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TOPOLOGICAL PROPERTIES OF SOME COHOMOGENEITY ONE RIEMANNIAN MANIFOLDS OF NONPOSITIVE CURVATURE

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ABSTRACT. In this paper we study some nonpositively curved Riemannian manifolds acted on by a Lie group of isometries with principal orbits of codimension one. Among other results it is proved that if the universal covering manifold satisfies some conditions then every nonexceptional singular orbit is a totally geodesic submanifold. When M is flat and is not toruslike, it is proved that either each orbit is isometric to $R^k \times T^m$ or there is a singular orbit. If the singular orbit is unique and nonexceptional, then it is isometric to $R^k \times T^m$.

1. Introduction

Recently, cohomogeneity one Riemannian manifolds have been studied from different points of view. A. Alekseevsky and D. Alekseevsky in [1] and [2] gave a description of such manifolds in terms of Lie subgroups of a Lie group G, F. Podesta and A. Spiro in [13] got some nice results in negatively curved case, C. Searle in [14] provided a complete classification of such manifolds in dimensions less than 6 when they are compact and of positive curvature. The aim of this paper is to deal with some nonpositively curved cohomogeneity one Riemannian manifolds. We generalize some of the theorems of [13] to the case where M is a product of negatively curved manifolds. Also in section 4 we study some cohomogeneity one flat Riemannian manifolds. Our main results are theorems 3.5, 3.7, 3.10 and 4.4.

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2. Preliminaries

DEFINITION 2.0. Let M be a complete Riemannian manifold and G a Lie group of isometries which is closed in the full group of isometries of M. We say that M is of cohomogeneity one under the action of G if G has an orbit of codimension one.

It is known (see [1] and [4], [11]) that the orbit space $\Omega = \frac{M}{G}$ is a topological Hausdorff space homeomorphic to one of the following spaces: $R, S^1, R^+ = [0, +\infty)$ and $[0, 1]$. In the following we will indicate by $k : M \to \Omega$ the projection to the orbit space. Given a point $x \in M$, the orbit $D = Gx$ is called principal (resp. singular) if the corresponding image in the orbit space is an internal (resp. boundary) point of Ω, and the point x is called a regular (resp. singular) point. We say that a singular orbit is exceptional if it has codimension one. Also note that the principal orbits are diffeomorphic to each other and M is diffeomorphic to $\Omega \times D$ if $\frac{M}{G} = R$.
If G_p is the isotropy subgroup of G at $p, (p \in M)$, then G_x and G_y are conjugate if both x, y are regular, while G_x is conjugate to a subgroup of G_y if x is regular and y is singular.

DEFINITION 2.1. A (complete) geodesic γ on a Riemannian manifold of cohomogeneity one is called a normal geodesic if it crosses each orbit orthogonally.

We know (see [2]) that a geodesic γ is a normal geodesic if and only if it is orthogonal to each orbit Gx at one point $x \in \gamma$, and that each regular point belongs to a unique normal geodesic.

DEFINITION 2.2. A differentiable real valued function F on a complete Riemannian manifold M is said to be convex (resp. strictly convex) if for each geodesic $\gamma : R \to M$ the composed function $Fo\gamma : R \to R$ is convex (resp. strictly convex), that is $(Fo\gamma)'' \geq 0$ (resp. $(Fo\gamma)'' > 0$).

Let φ be an isometry of a simply connected Riemannian manifold M, the squared displacement function of φ is the function defined by $d^2_\varphi(p) = d^2(p, \varphi(p)), p \in M$, where d denotes the distance on M.

In the next proposition, we list some known properties of cohomogeneity one Riemannian manifolds, which we will use in the sequel.

PROPOSITION 2.3 ([4], [8] and [13]). *Let M be a cohomogeneity one* Riemannian *manifold under the action of a connected* Lie *group G which is closed in the full isometry group of M, then*

(a) *If M is simply connected with nonpositive curvature, there is at most singular orbit.*
(b) *If M has* nonpositive curvature *and B is the unique singular orbit of M, $\pi_1(M) = \pi_1(B)$.*
(c) *If M is simply connected no exceptional orbit may exist.*
(e) *If M is simply connected and without singular orbit, then $\Omega \neq S^1$ i.e., $\Omega = R$.*
(d) *No exceptional orbit is simply connected.*
(f) *If γ is a normal geodesic then the map $k : \gamma \longrightarrow \Omega$ is surjective and it defines a covering over the set Ω^0 of internal points of Ω. When $\Omega = R^+$* or *R, we can endow Ω with the metric given by the covering k.*

The following proposition and theorems will be needed later.

PROPOSITION 2.4(see [3]). *Let M be a simply connected and complete* Riemannian *manifold of* nonpositive *curvature, then*

(a) *If the minimum point set C of a real valued convex function F defined on M is a submanifold of M then C is totally geodesic in M, and each critical point of F belongs to C.*
(b) *d^2_φ is a convex function for each isometry φ of M and if M has negative curvature it is strictly convex except at the minimum point set C which is at most the image of a geodesic.*

THEOREM 2.5 ([15]). *Let M be a connected homogeneous* Riemannian *manifold with nonpositive curvature, then M is diffeomorphic to the product of a torus and a* Euclidean *space.*

THEOREM 2.6 ([9]). *Let M be a homogeneous* Riemannian *manifold with nonpositive curvature and negative definite* Ricci *tensor then M is simply connected.*

3. Cohomogeneity one UND manifolds

Throughout the following M will denote a complete Riemannian manifold of dimension n with nonpositive curvature and of cohomogeneity one under the action of G, a connected Lie group which is closed in the full group of isometries of M. If M is not simply connected then $\tilde{M}$ will denote the universal Riemannian covering manifold of M endowed with the pulled back metric and $\pi : \tilde{M} \to M$ will be the covering projection, with the symbol Δ we will denote the deck transformation group of the universal covering of M. We know (see [4] page 63) that the group G always admits a connected covering group $\tilde{G}$ which acts on $\tilde{M}$ by isometries and of cohomogeniety one, the projection $\tilde{\pi} : \tilde{G} \to G$ is such that $\tilde{\pi}(\tilde{g})(x) = \pi(\tilde{g}(y))$ for all $\tilde{g} \in \tilde{G}, x \in M$ and $y \in \pi^{-1}(x)$. Moreover Δ centeralizes $\tilde{G}$, so that it maps $\tilde{G}$-orbits onto $\tilde{G}$-orbits, so for each $\varphi \in \Delta, d^2_{\varphi}$ is constant along orbits.

DEFINITION 3.0. We say that a Riemannian manifold M is universally and negatively decomposable(UND) when its universal covering manifold $\tilde{M}$ decomposes as $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times ... \times \tilde{M}_k$ and for each i, $\tilde{M}_i$ has negative curvature and each $\varphi \in \Delta$ decomposes as $\varphi = \varphi_1 \times \varphi_2 \times ... \times \varphi_k$ where φ_i is an isometry of $\tilde{M}_i$.

EXAMPLE 3.0(A). If $M = M_1 \times M_2 \times ... \times M_k$ and for each i, M_i has negative curvature then M is a UND manifold.

EXAMPLE 3.0(B). If the factors of the de Rham decomposition of $\tilde{M}$ have negative curvature and $\Delta \subset I^{\circ}(\tilde{M})$ ($I^{\circ}(\tilde{M})$ is the connected component of the identity in the Lie group of isometries of $\tilde{M}$), then M is a UND manifold (see [9] vol 1 page 240).

LEMMA 3.1. *If* $M = M_1 \times M_2$ *is a complete simply connected* Riemannian *manifold of nonpositive curvature such that for a geodesic* $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ *and for an isometry* $\varphi = \varphi_1 \times \varphi_2$, $d^2_{1\varphi_1} o \gamma_1 : R \to R$ *is strictly convex, then* $d^2_{\varphi} o \gamma : R \to R$ *is a strictly convex function.*

Proof. let d, d_1 and d_2 be distance functions of M, M_1 and M_2 and $m_1 = (x_1, y_1), m_2 = (x_2, y_2)$ be two points of M. It is easy to show that $d^2(m_1, m_2) = d_1^2(x_1, x_2) + d_2^2(y_1, y_2)$, so we have $d_\varphi^2 o\gamma(t) = d^2(\gamma(t), \varphi\gamma(t)) = d_{1\varphi_1}^2 o\gamma_1(t) + d_{2\varphi_2}^2 o\gamma_2(t)$, $d_{2\varphi_2}^2 o\gamma_2$ is convex by theorem 2.4(b) and $d_{1\varphi_1}^2 o\gamma_1$ is strictly convex by assumption. Since the sum of a convex function with a strictly convex function is strictly convex, we get that $d_\varphi^2 o\gamma(t)$ is strictly convex. □

LEMMA 3.2. *If $\varphi \in \Delta$ is nontrivial and for a normal geodesic γ, $d_\varphi^2 o\gamma : R \to R$ does not have any minimum point then, φ maps each orbit $\tilde{D}$ onto itself.*

Proof. Since $F(t) = d_\varphi^2 o\gamma(t)$ does not have any minimum point and it is a convex function, for each $t_1 \neq t_2$ we have $F(t_1) \neq F(t_2)$ (since otherwise, there exists a minimum point between t_1 and t_2). Let $\tilde{D}_1 = \tilde{G}\gamma(t_1)$ and $\tilde{D}_2 = \tilde{G}\gamma(t_2)$ be two distinct orbits such that $\varphi(\tilde{D}_1) \subseteq \tilde{D}_2$, and let $\varphi(\gamma(t_1)) = x \in \tilde{D}_2$. We have $d^2(\gamma(t_1), \varphi\gamma(t_1)) = d^2(\varphi\gamma(t_1)), \varphi^2\gamma(t_1)) = d^2(x, \varphi(x)) = d^2(\gamma(t_2), \varphi\gamma(t_2))$, where the last equality comes from the fact that d_φ^2 is constant along orbits. Therefore we get that $F(t_1) = F(t_2)$, which is a contradiction. □

LEMMA 3.3. *Let M be a* UND *cohomogeneity one* Riemannian *manifold and let $\varphi \in \Delta$ be nontrivial, then there exists a normal geodesic γ on $\tilde{M}$ such that $d_\varphi^2 o\gamma : R \to R$ is a strictly convex function.*

Proof. Let $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \ldots \times \tilde{M}_k$ be the decomposition of $\tilde{M}$ and $\varphi = \varphi_1 \times \varphi_2 \times \ldots \times \varphi_k$ Without lose of generality let φ_1 be nontrivial, and let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_k(t))$ be a normal geodesic in $\tilde{M}$ such that the image of γ_1 is not the minimum point set of $d_{1\varphi_1}^2 : \tilde{M}_1 \to R$ (Since the union of the images of all normal geodesics equals to $\tilde{M}$, we can find such a γ by 2.4(b)). As $d_{1\varphi_1}^2 o\gamma_1 : R \to R$ is strictly onvex by 2.4(b), we get the result by 3.1. □

LEMMA 3.4. *Let γ be a normal geodesic in $\tilde{M}$ and $\varphi \in \Delta$ be such that $d_\varphi^2 o\gamma : R \to R$ is strictly convex and $t_1 \in R$ is not a minimum point of the function $F(t) = d_\varphi^2 o\gamma(t)$, then the orbit $\tilde{B} = \tilde{G}\gamma(t_1)$ is a hypersurface in $\tilde{M}$.*

Proof. Since d_φ^2 is constant along orbits we conclude that $\tilde{G}\gamma(t_1)$ does not have any minimum point of d_φ^2, so by 2.4(a) it does not have any critical point of d_φ^2. Since $\tilde{G}\gamma(t_1)$ is a component of $(d_\varphi^2)^{-1}(F(t_1))$ we get the result by regular value theorem. □

THEOREM 3.5. *If M is a nonsimply connected* UND *cohomogeneity one* Riemannian *manifold with only one singular orbit B, and B is not exceptional, then it is a totally geodesic submanifold of M diffeomorphic to $R^k \times T^m$ and $\pi_1(M) = Z^m$.*

Proof. First note that since dim $\pi^{-1}(B) = \dim B < n-1$, each component of $\pi^{-1}(B)$ must be a nonexceptional singular orbit in $\tilde{M}$. Therefore by 2.3(a), $\pi^{-1}(B)$ has only one component $\tilde{B}$. Now let $\varphi \in \Delta$ be a nontrivial deck transformation and γ a normal geodesic in $\tilde{M}$ such that $F = d_\varphi^2 o\gamma : R \to R$ is a strictly convex function (see 3.3), then we have two cases.

Case 1: F has only one minimum point $t_0 \in R$.

In this case since d_φ^2 is constant along orbits, we get that $\tilde{G}\gamma(t_0)$ is the minimum point set of d_φ^2, so by 2.4(a) it is a totally geodesic submanifold of $\tilde{M}$. We show that $\tilde{B} = \tilde{G}.\gamma(t_0)$. If not, then $\tilde{B} = \tilde{G}\gamma(t_1), t_1 \neq t_0$, so by 3.4 $\tilde{B}$ must be a hypersurface in $\tilde{M}$, since $dim\tilde{B} < n-1$ this is a contradiction, therefore $\tilde{B} = \tilde{G}\gamma(t_0)$ and $\tilde{B}$ is a totally geodesic submanifold of $\tilde{M}$. Consequently $B = \pi(\tilde{B})$ is totally geodesic in M, so is of nonpositive curvature. Since B is homogeneous we get by 2.5 that B is diffeomorphic to $R^k \times T^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = Z^m$.

Case 2: F has not any minimum point.
This case can not occur because by 3.4 each orbit of $\tilde{M}$ must be a hypersurface, so $\tilde{B}$ is a hypersurface, which is in contrast with the fact dim $\tilde{B} < n-1$. □

LEMMA 3.6. *If for each deck transformation $\varphi \in \Delta$ and each orbit $\tilde{D}$ in $\tilde{M}, \varphi$ maps $\tilde{D}$ onto itself and if there is no singular orbit in M, then each orbit D in M is diffeomorphic to $R^{k_1} \times T^{m_1}$.*

Proof. The proof of this lemma in given in a portion of the proof of theorem 3.7 in [4] and the sketch of the proof is as follows: for an orbit D in M, $\pi^{-1}(D)$ has only one component $\tilde{D}$ and $\tilde{D} = \frac{\tilde{G}}{\tilde{K}}$ with $\tilde{K}$ maximal compact in $\tilde{G}$. So there is a solvable subgroup H acting transitively on $\tilde{D}$. Since $D = \frac{\tilde{D}}{\Delta}$ and Δ centeralizes $\tilde{G}$ (and hence H too). we obtain that H acts transitively on D, so D is a solvmanifold and diffeomorphic to a product $R^{k_1} \times T^{m_1}$ (see [13] page 76 and [6]). □

THEOREM 3.7. *If M is a nonsimply connected UND cohomogeneity one* Riemannian *manifold without any singular orbit, then each orbit is diffeomorphic to $R^{k_1} \times T^{m_1}$. In this case if $\frac{M}{G} = R$, then M is diffeomorphic to $R^k \times T^m$, $k = k_1 + 1$.*

Proof. By 3.3 for each nontrivial $\varphi \in \Delta$, there is a normal geodesic γ (related to φ) such that $d_\varphi^2 o\gamma$ is a strictly convex function. We have two cases.

Case 1: There exists a $\varphi \in \Delta$ such that $d_\varphi^2 o\gamma$ has a minimum point $t_0 \in R$.

In this case the orbit $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of the function d_φ^2. Therefore by 2.4 (a) it is totally geodesic and so $B = \pi(\tilde{B})$ is totally geodesic in M, hence is of nonpositive curvature. Since B is homogeneous, it is diffeomorphic to $R^{k_1} \times T^{m_1}$ by 2.5. From the fact that the (principal) orbits are diffeomorphic we get that each orbit is diffeomorphic to $R^{k_1} \times T^{m_1}$.

Case 2: For each nontrivial $\varphi \in \Delta, d_\varphi^2 o\gamma$ does not have any minimum point.

In this case by 3.2, φ maps each orbit $\tilde{D}$ onto itself. Therefore by 3.6 each orbit D in M is diffeomorphic to $R^{k_1} \times T^{m_1}$.

If $\frac{M}{G} = R$, from the fact that M is diffeomorphic to $\frac{M}{G} \times D$ we get that M is diffeomorphic to $R \times R^{k_1} \times T^{m_1} = R^k \times T^{m_1}$. □

LEMMA 3.8. *Let $M = M_1 \times M_2$ and $X = X_1 + X_2, Z$ be two vectors at the point $p = (p_1, p_2)$ such that X_1, Z are tangent to M_1 and X_2 is tangent to M_2, then $K_M(X, Z) = K_{M_1}(X_1, Z)$.*

Proof. $M = M_1 \times M_2$ is a warped product by the warping function $f = 1$, so by using theorem 42 in [12] page 210 we have $R_X ZX =$

$R_{X_1+X_2}Z(X_1+X_2) = R_{X_1}ZX_1+R_{X_1}ZX_2+R_{X_2}ZX_1+R_{X_2}ZX_2 = R_{X_1}ZX_1 \in T_{P_1}M_1$. Therefore $K_M(X,Z) = K_{M_1}(X_1,Z)$. □

LEMMA 3.9. *If $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times ... \times \tilde{M}_k$, where for each i, $\tilde{M}_i$ is negatively curved with dim $\tilde{M}_i \geq 3$, then each totally geodesic hypersurface S of $\tilde{M}$ has negative definite Ricci tensor.*

Proof. Let X be a unit vector tangent to S at a point p,and let Y be another unit vector normal to S at this point. We have $X = X_1 + X_2 + ... + X_k, Y = Y_1 + Y_2 + ... + Y_k$, where for each i, X_i, Y_i are tangent to $\tilde{M}_i$. Without lose of generality, let $X_1 \neq 0$. Since dim $\tilde{M}_1 \geq 3$, there is a unit vector Z_1 which is tangent to $\tilde{M}_1$ and normal to X_1, Y_1. Now consider a frame $\{E_1, E_1', E_2, ..., E_{n-1}\}$ for $T_p\tilde{M}$ where $E_1 = X, E_1' = Y, E_2 = Z_1$, since $E_1'(= Y)$ is normal to S we get that $\{E_1, E_2, ..., E_{n-1}\}$ is a frame for T_pS, and we have the following relations.
(1) $\text{Ric}_S(X,X) = \sum_{i\geq 2} K_S(X,E_i)$, see [12] page 88.
(2) $K_S(X,E_i) = K_{\tilde{M}}(X,E_i)$, since S is totally geodesic in $\tilde{M}$.
(3) $K_{\tilde{M}}(X,E_i) \leq 0$.
(4) $K_{\tilde{M}}(X,E_2) = K_{\tilde{M}_1}(X_1,Z_1)$, this is a consequence of the lemma 3.8.
(5) $K_{\tilde{M}_1}(X_1,Z_1) < 0$, since $\tilde{M}_1$ is negatively curved.
By using this relations we get that $\text{Ric}_S(X,X) < 0$, so S has negative definite Ricci tensor. □

THEOREM 3.10. *If M is a nonsimply connected UND cohomogeneity one* Riemannian *manifold and $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times ... \times \tilde{M}_k$, where for each i, dim$\tilde{M}_i \geq 3$, then*
(a) *There is at most one singular orbit.*
(b) *If there is a singular orbit B, it is nonexceptional and diffeomorphic to $R^{K_1} \times T^{m_1}$ and $\pi_1(M) = Z^{m_1}$.*

Proof. (a): We prove the theorem in two steps.
Step 1: M does not have two exceptional singular orbits.
Proof: If M has two exceptional singualr orbits, then the dimension of each orbit of M (and so the dimension of each orbit of $\tilde{M}$) is $n-1$, so by 2.3 (c), (e), $\tilde{M}$ does not have any singualr orbit and $\frac{\tilde{M}}{\tilde{G}} = R$. Therefore each normal geodesic γ in $\tilde{M}$ intersects an orbit $\tilde{D}$ exactly once. But

since $\frac{M}{G} = [0,1]$, the normal geodesic $\pi o \gamma$ intersects a principal orbit D in M infinitely many times, so π^{-1}(D) has more than one connected component. Therefore if $\tilde{D}$ is a component of π^{-1} (D), there exist a nontrivial $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$ thus by lemmas 3.2, 3.3, for a normal geodesic $\gamma, d_{\varphi}^2 o \gamma$ is strictly convex with a minimum point $t_0 \in R$, and since d_{φ}^2 is constant along orbits, $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of d_{φ}^2. So it is totally geodesic by 2.4 (a). Now since each factor of the decomposition of $\tilde{M}$ is negatively curved with dim $\tilde{M}_i \geq 3$, we get by 3.9 that every totally geodesic hypersurface of $\tilde{M}$ has negative definite Ricci tensor, so $\tilde{B}$ (hence $B = \pi(\tilde{B})$) has negative definite Ricci tensor, thus by 2.6, B is simply connected. Since dim $B = n - 1$, we get by 2.3 (d) that B is not a singular orbit. As B is simply connected, $B = \frac{G}{K}(K = G_x, x \in B)$, where K is maximal compact subgroup of G (See [10] vol II page 112), which is in contrast with the fact that there exists singular orbit.

Step 2: M does not have two singular orbits, at least one orbit nonexceptional.

Proof: Let B_1 be a nonexceptional singular orbit of M then $\tilde{B} = \pi^{-1}(B_1)$ is the unique singular orbirt of $\tilde{M}$. Because of dimensional reasons for each $\varphi \in \Delta$ we have $\varphi(\tilde{B}) = \tilde{B}$. The isometry φ induces an isometry φ^* on the orbit space R^+ of $\tilde{M}$ such that for each orbit $\tilde{D}$ we have $\varphi^*(k(\tilde{D}) = k(\varphi(\tilde{D}))$. Since $\varphi(\tilde{B}) = \tilde{B}$, we get that $\varphi^*(0) = \varphi^*(k(\tilde{B})) = k\varphi(\tilde{B}) = k(\tilde{B}) = 0$, so for each $t \in R^+$ we have $\varphi^*(t) = t$. Thus $\varphi(\tilde{D}) = \tilde{D}$. Now we have a contradiction because a normal geodesic γ in $\tilde{M}$ intersects each principal orbit in two points ($\frac{\tilde{M}}{\tilde{G}} = R^+$) while $\pi o \gamma$ intersects a principal orbit infinitely many times ($\frac{M}{G} = [0,1]$). So there exists $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$.

(b): We need only to show that B can not be an exceptional orbit, the other parts of the claim is a simple consequence of theorem 3.5. To prove the claim observe that if it were the case, $\tilde{M}$ would admit only principal orbits and a normal geodesic intersects each orbit in $\tilde{M}$ exactly in one point while since $\frac{M}{G} = R^+$, a normal geodesic in M intersects each principal orbit in two points, and a contradiction arises as in the step 1 of the proof of (a). □

4. Cohomogeneity one flat manifolds

In this section we study cohomogeneity one flat Riemannian manifolds which are not toruslike.

It is known that every isometry $\varphi \in Iso(R^n)$ is of the form $\varphi = (A, b), A \in O(n), b \in R^n$, that is, $\varphi(x) = Ax + b$, $x \in R^n$. We say that φ is an ordinary translation when $A = I$ (I is the identity map on R^n).

Note that R^n is the universal Riemannian covering manifold of each flat manifold M of dimension n.

DEFINITION 4.1. We say that a flat Riemannian manifold M is "toruslike" if each deck transformation of the universal covering manifold of M is an ordinary translation.

In the following $V.W$ denotes the inner product of the vectors V and W in R^n and $|V|$ is the length of V.

LEMMA 4.2. *Let R^n be of cohomogeneity one under the action of a closed Lie subgroup $G \subset Iso(R^n)$ and let $\varphi = (A, b) \in G, A \neq I$. Then there is a normal geodesic γ on R^n such that the function $F(t) = d^2_{\varphi o}\gamma(t)$ is a strictly convex function with the minimum point $t_0 \in R$.*

Proof. If for each normal geodesic γ, φ acts as ordinary translation on the image of γ then by the fact that each point of R^n belongs to a normal geodesic, we get that φ acts as an ordinary translation on R^n, which is in contrast with the fact that $A \neq I$. So there is a normal geodesic $\gamma(t) = at + c$ $(a, c \in R^n)$ such that the action of φ on the image of γ is not an ordinary translation. Let $F(t) = d^2_{\varphi o}\gamma(t) = |(A, b)(at + c) - (at + c)|^2$. It is easy to see that

$$F'(t) = 2(A - I)a.[(A - I)(at + c) + b]$$

and

$$F''(t) = |2(A - I)a|^2$$

we have $(A - I)a \neq 0$ (because if not then $Aa = a$ and (A, b) acts as ordinary translation on γ), so we have $F''(t) > 0$. Therefore F is a strictly convex function. Now let t_0 be such that $F'(t_0) = 0$ (since $(A - I)a \neq 0$ such a t_0 exists), so t_0 is the minimum point of F. □

LEMMA 4.3. *If R^n is of cohomogeneity one under the action of a closed* Lie *subgroup G of $Iso(R^n)$ and if all the orbits are regular and one orbit is isometric to R^{n-1}, then other orbits are isometric to R^{n-1}.*

Proof. Let D_1 be an orbit which is isometric to R^{n-1} and D_2 be another orbit, also let $L = \gamma(t)$ be a normal geodesic (so L is a line orthogonal to the orbits). As the group G acts transitively on normal geodesics, for each $x \in D_2$ there exists a $g \in G$ such that gL is another line orthogonal to the orbits, and intersects D_2 at x. Since L and gL are orthogonal to D_1 they are parallel. Also D_2 is a hypersurface in R^n where at each point $x \in D_2$ there exists a line gL normal to D_2 at x and parallel to L. Thus D_2 is a hyperplane ($\simeq R^{n-1}$). □

THEOREM 4.4. *If M is a flat cohomogeneity one* Riemannian *manifold under the action of a closed* Lie *group $G \subset Iso(M)$ and M is not toruslike, then*
(a) *Either each orbit D of M is isometric to $R^k \times T^m$ for some $m, k, m + k = n - 1$, or there is a singular orbit B in M.*
(b) *If there is a unique singular orbit B which is nonexceptional, then B is isometric to $R^k \times T^m$ for some m, k and $\pi_1(M) = Z^m$.*

Proof. Let $\tilde{M} = R^n$ be the universal covering manifold of M and let $\tilde{G}$ be the corresponding covering Lie group of G, which acts on $\tilde{M} = R^n$ by cohomogeneity one.
(a): Since M is not toruslike there is a deck transformation φ such that $\varphi = (A, b), A \neq I$. By lemma 4.2 there is a normal geodesic γ in $\tilde{M}$ such that the function $F(t) = d^2_{\varphi} \circ \gamma(t)$ is a strictly convex function with a minimum point t_0. Since d^2_{φ} is constant along orbits we get that the orbit $\tilde{D}_0 = \tilde{G}\gamma(t_0)$ is the minimum point set of d^2_{φ} Thus by 2.4(a) it is totally geodesic in $\tilde{M} = R^n$, so it is flat and therefore isometric to R^r for some r. Now let there is not any singular orbit in M. So $\tilde{M} = R^n$ does not have any singular orbit, therefore $r = n - 1$ and $\tilde{D}_0$ is isometric to R^{n-1}, so by lemma 4.3 we get that each orbit $\tilde{D}$ of $\tilde{M}$ is isometric to R^{n-1}, therefore each orbit $D(= \pi(\tilde{D}))$ of M is flat, and since it is homogeneous we get by theorem 2.5 that D is isometric to $R^k \times T^m$, for some $m, k, m + k = n - 1$.

This proves the part (a).
(b): Let B be the unique nonexceptional singular orbit of M and $\tilde{B} = \pi^{-1}(B)$ and let $F(t)$ be the function obtained in the proof of part (a) with the minimum point t_0. For each $t \in R$ we have $\tilde{G}\gamma(t) = g^{-1}(F(t))$, where $g = d_\varphi^2$. If c and b are regular values of g then $g^{-1}(c)$ and $g^{-1}(b)$ are diffeomorphic (see [3] page 10 corollary 3.11), from these facts we get that $\tilde{B} = g^{-1}(F(t_0))$ (because if not, then $\tilde{B} = g^{-1}(b)$ where b is a regular value of g, and so $\tilde{B}$ must be diffeomorphic to principal orbits which is a contradiction). So $\tilde{B}$ is the minimum point set of g and therefore by 2.4(a) it is totally geodesic in M and is flat, thus B is flat. Since it is homogeneous we get by 2.5 that B is diffeomorphic to $R^k \times T^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = Z^m$. □

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