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STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS FOR CATALYTIC SUPER-BROWNIAN MOTIONS

YOUNGMEE KWON* AND HYE-JEONG KANG

ABSTRACT. We study a class of catalytic super Brownian motion X in 1-dimension. We show under some conditions of catalyst, the process X is absolutely continuous and we get a stochastic partial differential equation for X.

1. Introduction

Super Brownian motion(SBM) on R^d is a measure-valued process which can be obtained as a limit of branching Brownian particle system on R^d. We refer to [2] for such an approximation in a more general setting. In the last 25 years, spatial branching process have been extensively investigated, i.e., the case of the additive functional of Brownian motion corresponding to the branching rate $\rho(dy) = dy$, the Lebesgue measure. But when ρ may vary in time and space, SBM is called a catalytic super Brownian motion(CSBM). For example in 1-dimensional case, $\rho(t, dz) = \sum_i a_i \delta_{z(i,t)}$, where δ_z denotes the Dirac-δ function at $z \in R$, corresponds to the case where branching is allowed only at the position of these moving catalysists with unbounded rates. Dawson and Fleschmann[4] studied CSBM with a single point catalyst in detail. On the other hand, if $D \subset R^d$ and $\rho(t, dz) = \chi_D(z)dz$, ρ corresponds to the additive functional $A_t = \int_0^t \chi_D(B_s)ds$, where B_s is a Brownian motion. In this case branching occurs only when particles are in D. In this paper we consider $\rho(t, dz) = \rho(dz)$ for some σ-finite measure ρ on

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R satisfying Condition [A] given below. It was introduced by Delmas[3]. Let $M_F(R^d)$ denote the set of finite measure on R^d. Delmas showed that under Condition [A], CSBM exists on $C([0,\infty), M_F(R^d))$ and is unique in law by its unique representation of Laplace functional(Theorem 2.1). In this paper we show that in 1-dimension, CSBM under the condition [A] is absolutely continuous with respect to Lebesgue measure and give its stochastic differential equation for the density of CSBM in Theorem 2.2. Finally we get a limit theorem when branching rate $\rho(dy)$ varies using martingale measure theory.

2. The Main Result

Let $C([0,\infty), M_F(R))$ be the space of $M_F(R)$-valued continuous paths $\{\omega_t; t \geq 0\}$ with the coordinate process denoted by $X_t(\omega) = \omega_t$. Let $B_b(R)$, $C_b(R)$ be the space of bounded, bounded and continuous functions on R respectively with the supremum norm $||\cdot||$. For $f \in B_b(R)$, and $\mu \in M_F(R)$, let $\mu(f) = \langle \mu, f\rangle = \int_R f d\mu$. Note $B \equiv (\Omega, \mathcal{F}, \mathcal{F}_t, B_t, \theta_t, (P_x)_{x\in R})$, the canonical realization of Brownian motion on Ω, P_s the semigroup of Brownian motion and p, the density, i.e., for $(s,x) \in (0,\infty) \times R$

$$p(s,x) = \frac{1}{\sqrt{2\pi s}} \exp(-\frac{x^2}{2s}).$$

CONDITION [A] : Hypothesis of integrability

We say ρ is a σ-finite measure on R in H and denote $\rho \in H$ if there exists $\beta \in (0,1)$ such that

$$\sup_{x\in R} \int_{B(x,1)} \frac{\rho(dy)}{|x-y|^{2\beta-1}} < \infty. \tag{1}$$

Delmas showed the following lemmas.

LEMMA 2.1 OF [3]. *Let* $\rho \in H$ *in* Condition [A]. *Then for all* $(s,x) \in [0,\infty) \times R$

$$\int p(s,x-y)\rho(dy) \leq c\frac{1}{(s\wedge 1)^{1-\beta}} \tag{2}$$

for some constant $c > 0$ independent of s and there exists an additive functional A_t of B_t such that for $f \in B_b(R^+ \times R^1)$

$$E_x \int_0^\infty f(s, B_s) dA_s = \int_0^\infty ds \int \rho(dy) p(s, x-y) f(s, y), \tag{3}$$

$$a_t \equiv \sup_{x \in R} E_x A_t \leq C(t \vee t^\beta). \tag{4}$$

LEMMA 2.2 (Théorèm II 3.2 of [3]). *Let $\mu \in M_F(R)$ and $\rho \in H$. Then the CSBM $\{X_t\}$ exists on $C\big([0,\infty), M_F(R)\big)$, P_μ^X, the law of $\{X_t\}$ is unique where $X_0 = \mu$, P_μ^X-a.s. and for $\phi \in C_b(R)$, $\phi \geq 0$ and $a_t \|\phi\| < 1$*

$$E_\mu^X e^{-\langle X_t, \phi \rangle} = e^{-\langle \mu, w(t,\cdot) \rangle},$$

where $w(t,x)$ is the unique solution of the integral equation

$$w(t,x) = P_t \phi(x) - \frac{1}{2} E_x \int_0^t w(t-s, B_s)^2 dA_s. \tag{5}$$

LEMMA 2.3. $\big($(22), (23) of [3]$\big)$ *If $\mu \in M_F(R)$ and $\phi \in B(R)$*

$$E_\mu\big(\langle X_t, \phi \rangle\big) = E_\mu\big(\phi(B_t)\big) \tag{6}$$

$$E_\mu\big(\langle X_t, \phi \rangle^2\big) = \big(E_\mu \phi(B_t)\big)^2 + E_\mu \int_0^t \big(P_{t-s}\phi(B_s)\big)^2 dA_s. \tag{7}$$

THEOREM 2.1. *Let $\rho \in H$ and $\mu \in M_F(R)$. Then*

(i) *$P_\mu(X_t(dx)$ is absolutely continuous with respect to dx for a.e. $t > 0) = 1$*

(ii) *if $X_t(x)$ be the density of X_t, then for every $\phi \in C_c^\infty(R)$,*

$$\langle X_t, \phi \rangle - \langle X_0, \phi \rangle = \int_0^t \int \sqrt{X_s(x)} \phi(x) W(dxds) + \int_0^t \langle X_s, \phi'' \rangle ds,$$

where $W(x,t)$ is a space time white noise with covariance measure $\langle\, W(dxdt)\, \rangle = \rho(dx)dt$.

Proof of (i). By (6) and (7),

$$E_\mu^X \langle X_t, \phi \rangle = \langle \mu, P_t \phi \rangle$$

$$\begin{aligned}(8)\qquad & E_\mu^X \langle X_t, \phi\rangle^2 \\ = \;& \langle \mu, P_t\phi\rangle^2 + \langle \mu, E_x \int_0^t (P_{t-s}\phi(B_s))^2 dA_s\rangle \\ = \;& \langle \mu, P_t\phi\rangle^2 + \langle \mu, \int_0^t ds \int \rho(dy) p(s, x-y)(P_{t-s}\phi(y))^2\rangle \\ = \;& \langle \mu, P_t\phi\rangle^2 + \int_0^t ds \langle \mu P_s^\rho, (P_{t-s}\phi)^2\rangle,\end{aligned}$$

where we denote

$$P_s^\rho \phi(x) = \int p(s, x-y)\phi(y)\rho(dy).$$

Let $p_t^x(y) = p(t, y-x)$ and $X_t^h(x) = \langle\ X_t,\ p_h^x\ \rangle$. We first show that for any $T > 0$,

$$\int_0^T \int_R E_\mu^X (X_t^h(x)^2) t^{\frac{1}{2}} dx dt < \infty \tag{9}$$

and

$$\lim_{h\downarrow 0} \lim_{h'\downarrow 0} \int_0^T \int_R E_\mu^X (X_t^h(x) - X_t^{h'}(x))^2 t^{\frac{1}{2}} dx dt = 0. \tag{10}$$

Noting that $P_t p_h^x(y) = p_{t+h}^x(y)$, we get from (8)

$$E_\mu^X (X_t^h(x)^2) = \langle \mu, p_{t+h}^x\rangle^2 + \int_0^t ds \langle \mu P_{t-s}^\rho, (P_s p_h^x)^2\rangle.$$

Since $t^{\frac{1}{2}} \langle \mu, p_{t+h}^x\rangle \le \mu(1)$,

$$\begin{aligned}\int_0^T \int \langle \mu, p_{t+h}^x\rangle^2 t^{\frac{1}{2}} dx dt \;&\le\; \mu(1) \int_0^T \int \langle \mu, p_{t+h}^x\rangle dx dt \\ &=\; \mu(1)^2 T \;<\; \infty.\end{aligned}$$

The fact that $p_{s+h}(y) \le \frac{C}{(s+h)^{\frac{1}{2}}}$ implies

$$
\begin{aligned}
&\int_0^T \int_R \int_0^t ds\langle \mu P_{t-s}^{\rho}, (p_{s+h}^x)^2\rangle t^{\frac{1}{2}} dxdt \\
&\leq C\int_0^T \int_R \int_0^t ds\langle \mu P_{t-s}^{\rho}, p_{s+h}^x\rangle \frac{t^{\frac{1}{2}}}{(s+h)^{\frac{1}{2}}} dxdt \\
&= C\int_0^T \int_R \int_0^t ds \int\int p(t-s, y-z)p(s+h, y-x)\rho(dy)\mu(dz) \\
&\qquad\qquad \frac{t^{\frac{1}{2}}}{(s+h)^{\frac{1}{2}}} dxdt.
\end{aligned}
$$

By integrating with respect to x first and then applying Lemma 2.1, the right hand side equals

$$
\begin{aligned}
&C\int_0^T \int_0^t ds \int\int p(t-s, y-z)\rho(dy)\mu(dz) \frac{t^{\frac{1}{2}}}{(s+h)^{\frac{1}{2}}} dt \\
&\leq C(\mu)\int_0^T \int_0^t \frac{1}{((t-s)\wedge 1)^{1-\beta}} \frac{t^{\frac{1}{2}}}{(s+h)^{\frac{1}{2}}} dsdt \\
&\leq C(T,\mu)\int_0^T \int_0^t \frac{t^{\frac{1}{2}}}{(t-s)^{1-\beta} s^{\frac{1}{2}}} dsdt + C'(T) \\
&= C(T,\mu)\int_0^T \int_0^t (\frac{s}{t})^{-\frac{1}{2}} t^{\beta-1}(1-\frac{s}{t})^{\beta-1} dsdt + C'(T) \\
&= C(T,\mu)\int_0^T t^{\beta}\int_0^1 y^{-\frac{1}{2}}(1-y)^{\beta-1} dydt + C'(T) < \infty,
\end{aligned}
$$

since $0<\beta<1$ and we get (9). Also

$$
\begin{aligned}
&E_{\mu}^X(X_t^h(x) - X_t^{h'}(x))^2 \\
&= E_{\mu}(\langle X_t, p_h^x - p_{h'}^x\rangle^2) \\
&= \langle \mu, p_{t+h}^x - p_{t+h'}^x\rangle^2 + \int_0^t ds\langle \mu P_{t-s}^{\rho}, (p_{s+h}^x - p_{s+h'}^x)^2\rangle.
\end{aligned}
$$

Then by the same argument to show (9), we have (10). Therefore there exists a jointly measurable function $X_t(x,\omega) : [0,\infty)\times R\times\Omega \to [0,\infty)$

satisfying

$$\int_0^T \int_R E_\mu^X (X_t(x)^2) t^{\frac{1}{2}} dtdx < \infty \quad \text{for any } T > 0$$

and

$$\lim_{h \downarrow 0} \int_0^T \int_R E_\mu^X (X_t^h(x) - X_t(x))^2 t^{\frac{1}{2}} dtdx = 0. \tag{11}$$

Moreover, for every $\phi \in C_c^\infty(R)$

$$\begin{aligned} & E_\mu^X |\langle X_t, \phi\rangle - \int_R X_t(x)\phi(x)dx|^2 \\ = \; & \lim_{h\downarrow 0} E_\mu^X |\langle X_t^h, \phi\rangle - \int X_t(x)\phi(x)dx|^2 \\ \le \; & \lim_{h\downarrow 0} \int_R E_\mu^X (X_t^h(x) - X_t(x))^2 dx \int_R \phi^2(x)dx \\ \to \; & 0. \end{aligned}$$

Therefore we prove (i).
proof of (ii). For $\phi \in B_b([0,\infty) \times R)$, there exists $\Gamma_\rho(ds, dy)$ such that

$$\int_0^t ds \int \rho(dy) \int X_s(dz) p(\varepsilon, z-y)\phi(s,y) \to \int_0^t \Gamma_\rho(ds, dy)\phi(s,y) \quad \text{a.s. } P_\mu^X$$

as $\varepsilon \to 0$ by Proposition 5.1 of [3] and for a.e. s and a.e. y,

$$X_s^\varepsilon(y) = \int X_s(dz)p(\varepsilon, y-z) \to X_s(y) \quad \text{a.s. } P_\mu^X$$

by (i). Therefore for all $\phi \in B_b([0,\infty) \times R)$ if $\rho(dy) = f(y)dy$ for some $f \in B_b(R)$,

$$\int_0^t \Gamma_\rho(ds, dy)\phi(s,y) = \int_0^t \int X_s(y)\phi(s,y)\rho(dy)ds. \tag{12}$$

For all $\phi \in C_c^2(R)$, define

$$(M\phi)_t = \langle X_t, \phi\rangle - \langle X_0, \phi\rangle - \int_0^t \langle X_s, \frac{1}{2}\phi''\rangle ds$$

then by Proposition 9.1 of [3] and (12), if $\rho(dy) = f(y)dy$, $(M\phi)_t$ is a continuous martingale with

$$(13) \qquad \langle M\phi\rangle_t = \int_0^t \Gamma_\rho(ds, dy)\phi^2(y) = \int_0^t \int X_s(y)\phi^2(y)\rho(dy)ds.$$

Take $\overline{W}_t$, a white noise independent of $X_t(dx)$ with $\langle \overline{W}(dxds)\rangle = \rho(dx)ds$. Set

$$(14) \qquad W_t(\phi) = \int_0^t \int_R \frac{1}{\sqrt{X_s(x)}} I_{(X_s(x)\neq 0)}\phi(x)M(dxds) + \int_0^t \int_R I_{(X_s(x)=0)}\phi(x)\overline{W}(dxds).$$

Then $W_t(\phi)$ is a continuous martingale such that

$$\begin{aligned}\langle W(\phi)\rangle_t &= \int_0^t \int I_{(X_s(x)\neq 0)}\phi^2(x)\rho(dx)ds + \int_0^t \int I_{(X_s(x)=0)}\phi^2(x)\rho(dx)ds \\ &= \int_0^t \int \phi^2(x)\rho(dx)ds.\end{aligned}$$

Therefore

$$\langle X_t, \phi\rangle = \langle X_0, \phi\rangle + \int_0^t \langle X_s, \frac{1}{2}\phi''\rangle ds + \int_0^t \int \sqrt{X_s(x)}\phi(x)W(dxds),$$

where $W(dxds)$ is a continuous martingale measure such that $\langle\, W(dxds)\rangle = \rho(dx)ds$, i.e., a white noise. □

By Proposition 9.1 of [3], X_t is a solution of martingale problem (MP): For all $\psi \in C_b^{1,2}([0,\infty) \times R)$,

$$(M\psi)_t = \langle X_t, \psi(t)\rangle - \langle X_0, \psi(0)\rangle - \int_0^t \langle X_s, \frac{\partial\psi}{\partial s}(s) + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2}(s)\rangle ds$$

is a martingale such that

$$\langle M\psi\rangle_t = \int_0^t \int X_s(x)\psi^2(s,x)\rho(dx)ds.$$

Now we show that the above martingale problem has a unique solution if $\rho(dy) = f(y)dy$ for some $f \in C_b(R)$.

THEOREM 2.2. *Let $\rho \in H$ and $\rho(dy) = f(y)dy$ for some $f \in C_b(R)$. Then the solution of* (MP) *is unique.*

To prove this theorem, we need the following result.

LEMMA 2.4. *Under the same conditions as* Theorem 2.2, *$w(t,x)$ defined by* (5) *satisfies the following differential equation*

$$\frac{\partial}{\partial t}w(t,x) - \frac{\partial^2}{\partial x^2}w(t,x) - \frac{1}{2}w(t,x)f(x) = 0. \tag{14}$$

Proof. Let $A = \dfrac{\partial^2}{\partial x^2}$. First we show that $\int_0^t P_s(w^2(t-s)f)(x)ds \in \mathcal{D}(A)$. Put

$$M = \sup_{0 \le s \le t, y \in R} |w^2(s,y)f(y)|.$$

Then by (4) of [3] we have

$$\frac{1}{h}[P_s(w^2(t-s)f)(x+h) - P_s(w^2(t-s)f)(x)] \le M\frac{1}{\sqrt{s}}. \tag{15}$$

We claim that for each x, the left hand side of (15) has a finite limit. The left hand side of (15) can be decomposed as follows.

$$\begin{aligned}
&\frac{1}{h}[P_s(w^2(t-s)f)(x+h) - P_s(w^2(t-s)f)(x)] \\
&= \int w^2(t-s,y)f(y)\frac{1}{\sqrt{2\pi s}} \cdot \frac{1}{h}[e^{-\frac{(x+h-y)^2}{2s}} - e^{-\frac{(x-y)^2}{2s}}]dy \\
&= \frac{1}{\sqrt{2\pi s}}\int w^2(t-s,x-u)f(x-u) \cdot \frac{1}{h}[e^{-\frac{(u+h)^2}{2s}} - e^{-\frac{u^2}{2s}}]du \\
&= \frac{1}{\sqrt{2\pi s}}\int_{-\frac{h}{2}}^{\infty} + \int_{-\infty}^{-\frac{h}{2}} du\, w^2(t-s,x-u)f(x-u) \cdot \frac{1}{h}[e^{-\frac{(u+h)^2}{2s}} - e^{-\frac{u^2}{2s}}].
\end{aligned}$$

Since $1 - e^{-x} \le x$ for $x \ge 0$, if $u \ge -\frac{h}{2}$,

$$|e^{-\frac{(u+h)^2}{2s}} - e^{-\frac{u^2}{2s}}| = e^{-\frac{u^2}{2s}}(1 - e^{-\frac{2hu+h^2}{2s}}) \le \frac{2hu+h^2}{2s} \cdot e^{-\frac{u^2}{2s}},$$

and if $u \le -\frac{h}{2}$,

$$|e^{-\frac{(u+h)^2}{2s}} - e^{-\frac{u^2}{2s}}| = e^{-\frac{(u+h)^2}{2s}}(1 - e^{\frac{2hu+h^2}{2s}}) \le -\frac{2hu+h^2}{2s} \cdot e^{-\frac{(u+h)^2}{2s}}.$$

Hence

$$|w^2(t-s,x-u)f(x-u)\cdot\frac{1}{h}(e^{-\frac{(u+h)^2}{2s}}-e^{-\frac{u^2}{2s}})|$$

$$\leq \begin{cases} M\frac{2u+h}{2s}e^{-\frac{u^2}{2s}} & \text{if } u\geq -\frac{h}{2} \\ -M\frac{2u+h}{2s}e^{-\frac{(u+h)^2}{2s}} & \text{if } u< -\frac{h}{2} \end{cases}$$

$$\to \begin{cases} \frac{u}{s}e^{-\frac{u^2}{2s}} & \text{if } u\geq 0 \\ -\frac{u}{s}e^{-\frac{u^2}{2s}} & \text{if } u<0 \end{cases} \quad \text{as} \quad h\to 0.$$

Furthermore, we have

$$\lim_{h\to 0}\int_{-\frac{h}{2}}^{\infty}(2u+h)e^{-\frac{u^2}{2s}}du = \int_0^{\infty} 2ue^{-\frac{u^2}{2s}}du$$

$$\lim_{h\to 0}\int_{-\infty}^{-\frac{h}{2}} -(2u+h)e^{-\frac{(u+h)^2}{2s}}du = \int_{-\infty}^{0} -2ue^{-\frac{u^2}{2s}}du$$

and so we arrive at

$$\lim_{h\to 0}\frac{1}{h}[P_s(w^2(t-s)f)(x+h)-P_s(w^2(t-s)f)(x)]$$
$$= \int w^2(t-s,y)f(y)\frac{1}{\sqrt{2\pi s}}\cdot\frac{x-y}{s}e^{-\frac{(x-y)^2}{2s}}dy.$$

Combining the last equality with (15), we conclude that $\frac{\partial}{\partial x}\int_0^t P_s\big(w^2(t-s)\,f\big)(x)\,ds$ exists and is equal to

$$\int_0^t\int w^2(t-s,y)f(y)\frac{1}{\sqrt{2\pi s}}\cdot\frac{x-y}{s}e^{-\frac{(x-y)^2}{2s}}\,dyds.$$

In a similar way but with longer calculation, we can show that $\frac{\partial^2}{\partial x^2}\int_0^t P_s\big(w^2(t-s)\,f\,\big)(x)\,ds$ exists, in other words, $\int_0^t P_s(w^2(t-s)f)(x)ds\in\mathcal{D}(A)$. Now,

$$(16)\quad \frac{1}{h}[\int_0^{t+h}P_s(w^2(t+h-s)f)(x)ds-\int_0^t P_s(w^2(t-s)f)(x)\,ds]$$
$$=\frac{1}{h}[\int_h^{t+h}P_s(w^2(t+h-s)f)(x)ds-\int_0^t P_s(w^2(t-s)f)(x)\,ds]$$
$$+\frac{1}{h}\int_0^h P_s(w^2(t+h-s)f)(x)ds$$

$$= \frac{P_h - I}{h} \int_0^t P_s(w^2(t-s)f)(x)ds + \frac{1}{h}\int_0^h P_s(w^2(t+h-s)f)(x)ds$$

On the other hand,

$$\frac{1}{h}\int_0^h P_s(w^2(t+h-s)f)(x)ds$$
$$= \frac{1}{h}\int_0^h P_s(w^2(t+h-s)f - w^2(t-s)f)(x)ds + \frac{1}{h}\int_0^h P_s(w^2(t-s)f)(x)ds.$$

Since $w(s,y)$ is uniformly continuous on $[\delta, T] \times R$ for any $T > 0$ any $\delta > 0$ and f is bounded, we can make the first term arbitrary small. Also the continuity and boundedness of $w^2(t-s,y)f(y)$ implies that

$$\lim_{h \downarrow 0} \frac{1}{h}\int_0^h P_s(w^2(t+h-s)f)(x)ds = w^2(t,x)f(x). \tag{17}$$

We obtain (11) from (12) and (17). □

Proof of Theorem 2.2. It is enough to show that $E_\mu^X(\exp(-\langle X_t, \phi\rangle))$ is uniquely determined for all $\phi \in C_b^2(R)$. Since $\rho(dy) = f(y)dy$, we have

$$w(t,x) = P_t\phi(x) - \frac{1}{2}\int_0^t \int w^2(t-s,y)p(s,x-y)f(y)dyds.$$

For $T > 0$, put $\psi(t,x) = w(T-t,x)$. Then by Lemma 2.4 $\psi(t,x)$ satisfies the following partial differential equation

$$\frac{\partial}{\partial t}\psi(t,x) + \frac{\partial^2}{\partial x^2}\psi(t,x) + \frac{1}{2}\psi^2(t,x)f(x) = 0.$$

By Ito's formula

$$\begin{aligned} &e^{-\langle X_t, \psi(t,\cdot)\rangle} - e^{-\langle X_0, \psi(0,\cdot)\rangle} \\ &- \int_0^t e^{-\langle X_s, \psi(s,\cdot)\rangle}\big[\langle X_s, \frac{\partial}{\partial s}\psi(s,\cdot) + \frac{\partial^2}{\partial x^2}\psi(s,\cdot)\rangle \\ &+ \frac{1}{2}\int X_s(x)\psi^2(s,x)\rho(dx)\big]ds \end{aligned}$$

is a martingale. Therefore if $\rho(dx) = f(x)dx$, by (14) and letting $T = t$,

$$E_\mu^X e^{-\langle X_t, \phi\rangle} = E_\mu^X e^{-\langle \mu, w(t,\cdot)\rangle} = e^{-\langle \mu, w(t,\cdot)\rangle}$$

and the theorem is proved. □

We denote that $\rho_n(dy) \to \rho(dy)$ as $n \to \infty$ if for $\phi \in C_c^\infty(R)$, $\int \phi(y)\rho_n(dy) \to \int \phi(y)\,\rho(dy)$.

THEOREM 2.3. *Let X^n and X be the* CSBM *with respect to ρ_n and ρ respectively and ρ_n, ρ are in H in* Condition [A] *with a uniform bound and a common β. If $\rho_n(dy) = f_n(y)dy$, $\rho(dy) = f(y)dy$ for some $f_n, f \in C_b(R)$ and $\rho_n(dy) \to \rho(dy)$, there exists an $M_F(R)$-valued process Y such that $X_t^n(x)$ converges to $Y_t(x)$ in L^2 for almost all $t \in [0,\infty)$ and $x \in R$, and $Y = X$ if $f \in C_b(R)$.*

Proof. By Theorem 2.1, for $\phi \in C_c^\infty(R)$,

$$\langle X_t^n, \phi\rangle = \langle X_0^n, \phi\rangle + \int_0^t \langle X_s^n, \phi''\rangle ds + \int_0^t \int \sqrt{X_s^n(x)}\phi(x) W_n(dxds),$$

where W_n is a white noise with $\langle W_n(dxds)\rangle = \rho_n(dx)ds$. Denote $\langle X_t^n, \phi\rangle$ by $X_t^n(\phi)$. Then first we show that $\{X_t^n(\phi)\}$ is tight, i.e.,

(i) for any $T > 0$, $\sup_{0\le t\le T} E_\mu^{X^n}[X_t^n(\phi)^2] < \infty$,

(ii) for $0 \le t \le T$, $0 \le u \le \delta$

$$E_\mu^{X^n}((X_{t+u}^n(\phi) - X_t^n(\phi))^2 | \mathcal{F}_t^n) \to 0, \quad \text{as} \quad \delta \to 0$$

where $\mathcal{F}_t^n$ is a σ-field generated by X^n.

By Corollaire 4.5 of [3],

$$\sup_{n,0\le t\le T} E_\mu^{X_n}[X_t^n(\phi)]^p < \infty, \tag{18}$$

for given $p \ge 1$, therefore (i) holds. Now for (ii)

$$\begin{aligned} X_{t+u}^n(\phi) - X_t^n(\phi) &= \int_t^{t+u} \langle X_s^n, \phi''\rangle ds + \int_t^{t+u} \int \sqrt{X_s^n(x)}\phi(x) W^n(dxds) \\ &= I + II. \end{aligned}$$

From the Lemma 5.2 of [3], we have the followings.

$$\begin{aligned} E_\mu^{X^n}(I^2) &= E_\mu^{X_n}\Big[\int_t^{t+u} X_s^n(\phi'')\,ds\Big]^2 \le ||\phi''||^2 \mu(1) u \\ E_\mu^{X_n}(II^2) &= E_\mu^{X_n}\Big[\int_t^{t+u} \int X_s^n(x)\phi^2(x)\rho_n(dx)\,ds\Big] \\ &= E_\mu^{X_n}\Big[\int_t^{t+u} \Gamma_\rho^n(ds, dx)\phi^2(x)\Big] \\ &\le M||\phi^2|| u^\beta. \end{aligned}$$

Therefore we get a limit and let $Y_t(\phi)$ be a limit of $X_t^n(\phi)$. By (18), $\{X_t^n(\phi)\}$ is uniformly integrable and this implies

$$E_\mu^{X_n}[X_t^n(\phi)]^2 \to E_\mu^Y(Y_t(\phi))^2 \quad \text{as} \quad n \to \infty,$$

and (9) and (10) hold with Y_t. These are all we need to have the density of Y_t, $Y_t(x)$ satisfying

$$\lim_{h \downarrow 0} \int_0^T \int_R E_\mu(Y_t^h(x) - Y_t(x))^2 t^{\frac{1}{2}} dxdt = 0,$$

where $Y_t^h(x) = \langle Y_t, p_h^x \rangle$. Let $X_{t,n}^h(x) = \langle X_t^n, p_h^x \rangle$. By Skorohod representation, we can write the following in some appropriate space,

$$\begin{aligned} &\int_0^T \int_R E_\mu(X_t^n(x) - Y_t(x))^2 t^{\frac{1}{2}} dxdt \\ \le\ & const. \Big[\int_0^T \int_R E_\mu(X_t^n(x) - X_{t,n}^h(x))^2 t^{\frac{1}{2}} dxdt \\ &+ \int_0^T \int_R E_\mu(X_{t,n}^h(x) - Y_t^h(x))^2 t^{\frac{1}{2}} dxdt \\ &+ \int_0^T \int_R E_\mu(Y_t^h(x) - Y_t(x))^2 t^{\frac{1}{2}} dxdt \Big] \\ =\ & (I) + (II) + (III). \end{aligned}$$

Since for fixed $h > 0$, $(II) \to 0$ as $n \to \infty$, for fixed n, $(I) \to 0$ as $h \to 0$ and $(I), (II), (III)$ are uniformly bounded for all n and $0 < h < 1$. Therefore $X_t^n(x) \Rightarrow Y_t(x)$ in L^2 for almost all t. Moreover, for each $\phi \in C_c^\infty(R)$, $W_t^n(\phi)$ and $W_t(\phi)$ are continuous martingales with covariance $\rho_n(\phi)t$ and $\rho(\phi)t$ respectively. Therefore $W_t^n(\phi) = B_{\rho_n(\phi)t}$ a.s. for $0 \le t < \infty$ and this implies that $W_t^n(\phi)$ has a unique limit, $W_t(\phi)$. That is, $W_n(dxds) \Rightarrow W(dxds)$ in $C_{S'(R^d)}[0, \infty)$, where $S'(R^d)$ is the dual of Schwartz space. Also

$$\begin{aligned} &\sup_n \int_0^t \int \frac{1}{(1+|x|)^{-1+2\beta}} \rho_n(dx)ds \\ \le\ & \sup_n \int_0^t \sup_x \int_{B(x,1)} \frac{1}{|x-y|^{-1+2\beta}} \rho_n(dy)ds < \infty. \end{aligned}$$

Therefore by Theorem 2.2 in [1], we get

$$\int_0^t \int_R X_s^n(x)\phi(x)W_n(dxds) \Rightarrow \int_0^t \int_R Y_s(x)\phi(x)W(dxds)$$

and

$$Y_t(\phi) = \mu(\phi) + \int_0^t Y_s(\phi'')ds + \int_0^t \int \sqrt{Y_s(x)}W(dxds).$$

If $\rho(dx) = f(x)dx$ for some $f \in C_b(R)$, then by the uniqueness of the solutions of martingale problem(Theorem 2.2), $Y = X$, i.e., the CSBM with respect to ρ. □

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Youngmee Kwon, Department of Computer Science and Statistics, Hansung University, 389, 2Ga Samsun-dong, Sungbuk-gu, 136-792, Seoul, Korea

Hye-Jeong Kang, UNIBOSS Ltd. 25F Hanhwa Securities Bldg, Yoido-dong Youngdeungpo-gu, 150-010 Seoul, Korea