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m-CANONICAL IDEALS IN SEMIGROUPS

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ABSTRACT. For a grading monoid S, we prove that (1) if (S, M) is a valuation semigroup, then M is an m-canonical ideal, that is, an ideal M such that $M : (M : J) = J$ for every ideal J of S. (2) if S is an integrally closed semigroup and S has a principal m-canonical ideal, then S is a valuation semigroup, and (3) if S is a completely integrally closed and S has an m-canonical ideal I, then every ideal of S is I-invertible, that is, $J + (I : J) = I$ for every ideal J of S.

1. Introduction

A non-zero subsemigroup with 0 of a torsion-free abelian (additive) group is called a *grading monoid*. Throughout this paper, S denotes a grading monoid. An *ideal* of S is a non-empty subset I of S such that $S + I \subseteq I$. An ideal I of S is *prime* if $I \neq S$ and if $x + y \in I$ implies $x \in I$ or $y \in I$ for $x, y \in S$. For $x \in S$, set $(x) = x + S$. An ideal I of S is *principal* if $I = (x)$ for some $x \in S$. Also, let $M = \{m \in S \mid m$ is a non-unit element of $S\}$. If M is a non-empty set, then M is the unique maximal ideal of S. We denotes by $q(S) = \{s - t \mid s, t \in S\}$ the quotient group of S.

A non-empty subset I of $q(S)$ is called a *fractional ideal* of S if (1) $S + I \subseteq I$ and (2) $s + I \subseteq S$ for some $s \in S$. For a fractional ideal I of S, set $I^{-1} = (S : I) = \{x \in q(S) \mid x + I \subseteq S\}$. A fractional ideal I of S is said to be *invertible* if $I + I^{-1} = S$. For a fractional ideal I $S : (S : I) = (I^{-1})^{-1}$ is defined by I_v and if $I = I_v$, then I is called *divisorial*. A fractional ideal I is said to be *principal* if $I = x + S$ for

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some $x \in q(S)$. A semigroup S is said to be *reflexive* if every ideal of S is divisorial. Note that a fractional ideal of S is invertible if and only if it is principal. Hence each invertible ideals of S is divisorial ideals.

Let I and J be fractional ideals of S. Then $I : J$ is defined to be $\{x \in q(S) \mid x + J \subseteq I\}$ and it is also a fractional ideal of S, where all colons are taken over $q(S)$, and in case $I = S$ we abide by the usual convention and say J is divisorial. We define an ideal I of S to be *m-canonical* (multiplicative canonical) if every ideal of S is I -divisorial. Hence the ideal S of S is m-canonical if and only if every ideal of S is divisorial. Hence S is reflexive if and only if S is m-canonical. In this paper, we study a semigroup version of [3]. In particular, we study m-canonical ideals in valuation semigroups.

2. General results of m-canonical ideals

In this section, we present several results concerning m-canonical ideals in semigroups.

LEMMA 2.1. ([2, Theorem 16.4]) *Let I and J be fractional ideals of a semigroup S. Then*

(1) $(u + I : J) = u + (I : J)$;
(2) $(I : u + J) = -u + (I : J)$.

LEMMA 2.2. *Let I be an m-canonical ideal of a semigroup S. Then*

(1) $(I : I) = S$;
(2) *If M is a maximal ideal of S and $I \subsetneq M$, then $I \subsetneq (I : M) \subsetneq S$ and there is no ideal properly between I and $(I : M)$;*
(3) *For each element a of S, $a + I$ is m-canonical;*
(4) *$I : (I : J) = J$ for each fractional ideal J of S;*
(5) *If $\{J_\alpha\}$ is a non-empty set of fractional ideals of S such that $\cap_\alpha J_\alpha \neq \emptyset$, then $I : (\cap_\alpha J_\alpha) = \cup_\alpha (I : J_\alpha)$; moreover it is also true in general,without assuming that I is an m-canonical ideal, then $(I : \cup_\alpha J_\alpha) = \cap_\alpha (I : J_\alpha)$;*
(6) *If J is an ideal of S with $J^{-1} = S$, then $I + J = I$ (in particular, $I \subseteq J$);*
(7) *If I is a divisorial ideal, then I is invertible;*
(8) *If I is a maximal ideal of S, then I is either invertible or $I = I + I$ and I is nonfinitely generated;*

(9) *If I is a prime ideal, then I is a maximal ideal.*

Proof. (1) Since I is m-canonical, $S = I : (I : S)$. Since $(I : S) = I$, we have $(I : I) = S$.

(4) We observe that if J is a fractional ideal of S, then $J = -s + A$ where $s \in S$ and A is an ideal of S. Hence $I : (I : J) = I : (I : -s + A) = -s + (I : (I : A)) = -s + A = J$ by Lemma 2.1.

(5) We have $(I : J_\alpha) \subseteq I : (\cap_\alpha J_\alpha)$ for each α, so that $\cup_\alpha (I : J_\alpha) \subseteq I : (\cap_\alpha J_\alpha)$. Set $J = \cup_\alpha (I : J_\alpha)$. Then we have $(I : J) \subseteq I : (I : J_\alpha) = J_\alpha$ for each α, so $(I : \cap_\alpha J_\alpha) = I : (I : J) = J$. Therefore $(I : J_\alpha) = \cup_\alpha (I : J_\alpha)$.

(8) If I is not invertible of S, then $I + I^{-1} \subsetneq S$. Thus $I + I^{-1} = I$ and so $I^{-1} = I : I = S$ by (1). Since I is m-canonical, we have $I + I = I : (I : I + I)$. We observe that $(I : I + I) = S$. Indeed, if $x + I + I \subseteq I$, then $x + I \subseteq (I : I) = S$, and hence $x \in I^{-1} = S$. Therefore $I + I = (I : S) = I$. Suppose now that I is finitely generated, say $I = (a_1, ..., a_n)$. Since $I + I = I$, each $a_i = x_i + y_i$ for some $x_i, y_i \in I$. We observe that $x_i \notin (a_i)$ and $y_i \notin (a_i)$. Indeed, if $x_i \in (a_i)$, $y_i \in (a_i)$, then $x_i = a_i + s$, $y_i = a_i + t$ for some $s, t \in S$. Hence $-(s + t) = a_i \in S$, and so $I = S$, a contradiction. If $x_i \in (a_i)$, $y_i \notin (a_i)$, then $x_i = a_i + s$, $y_i = a_j + t$ for some $s, t \in S$ and $i \neq j$. Hence $-(s + t) = a_j \in S$, and so $I = S$, a contradiction. Therefore $x_i \in (a_j)$, $y_i \in (a_k)$ for some j, k and $i \neq j$, $i \neq k$. Then $a_i = a_j + s + a_k + t$ for some $s, t \in S$. Hence $I = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$. By repeating this process, we get $I = (a_t)$ for some t. But I is not invertible, this is a contradiction. Hence I is nonfinitely generated. □

PROPOSITION 2.3. *Suppose I is an m-canonical ideal of S. Then I is not the intersection of any set of fractional ideals of S properly containing I. Thus I admits a unique minimal proper fractional overideal.*

Proof. Let $\{J_\alpha\}$ be a set of fractional ideals of S such that $I \subsetneq J_\alpha$ for each α. We claim that $I \neq \cap_\alpha J_\alpha$. Assume otherwise, then $S = (I : I) = (I : \cap_\alpha J_\alpha) = \cup_\alpha (I : J_\alpha)$, by Lemma 2.2. Since $0 \in S$, $0 \in (I : J_\alpha)$ for some α, and so $J_\alpha \subseteq I$, a contradiction. □

PROPOSITION 2.4. *Suppose I is an m-canonical ideal of (S, M) such that $I \subsetneq M$. Then every fractional ideal of S properly containing I contains $(I : M)$.*

Proof. Let $\{J_\alpha\}$ be a set of fractional ideals of S such that $I \subsetneq J_\alpha$ for each α. By Proposition 2.3, we have $I \subsetneq \cap_\alpha J_\alpha \subseteq (I : M)$. It follows that $\cap_\alpha J_\alpha = (I : M)$. Hence $(I : M) \subseteq J_\alpha$ for each α. □

PROPOSITION 2.5. *Let* (S, M) *be a* Noetherian *semigroup. If* S *has an* m*-canonical ideal, then* $dim(S) \leq 1$.

Proof. Assume $dim(S) \geq 2$. Then $ht(M) \geq 2$. Let P, Q be distinct height-one prime ideals of S and choose $a \in P \cap Q$. If I is an m-canonical ideal of S, then $a + I$ is an m-canonical ideal of S. By Proposition 2.3, $a+I$ is irreducible. Since S is Noetherian, it follows that $a+I$ is primary. This is contradiction, as $a+I$ has more than one minimal prime ideal. □

PROPOSITION 2.6. *If* I *is an* m*-canonical ideal of* (S, M)*, then* $I+S_M$ *is an* m*-canonical ideal of* S_M*, where* $S_M = S_{S-M} = \{s - u \mid s \in S, u \in S - M\}$.

Proof. We note that if J is an ideal of S, then $J = J+S_M$. Let $J+S_M$ be any ideal of S_M, where J is an ideal of S. Then $I + S_M : (I + S_M : J + S_M) = I : (I : J) = J = J + S_M$. Hence $I + S_M$ is an m-canonical ideal of S_M. □

REMARK 2.7. If I and J are ideals of a semigroup (S, M), then $(I : J) + S_M = (I + S_M : J + S_M)$.

The following proposition is an analogy for semigroups of [3, Proposition 5.1].

PROPOSITION 2.8. *Let* $S \subseteq T \subseteq q(S)$*. Assume* $(S : T) \neq \emptyset$*. If* I *is an* m*-canonical ideal of* S*, then* $(I : T)$ *is an* m*-canonical ideal of* T.

3. Connections with star-operations

Let $F(S)$ be the set of all fractional ideals of S. We recall that a star-operation on S is mapping $J \to J^*$ of $F(S)$ into $F(S)$ satisfying the following conditions for all $J, L \in F(S)$ and $0 \neq u \in q(S)$:

(1) $(u)^* = (u)$ and $(u + J)^* = u + J^*$.

(2) $J \subseteq J^*$, and $J^* \subseteq L^*$ if $J \subseteq L$.
(3) $(J^*)^* = J^*$.

Consider the mapping $J \to I : (I : J)$, where I is an ideal of S with $(I : I) = S$ and J is a fractional ideal of S. We prove in Proposition 3.2 that this mapping is a star-operation. First, we need a lemma.

LEMMA 3.1. *Let I be an ideal of S and J a fractional ideal of S. Then $I : (I : J) = \cap_u I + u$, where $u \in q(S)$ and $J \subseteq I + u$.*

Proof. Suppose $x \in I : (I : J)$ and $J \subseteq I + u$. Then $x + (I : J) \subseteq I$ and $-u \in (I : J)$, so $x + (-u) \in I$. Hence $x \in I + u$. Conversely, suppose $x \in \cap_u I + u$ for all $u \in q(S)$ such that $J \subseteq I + u$. We need to show $x + (I : J) \subseteq I$. Note that $-u \in (I : J) \Leftrightarrow -u + J \subseteq I \Leftrightarrow J \subseteq I + u$. Hence $x \in I + u$ implies $x + (-u) \in I$ as required. □

PROPOSITION 3.2. *Let I be an ideal of S with $(I : I) = S$. Then the mapping $J \to I : (I : J)$ is a star-operation on S.*

Proof. Conditions (1) and (2) of the definition of a star-operation are easily verified. We show the condition (3). Since $(I : I) = S$, we have $I^* = I : (I : I) = I : S = I$. This implies that $J^* \subseteq (I + u)^* = I^* + u = I + u$, and so $J^* = (J^*)^*$ by Lemma 3.1. □

COROLLARY 3.3. *Let I be an ideal of S and let f denote the mapping on $F(S)$ defined by $f(J) = (I : J)$. Then the following conditions are equivalent:*

(1) *I is m-canonical;*
(2) *f is one-to-one;*
(3) *f is onto.*

The next corollary helps make the search for m-canonical ideals tractable.

COROLLARY 3.4. *Let I be an ideal of S with $(I : I) = S$. If J is a divisorial fractional ideal, then J is I-divisorial.*

REMARK 3.5. If I is an m-canonical ideal of S, then the mapping $J \to J^{*_I} = I : (I : J)$ is a star-operation on S with finite character, where $J \in F(S)$, in fact, $*_I = d$, where d is the identity star-operation on S. Indeed, since $(I : I) = S$, $*_I$ is a star-operation on S. Since I is m-canonical, we have $J^{*_I} = I : (I : J) = J$. Therefore $*_I = d$ and $*_I$ is of finite character.

LEMMA 3.6. *Let I be an ideal of S with $(I : I) = S$ and define $J^* = I : (I : J)$ for each $J \in F(S)$. If I is divisorial, i.e., $I = I_v$, then $J^* = J_v$ for every $J \in F(S)$.*

Proof. Let J be an ideal of S and recall that $J^* = \cap_u I + u$, where the intersection is over all $u \in q(S)$ such that $J \subseteq I + u$ by Lemma 3.1. Since I is divisorial, $(I+u)_v = I_v + u = I + u$ [2, Theorem 16.4(6)], and so J^* is divisorial. Therefore $J_v \subseteq J^*$. Since the map $J \to J^*$ is a star-operation, we also have $J^* \subseteq J_v$, so $J^* = J_v$. □

PROPOSITION 3.7. *The following statements are equivalent:*
(1) *Each ideal of S is divisorial;*
(2) *S has a principal m-canonical ideal;*
(3) *S has an invertible m-canonical ideal;*
(4) *S has a divisorial m-canonical ideal.*

Proof. It suffices to prove the implication (4) ⇒ (1). Let I be an m-canonical ideal such that $I = I_v$. Hence, $J = I : (I : J) = J^* = J_v$ by Lemma 2.2(1) and Lemma 3.6. □

With corollary 3.4 in mind, it is natural to make the following definition.

DEFINITION 3.8. Let I be an ideal of S and $J \in F(S)$. Then J is said to be I*- invertible* if $J + (I : J) = I$.

REMARK 3.9. (1) If J is invertible, then J is I-invertible. To see this, write $J = x + S$ for some $x \in S$. Then $(I : J) = (I : x + S) = -x + (I : S) = -x + I$ by Lemma 2.1. Since $x + I \subseteq J + I$, we have $I \subseteq J + (-x) + I = J + (I : J)$. The reverse inclusion is clear. So $J + (I : J) = I$.

(2) If I is a principal ideal of S, then I-invertible implies invertible. Hence I-invertible implies I-divisorial.

We recall that an element $x \in q(S)$ is called *almost integral* over S if there exists $a \in S$ such that $a+nx \in S$ for all $n \geq 1$. S is called *completely integrally closed* if every almost integral element over S belongs to S.

PROPOSITION 3.10. *Let S be a completely integrally closed semigroup. If I is an m-canonical ideal of S, then every ideal of S is I-invertible.*

Proof. Let J be an ideal of S and set $L = J + J^{-1}$. Since S is completely integrally closed, $L^{-1} = S$. By Lemma 2.2 part (6), we have $I + L = I$. Since $I + J^{-1} \subseteq (I : J)$, $I \subseteq J + (I : J)$. Therefore $J + (I : J) = I$. □

4. m-canonical ideals in valuation semigroups

In this section, we introduce the properties of m-canonical ideals in valuation semigroups.

PROPOSITION 4.1. *If I and J are m-canonical ideals of S, then $J = I + u$ for some $u \in q(S)$.*

Proof. We have $J = I : (I : J) = \cap_u I + u$ (Lemma 3.1), since I is m-canonical. On the other hand, J is m-canonical, so Proposition 2.3 implies that $J = I + u$ for some $u \in q(S)$. □

COROLLARY 4.2. *If I and J are m-canonical ideals of S, then J is I-invertible.*

Proof. This follows directly from Proposition 4.1, Lemma 2.1 and Lemma 2.2(1). □

Let G be a torsion-free abelian group and Γ a totally ordered abelian group, where both G and Γ are additive groups. A *valuation* $v : G \to \Gamma$ is a function such that $v(a + b) = v(a) + v(b)$ for each a,b $\in G$. Set $V = \{a \in G \mid v(a) \geq 0\}$, then V is called a *valuation semigroup.* We recall that S is a valuation semigroup if and only if either $a \in S$ or $-a \in S$ for each a $\in G$ [9, Lemma 10]. Note that if (S,M) is a valuation semigroup, then $(M : M) = S$.

LEMMA 4.3. ([8]) *Suppose (S, M) is a valuation semigroup. If M is not principal, then the set of ideals which are non-divisorials is the form $\{a + M \mid a \in S\}$.*

THEOREM 4.4. *If (S, M) is a valuation semigroup, then M is an m-canonical ideal of S.*

Proof. Let J be an ideal of S. We recall that every ideal of S is divisorial if and only if M is principal [9]. So it suffices to assume J is non-divisorial (Corollary 3.4) and M is nonfinitely generated. In this case, $J = a + M$ for some a $\in S$ by Lemma 4.3. Hence $M : (M : J) = M : (M : a + M) = a + [M : (M : M)] = a + (M : S) = a + M = J$. So J is M-divisorial. □

COROLLARY 4.5. *If (S, M) is a valuation semigroup and M is nonfinitely generated, then $M = M + M$.*

Proof. Since M is m-canonical, we have $M + M = M : (M : M + M)$. We observe that $(M : M + M) = (S : M)$. Indeed, $x + M + M \subseteq M \Leftrightarrow x + M \subseteq (M : M) = S \Leftrightarrow x \in (S : M)$. Since M is nonfinitely generated, $M_v = S$, and so $(S : M) = M^{-1} = S$. Hence $M + M = M : (M : M + M) = M : (S : M) = (M : S) = M$. □

We recall that an element $x \in q(S)$ is called *integral* over S if $nx \in S$ for some non-negative integer n. S is called *integrally closed* if every integral element over S belongs to S.

THEOREM 4.6. *Suppose S is an integrally closed semigroup. If I is principal m-canonical, then S is a valuation semigroup.*

Proof. Let $I = x + S$ for some $x \in S$, and J an ideal of S. Then $J = I : (I : J) = x + S : (x + S : J) = S : (S : J) = J_v$. Thus every ideal of S is divisorial. By [9, proposition 17], S is a valuation semigroup. □

For another simple proof, since I is principal m-canonical, every ideal of S is divisorial by Proposition 3.7. So S is a valuation semigroup [9, Proposition 17].

COROLLARY 4.7. *Let S be an integrally closed semigroup. If S is m-canonical, then S is a valuation semigroup.*

COROLLARY 4.8. *Let S be a semigroup with a principal maximal ideal M. If S is m-canonical, then S is a valuation semigroup.*

We recall that a semigroup S is *coherent* if the intersection of two finitely generated ideals of S is again finitely generated.

Following proposition is an analogy for semigroups of [3, Proposition 2.5].

PROPOSITION 4.9. *Suppose (S, M) is an integrally closed coherent semigroup. If S has an m-canonical ideal and $\cap_{n=1}^{\infty} nM = \emptyset$, then S is a valuation semigroup.*

PROPOSITION 4.10. *Let S be a completely integrally closed semigroup. If (S, M) is a valuation semigroup, then every ideal of S is M-invertible.*

Proof. Let J be an ideal of S. We may assume that J is infinitely generated, since otherwise J is invertible and thus M-invertible. Since S is completely integrally closed, $(J + J^{-1})_v = S$, and so $(J + J^{-1})^{-1} = S$. Since M is m-canonical, we have $M + (J + J^{-1}) = M$, by Lemma 2.2 part(6). Then $M \subseteq J + J^{-1} \subseteq S$. If $J + J^{-1} = S$, then J is invertible, and so J is principal, a contradiction. Thus $M = J + J^{-1}$. Hence $J + J^{-1} = J + (M : M + J) = M$. To complete the proof, we note that $M + J = M : (M : M + J) = (M : J^{-1})$, and since $J + J^{-1} = M$, we get $J \subseteq M : J^{-1} = M + J \subseteq J$. Therefore, $M + J = J$ and $J + (M : J) = M$. □

For another proof, since S is a valuation semigroup, M is an m-canonical ideal of S. Hence every ideal of S is M-invertible, by Proposition 3.7.

COROLLARY 4.11. *If (S, M) is a 1-dimensional valuation semigroup, then every ideal of S is M-invertible.*

Proof. Let $\{V_\lambda\}$ be the set of valuation oversemigroups of S and let V_λ^* be the complete integral closure for each λ. By [7, (3.11) (1)], we have $\cap_\lambda V_\lambda^* = S^*$, where S^* is the complete integral closure of S. If $M + M^{-1} = S$, then M is a finitely generated ideal of S. It follows that S is a Noetherian semigroup, and hence $S^* = S$. If $M + M^{-1} = M$, then $S^* \subseteq M^{-1} = (M : M)$. For each ideal I of S, $I : I$ is an idempotent element of $F(S)$. Therefore M^{-1} is an idempotent. By [7, (3.9)], S^* is the maximum idempotent element of $F(S)$. Hence $S^* = M^{-1}$. Since M is an m-canonical ideal of S, $(M : M) = S$. So $S^* = S$. By Proposition 4.10, every ideal of S is M-invertible. □

COROLLARY 4.12. *If S is a valuation semigroup with an nonfinitely generated maximal ideal M, then every non-divisorial ideal of S is M-invertible.*

Proof. Let J be a non-divisorial ideal of S. Then $J = a + M$ for some $a \in S$. Hence $J + (M : J) = a + M : (M : a + M) = M : (M : M) = (M : S) = M$. Therefore J is M-invertible. □

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