU times are measured in seconds.

The k-block preconditioner M_j^k requires less storage than other types of block preconditioners (see Fig. 2 to 4), but its convergence rate is much worse than the standard $IC(0)$ preconditioner and hence CPU time of the PCG using M_j^k is larger than that of the PCG using the $IC(0)$. Here, the convergence rate is measured by the number *ITER*. The k-block preconditioner $(M_\beta)_j^k$ requires more storage than other types (see Fig. 2 to 4) and its convergence rate is better than other types, but CPU time of the PCG using this preconditioner is larger than that of the PCG using the $IC(0)$ because of more computational costs per iteration for $(M_\beta)_j^k$. For this reason, numerical results for M_j^k and $(M_\beta)_j^k$ are not provided. Only the discretized matrix A is of importance, so the right-hand side vector b is created artificially. Therefore, the right-hand side function $f(x, y)$ in equation (4) is not relevant.

EXAMPLE 5.1. We consider equation (4) over the square region $\Omega = (0,1) \times (0,1)$ with $a(x,y) = b(x,y) = \cos x$, $c(x,y) = 0$, and the Dirichlet boundary condition $u(x,y) = 0$ on $\partial\Omega$. That is, the following PDE problem is considered:

$$\begin{cases} \nabla \cdot (\cos x \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We have used two uniform meshes of $\Delta x = \Delta y = \frac{1}{129}$ and $\Delta x = \Delta y = \frac{1}{241}$, which leads to two matrices of order $n = 128 \times 128$ and $n = 240 \times 240$, where Δx and Δy refer to the mesh sizes in the x-direction and y-direction, respectively. Once the matrix A is constructed from five-point discretization of the PDE, the right-hand side vector b is chosen so that the exact solution is the discretization of $10xy(1-x)(1-y)\exp(x^{4.5})$. Numerical results for this problem are listed in Table 1.

EXAMPLE 5.2. We consider equation (4) over the square region $\Omega = (0,1) \times (0,1)$ with $a(x,y) = b(x,y)$, $c(x,y) = 0$, and the boundary conditions $u = 0$ for $y = 0$, $u_x = 0$ for $x = 0$ and $x = 1$, $u_y = 0$ for $y = 1$, where

$$a(x,y) = \begin{cases} 1000 & \text{if } 0.1 < x, y < 0.9 \\ 1 & \text{otherwise.} \end{cases}$$

We have used two uniform meshes of $\Delta x = \Delta y = \frac{1}{128}$ and $\Delta x = \Delta y = \frac{1}{240}$, which leads to two matrices of order $n = 129 \times 128$ and $n = 241 \times 240$. Once the matrix A is constructed from five-point discretization of the PDE, the right-hand side vector b is chosen so that the exact solution is the discretization of $10x^2y(1-x)^2(1-y)^2\exp(x^{4.5})$. Numerical results for this problem are listed in Table 2.

TABLE 1. Numerical results of the PCG using $(M_\alpha)_j^k$ for Example 5.1

	$n = 128 \times 128$								
	$j=0$			$j=1$			$j=2$		
k	ITER	P-time	I-time	ITER	P-time	I-time	ITER	P-time	I-time
1	143	0.051	7.41						
2	138	0.070	7.14	120	0.098	6.71	117	0.124	6.86
4	135	0.079	6.99	104	0.130	5.97	96	0.189	5.97
8	133	0.083	6.87	94	0.147	5.47	83	0.220	5.29
16	132	0.086	6.82	88	0.154	5.19	75	0.238	4.85
32	132	0.087	6.82	86	0.159	5.08	72	0.245	4.70
$IC(0)$	130	0.088	6.70						
	$n = 240 \times 240$								
	$j=0$			$j=1$			$j=2$		
k	ITER	P-time	I-time	ITER	P-time	I-time	ITER	P-time	I-time
1	266	0.180	48.46						
2	257	0.247	46.96	212	0.345	40.73	205	0.445	42.49
4	250	0.279	45.64	192	0.458	38.02	178	0.674	39.08
15	244	0.302	44.10	164	0.536	38.09	139	0.844	31.72
30	244	0.307	44.42	158	0.557	33.09	132	0.875	30.38
60	244	0.309	44.37	155	0.565	31.68	128	0.891	29.58
$IC(0)$	241	0.311	43.54						

TABLE 2. Numerical results of the PCG using $(M_\alpha)_j^k$ for Example 5.2

	$n = 129 \times 128$								
	$j = 0$			$j = 1$			$j = 2$		
k	ITER	P-time	I-time	ITER	P-time	I-time	ITER	P-time	I-time
1	226	0.051	11.93						
2	217	0.070	11.49	188	0.097	10.51	182	0.126	10.82
4	211	0.079	11.11	159	0.130	9.19	148	0.191	9.33
8	208	0.083	10.94	142	0.146	8.34	124	0.223	8.04
16	207	0.086	10.88	131	0.154	7.74	112	0.239	7.37
32	204	0.087	10.74	127	0.159	7.54	105	0.247	6.95
$IC(0)$	203	0.089	10.87						

	$n = 241 \times 240$								
	$j = 0$			$j = 1$			$j = 2$		
k	ITER	P-time	I-time	ITER	P-time	I-time	ITER	P-time	I-time
1	425	0.181	78.87						
2	407	0.244	74.39	352	0.343	68.89	341	0.443	71.55
4	397	0.275	72.46	298	0.456	60.13	277	0.670	61.36
15	388	0.299	71.15	251	0.538	51.29	214	0.831	48.95
30	385	0.304	70.49	240	0.556	49.47	198	0.864	46.01
60	383	0.305	69.89	236	0.564	48.72	191	0.880	44.72
$IC(0)$	383	0.310	70.96						

6. Conclusions

We presented in this paper three types of block incomplete factorization preconditioners which can be computed in parallel. The k-block incomplete factorization preconditioner M_j^k is not recommended for use because of its poor convergence rate, and the k-block preconditioner $(M_\beta)_j^k$ is recommended only for large k since it requires much more storage and arithmetic than other types of block preconditioners for small k. The k-block preconditioner $(M_\alpha)_j^k$ yielded good performance results as compared with the standard $IC(0)$ preconditioner, so this type of block preconditioner is strongly recommended for use. Notice that the number of arithmetic operations for constructing the block incomplete factorization preconditioners grows as j becomes large. It was observed that the optimal value of j usually ranges from 1 to 3 for test problems used in this paper. Future work will include parallel implementation of the PCG using the k-block incomplete factorization preconditioners.

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