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A UNIQUE COMMON FIXED POINT THEOREM FOR A SEQUENCE OF SELF-MAPPINGS IN MENGER SPACES

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ABSTRACT. In this paper, a unique common fixed point theorem for a sequence of mutually contractive and commuting sequence of self mappings on Menger spaces has been proved.

1. Introduction

DEFINITION 1.1 (**Menger spaces [3]**). A Menger space is a triplet (M, F, t) where M is a non-empty set and F is a mapping on $M \times M$ to the set of distribution functions such that,
(a) $F_{xy}(0) = 0$ for all $x, y \in M$
(b) $F_{xy}(s) = 1$ for all $s > 0$ if and only if $x = y$
(c) $F_{xy} = F_{yx}$ for all $x, y \in M$
(d) $F_{xy}(u+v) \geq t(F_{xz}(u), F_{zy}(v))$ for all $u, v \geq 0$ and $x, y, z \in M$,
where t is t-norm, that is a function $t : [0,1] \times [0,1] \to [0,1]$ which satisfies for all $a, b, c, d \in [0,1]$,
(i) $t(1, a) = a$
(ii) $t(a, b) = t(b, a)$
(iii) $t(c, d) \geq t(a, b)$ if $c \geq a$ and $d \geq b$
(iv) $t(t(a,b), c) = t(a, t(b,c))$.
By a distribution function F, we mean a left continuous non-decreasing function from R to $[0,1]$ with

$$\inf_{x \in R} F(x) = 0 \quad \text{and} \quad \sup_{x \in R} F(x) = 1.$$

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The following are a few necessary concepts associated with the concept of convergence in a Menger space([1], [2], [3]).

A sequence $\{p_n\} \subset M$ is said to converge to a point $p \in M$ if given $\epsilon > 0$ and $\lambda > 0$, we can find a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{p_n p}(\epsilon) > 1 - \lambda \text{ for all } \quad n > N_{\epsilon,\lambda}.$$

A sequence $\{p_n\} \subset M$ is said to be a Cauchy sequence in M if given $\epsilon > 0$ and $\lambda > 0$, we can find a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{p_m p_n}(\epsilon) > 1 - \lambda \text{ for all } \quad m, n > N_{\epsilon,\lambda}.$$

A Menger space is said to be complete if every Cauchy sequence in M is convergent in M.

DEFINITION 1.2. A sequence $\{T_i\}_1^\infty$ of self-mappings on a complete Menger space is said to be mutually contractive if for $s > 0$,

$$F_{T_i x T_j y}(s) \geq F_{xy}(s/p) \tag{1.1}$$

where $x, y \in M,\ 0 < p < 1,\ i \neq j$ and $x \neq y$.

2. Main results

In this section, we prove a unique common fixed theorem for a sequence of self-mappings on Menger spaces. We also obtain a corresponding result is metric spaces.

THEOREM 2.1. *Let (M, F, t) be a complete Menger space and $\{T_i\}_1^\infty$ be a sequence of self-mappings on M such that*

(a) *T_i is continuous for all $i = 1, 2, \ldots$.*

(b) *$\{T_i\}_1^\infty$ is mutually contractive* (DEFINITION 1.2)

(c) *$T_i T_j = T_j T_i$ for all $i, j = 1, 2, \ldots$*

Then $\{T_i\}_1^\infty$ has a unique common fixed point.

Proof. Starting with any $x_0 \in M$ we construct a sequence $\{x_n\} \subset M$ as follows:

$$(2.1) \qquad x_0 \in M,\ x_1 = T_1 x_0,\ x_2 = T_2 x_1, \ldots, x_n = T_n x_{n-1}, \ldots.$$

The following cases may arise.

Case-I No two consecutive x_ns are equal.
Then for any $s > 0$,

$$(2.2) \qquad F_{x_n x_{n+1}}(s) = F_{T_n x_{n-1} T_{n+1} x_n}(s) \geq F_{x_{n-1} x_n}(s/p) \quad (\text{by} \quad (1.1)).$$

By repeated application of (2.2), we obtain

$$(2.3) \qquad F_{x_n x_{n+1}}(s) \geq F_{x_0 x_1}(s/p^n).$$

Then

(2.4)

$$\begin{aligned}
&F_{x_n x_{n+k}}(s) \\
&\geq t\left(F_{x_n x_{n+1}}\left(\frac{s}{k}\right), F_{x_{n+1}, x_{n+k}}\left(\frac{k-1}{k}s\right)\right) \\
&\geq t\left(F_{x_n x_{n+1}}\left(\frac{s}{k}\right), t\left(F_{x_{n+1}, x_{n+2}}\left(\frac{s}{k}\right), \ldots\right)\right) \\
&\geq t\Bigg(F_{x_0 x_1}\left(\frac{s}{kp^n}\right), t\left(F_{x_0 x_1}\left(\frac{s}{kp^{n-1}}\right), \ldots,\right. \\
&\qquad\qquad \left.\left. t\left(F_{x_0 x_1}\left(\frac{s}{kp^{n+k-2}}\right), F_{x_0 x_1}\left(\frac{s}{kp^{n+k-1}}\right), \ldots\right)\right)\right) \\
&\geq F_{x_0 x_1}\left(\frac{s}{kp^n}\right).
\end{aligned}$$

(2.4) implies that $\{x_n\}$ is a Cauchy sequence in M[3]. Hence it is convergent in M. Let

$$(2.5) \qquad x_n \to z \ (\text{say}) \text{ as } n \to \infty.$$

Since two consecutive terms of $\{x_n\}$ are unequal, for an arbitrary integer $i, s > 0$ and $\lambda > 0$, we can find n such that $z \neq x_{n-1}, n > i$,

$$(2.6) \qquad F_{z,x_n}(s/2) > 1 - \lambda \text{ and } F_{zx_{n-1}}(s/2) > 1 - \lambda.$$

$$\begin{aligned} F_{zT_iz}(s) &\geq t\Big(F_{zx_n}(s/2), F_{x_nT_iz}(s/2)\Big) \\ &= t\Big(F_{zx_n}(s/2), F_{T_nx_{n-1}T_iz}(s/2)\Big) \\ &\geq t\Big(F_{zx_n}(s/2), F_{x_{n-1}z}(s/2)\Big) \text{ (since } z \neq x_{n-1}) \\ &\geq 1 - \lambda. \qquad \big(\text{by (2.6)}\big) \end{aligned}$$

Since $s > 0$ and $\lambda > 0$ are arbitrary, $F_{zT_iz}(s) = 1$ for all $s > 0$, that is, $z = T_iz$ for all $i = 1, 2, \dots$.

Case-II $x_i = x_{i-1}$ for some integer i. Then $x_{i-1} = T_ix_{i-1}$.
Let

$$(2.7) \qquad z = x_{i-1} \text{ i.e. } T_iz = z.$$

Let

$$z \neq T_jz \text{ for some } j.$$

Let further

$$(2.8) \qquad z \neq T_j^nz \text{ for all } n = 1, 2, \dots .$$

Then, for $s > 0$

$$\begin{aligned} F_{zT_j^2z}(s) &= F_{T_izT_j(T_jz)}(s) \\ &\geq F_{z,T_jz}(s/p) \text{ (since } z \neq T_jz) \end{aligned}$$

$$\begin{aligned} F_{zT_j^3z}(s) &= F_{T_izT_j(T_j^2z)}(s) \\ &\geq F_{z,T_j^2z}(s/p) \\ &\geq F_{z,T_jz}(s/p^2). \text{ (since } z \neq T_j^2z) \end{aligned}$$

In this way,

$$F_{zT_j^n z}(s) \geq F_{zT_jz}(s/p^{n-1}), \ n = 2, 3, \dots . \tag{2.9}$$

where $z \neq T_j^n z$ for all $n = 1, 2, \dots$.
Making $n \to \infty$ in (2.9), we see that

$$T_j^n z \to z \ \text{ as } \ n \to \infty. \tag{2.10}$$

Again T_j is continuous, therefore,

$$T_j(T_j^n z) = T_j^{n+1} z \to T_j z \ \text{ as } \ n \to \infty. \tag{2.11}$$

But in a Menger space any sequence can converge to at most one point.
Therefore, $z = T_j z$, $j = 1, 2, \dots$.
This is a contradiction, so $z = T_j^k z$ for some k. Let k be the smallest integer with this property. Then

$$z \neq T_j^m z \ \text{ for some } \ m = 1, 2, \dots, k-1. \tag{2.12}$$

For $s > 0$,

$$\begin{aligned} &F_{T_j^{k-1}zz}(s) \\ &= F_{T_j(T_j^{k-2}z)T_iz}(s) \\ &\geq F_{T_j^{k-2}z,z}(s/p) \ \text{ (by (1.2) and (2.12))} \\ &= F_{T_j(T_j^{k-3}z)T_iz}(s/p) \\ &\geq F_{T_j^{k-3}z,z}(s/p^2) \ \text{ (by (1.1) and (2.12))} \\ &\geq \cdots \\ &\geq F_{T_jzz}(s/p^{k-2}). \end{aligned} \tag{2.13}$$

(2.12) and (2.13) show that z, $T_j z$, $T_j^2 z$, $\dots$, $T_j^{k-1} z$ are all distinct.

Then for $s > 0$,

(2.14)

$$
\begin{aligned}
F_{zT_jz}(s) &= F_{T_j^k zT_jT_iz}(s) \\
&= F_{T_j(T_j^{k-1}z)T_i(T_jz)}(s) \quad (\text{since } T_iT_j = T_jT_i) \\
&\geq F_{T_j^{k-1}z,T_jz}(s/p) \quad (\text{by (1.1) and since } T_j^{k-1}z \neq T_jz) \\
&\geq F_{T_j^{k-2}z,T_jz}(s/p^2) \\
&\geq \dots \\
&\geq F_{T_j^2 zT_jz}(s/p^{k-2}) \\
&= F_{(T_j^2T_iz)T_jz}(s/p^{k-2}) \\
&= F_{T_i(T_j^2z)T_jz}(s/p^{k-2}) \\
&\geq F_{T_j^2zz}(s/p^{k-1}) \quad (\text{since } T_j^2z \neq z) \\
&= F_{T_j(T_jz)T_iz}(s/p^{k-1}) \\
&\geq F_{T_jzz}(s/p^k) \quad (\text{since by our assumption } T_jz \neq z).
\end{aligned}
$$

(2.14) thus gives a contradiction. Hence $z = T_jz$ for all $j = 1, 2, \dots$.
Next we prove the uniqueness. If possible let z_1 and z_2 be two common fixed points such that $z_1 \neq z_2$.
Then for $s > 0$,

$$
\begin{aligned}
F_{z_1z_2}(s) &= F_{T_iz_1T_jz_2}(s) \quad (\text{here } i \neq j) \\
&\geq F_{z_1z_2}(s/p)
\end{aligned}
\tag{2.15}
$$

(2.15) is a contradiction. Therefore, $z_1 \neq z_2$, that is, the fixed point is unique.
This completes the proof of the theorem. □

We state the following result in metric space as a corollary.

COROLLARY. *Let (X, d) be a complete metric space such that the following conditions are satisfied with a sequence $\{T_i\}_1^\infty$ of self-mappings* *defined on X such that*

$$(2.16)\qquad \begin{aligned} &(a)\ T_i \text{ is continuous for all } i = 1, 2, \ldots \\ &(b)\ T_iT_j = T_jT_i \text{ for all } i, j = 1, 2, \ldots \\ \text{and}\ &(c)\ d(T_ix, T_jy) \le pd(x, y) \text{ for all } x, y \in X \end{aligned}$$

with $x \neq y$*, for all* $i, j = 1, 2, 3, \ldots$ *with* $i \neq j$ *and* $0 < p < 1$.
Then the sequence of self-mappings has a unique common fixed point.

The proof of the corollary is complete in observing that a complete metric space (X, d) may be treated as a complete Menger space if we put $F_{xy}(s) = H\big(s - d(x, y)\big)$ where H is the heaviside function and $t(a, b) = \min\{a, b\}$[3]. It may be seen that (2.16) then implies (1.1). The corollary is then proved by the application of the theorem.

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