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AN INEQUALITY ON PERMANENTS OF HADAMARD PRODUCTS

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Abstract. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be $n \times n$ complex matrices and let $A \circ B$ denote the Hadamard product of A and B, that is $A \circ B = (a_{i,j} b_{i,j})$. We conjecture a permanental analog of Oppenheim's inequality and verify it for $n = 2$ and 3 as well as for some infinite classes of matrices.

1. Introduction

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be $n \times n$ complex matrices and let $A \circ B$ denote the Hadamard product of A and B, that is $A \circ B = (a_{i,j} b_{i,j})$. Some authors call this the Schur product. Oppenheim's inequality states that $\det(A \circ B) \geq \det A \det B$, when A and B are positive semidefinite matrices. In 1982, Chollet([2]) conjectured a permanental analog:

$$\text{per}(A \circ B) \leq \ \text{per}A \quad \text{per}B. \tag{1}$$

He showed that inequality (1) holds for all positive definite matrices A and B if and only if for any positive semidefinite matrix A,

$$\text{per}(A \circ \bar{A}) \leq (\ \text{per}A)^2$$

where $\bar{A}$ denotes the complex conjugate of A. Gregorac and Hensel([3]) proved that inequality (1) holds for $n = 2$ and 3.

In 1986, Bapat and Sunder([1]) conjectured a stronger result:

$$\text{per}(A \circ B) \leq \ \text{per}A \prod_{i=1}^{n} b_{i,i}. \tag{2}$$

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An $n \times n$ matrix is called a correlation matrix if it is positive definite and all its main diagonal entries are 1. In 1989, Zhang([4]) showed that inequality (2) is true for all positive definite matrices if and only if it is true for all correlation matrices, that is if and only if

$$\text{per }(A \circ B) \leq \text{per } A$$

for all correlation matrices A and B, and verified inequality (2) is true when $n = 2$. He also proved for any n that for any $n \times n$ correlation matrix and the real matrix

$$B_t = \begin{bmatrix} 1 & t & \cdots & \cdots & t \\ t & 1 & \cdots & \cdots & t \\ t & t & 1 & \cdots & t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & \cdots & 1 \end{bmatrix}, 0 \leq t \leq 1$$

we have that

$$\text{per }(A \circ B_t) \leq \text{per } A.$$

We now conjecture the following, which is stronger than Chollet's conjecture, inequality (1), and weaker than Bapat and Sunder's conjecture, inequality (2).

CONJECTURE 1.1. *If $A = (a_{i,j})$ and $B = (b_{i,j})$ are positive semidefinite* Hermitian *$n \times n$ matrices then*

$$\text{per }(A \circ B) \leq \max\left\{\text{per } A \prod_{i=1}^{n} b_{i,i},\ \text{per } B \prod_{i=1}^{n} a_{i,i}\right\}. \tag{3}$$

We shall show that this conjecture is true for $n = 2$ and 3, as well as showing that it is true if and only if it holds for all correlation matrices.

2. Preliminaries

As previous researchers have done, when trying to establish inequalities (1) and (2), we shall show that only correlation matrices need be considered. In fact, we shall show that one only needs to consider the permanent of one correlation matrix and its conjugate.

LEMMA 2.1. *Inequality* (3) *holds for all positive semidefinite* Hermitian *$n \times n$ matrices if and only if it holds for all $n \times n$ correlation matrices.*

Proof. Since the sufficiency is obvious, we only show the necessity. We assume that if A and B are any $n \times n$ correlation matrices we have that

$$\text{per}\,(A \circ B) \leq \max\{\text{per}\,A,\ \ \text{per}\,B\}\,. \tag{4}$$

Let R and S be $n \times n$ positive definite Hermitian matrices and let $D = (R \circ I)^{\frac{1}{2}}$ and $D' = (S \circ I)^{\frac{1}{2}}$ where I is the $n \times n$ identity matrix. That is, D and D' are diagonal matrices whose diagonal entries are the positive square roots of the corresponding diagonal entries of R and S respectively. If A and B are matrices such that $D^{-1}RD^{-1} = A$ and $D'^{-1}SD'^{-1} = B$ then A and B are correlation matrices with $a_{i,j} = \frac{r_{i,j}}{(r_{i,i})^{\frac{1}{2}}(r_{j,j})^{\frac{1}{2}}}$ and $b_{i,j} = \frac{s_{i,j}}{(s_{i,i})^{\frac{1}{2}}(s_{j,j})^{\frac{1}{2}}}$. Since for any matrix X and diagonal matrix E we have per XE = per EX = per X per E, it follows that per $(R \circ S)$ = per $[(DAD) \circ (D'SD')]$ = per $[DD'(A \circ B)DD']$ = $(\text{per}\,D)^2(\text{per}\,D')^2\text{per}\,(A \circ B)$. Since A and B are correlation matrices we have that

$$\begin{aligned} \text{per}\,(R \circ S) &\leq (\text{per}\,D)^2(\text{per}\,D')^2 \max\{\text{per}\,A,\ \ \text{per}\,B\} \\ &= \max\{\text{per}\,R(\text{per}\,D')^2,\ \ \text{per}\,S(\text{per}\,D)^2\} \\ &= \max\{\text{per}\,R\textstyle\prod_{i=1}^n s_{i,i},\ \ \text{per}\,S\prod_{i=1}^n r_{i,i}\} \end{aligned}$$

since per $R = (\text{per}\,D)^2\text{per}\,A$ and per $S = (\text{per}\,D')^2\text{per}\,B$.

The lemma now follows by the fact that the set of positive semidefinite Hermitian matrices is the closure of the set of positive definite Hermitian matrices. □

LEMMA 2.2. *Inequality* (3) *holds for all positive semidefinite* Hermitian $n \times n$ *matrices if and only if for all* $n \times n$ *correlation matrices* A,

$$\text{per}\,(A \circ \bar{A}) \leq \text{per}\,A \tag{5}$$

Proof. The sufficiency is obvious since $\bar{A}$ is positive semidefinite whenever A is.

Suppose A and B are correlation matrices. Then

$$\text{per}\,(A \circ B) = |\text{per}\,(A \circ B)| \leq \left|\sum_{\sigma\in S_n}\prod_{i=1}^n a_{i,\sigma(i)}b_{i,\sigma(i)}\right| \leq \sum_{\sigma\in S_n}\prod_{i=1}^n |\,a_{i,\sigma(i)}\,||\,b_{i,\sigma(i)}\,|\,.$$

By the Cauchy-Schwartz inequality we now have

$$\text{per } (A \circ B) \leq \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} \mid a_{i,\sigma(i)} \mid^2 \right)^{\frac{1}{2}} \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} \mid b_{i,\sigma(i)} \mid^2 \right)^{\frac{1}{2}}.$$

That is per $(A \circ B) \leq [\text{per } (A \circ \bar{A})]^{\frac{1}{2}} [per(B \circ \bar{B})]^{\frac{1}{2}}$. By hypothesis we have per $(A \circ B) \leq (per A)^{\frac{1}{2}} (per B)^{\frac{1}{2}} \leq \max\{\text{per } A, \text{ per } B\}$. The last inequality follows from the fact that $1 \leq \min\{\text{per } A, \text{ per } B\}$.

The lemma now follows from Lemma 2.1. □

3. The Main Theorems

Now we prove that for $n = 2$ or 3, Conjecture 1.1 is true. The case $n = 2$ follows from [4]. For $n = 3$ we need only show that inequality (5) holds for all correlation matrices.

THEOREM 3.1. *If A is a 3×3 correlation matrix, then*

$$\text{per } (A \circ \bar{A}) \leq \text{per } A.$$

Proof. Let

$$A = \begin{bmatrix} 1 & x & y \\ \bar{x} & 1 & z \\ \bar{y} & \bar{z} & 1 \end{bmatrix}$$

then per $A = 1 + 2\text{Re}(x\bar{y}z) + |x|^2 + |y|^2 + |z|^2$ and per $(A \circ \bar{A}) = 1 + 2|xyz|^2 + |x|^4 + |y|^4 + |z|^4$. Thus per $A -$ per $(A \circ \bar{A}) = 2\text{Re}(x\bar{y}z) + |x|^2 + |y|^2 + |z|^2 - 2|xyz|^2 - |x|^4 - |y|^4 - |z|^4$. We now show that per $A -$ per $(A \circ \bar{A}) \geq 0$. Since A is a correlation matrix, and hence positive semidefinite, so is $(A \circ \bar{A})$, and we have that $1 \geq |x|, |y|, |z|$.

Case 1. $\text{Re}(x\bar{y}z) > 0$.

If $2\text{Re}(x\bar{y}z) \geq |x|^2 + |y|^2 + |z|^2$, then $2\text{Re}(x\bar{y}z) \geq 2|xyz|^2$ since $|x|^2 \geq |xyz|^2$ and $|y|^2 \geq |xyz|^2$. Also, since $|x|^2 + |y|^2 + |z|^2 \geq |x|^4 + |y|^4 + |z|^4$, we have per $A -$ per $(A \circ \bar{A}) = (2\text{Re}(x\bar{y}z) - 2|xyz|^2) + (|x|^2 + |y|^2 + |z|^2) - (|x|^4 + |y|^4 + |z|^4) \geq 0$.

If $2\text{Re}(x\bar{y}z) < |x|^2 + |y|^2 + |z|^2$, then since A is positive semidefinite, $\det A \geq 0$ so that $1 \geq |x|^2 + |y|^2 + |z|^2 - 2\text{Re}(x\bar{y}z) \geq 0$. Hence $|x|^2 + |y|^2 + |z|^2 - 2\text{Re}(x\bar{y}z) \geq [|x|^2 + |y|^2 + |z|^2 - 2\text{Re}(x\bar{y}z)]^2$ so that $|x|^2 + |y|^2 + |z|^2 - [|x|^4 + |y|^4 + |z|^4] + 2\text{Re}(x\bar{y}z) \geq 4[\text{Re}(x\bar{y}z)]^2 + 2|xy|^2 + 2|yz|^2 + 2|xz|^2 -$

$4\mathrm{Re}(x\bar{y}z)[|x|^2+|y|^2+|z|^2]+4\mathrm{Re}(x\bar{y}z)$. But $2|xy|^2 \geq 2|xyz|^2$ so $|x|^2+|y|^2+|z|^2-[|x|^4+|y|^4+|z|^4]+2\mathrm{Re}(x\bar{y}z)-2|xyz|^2 \geq 4\mathrm{Re}(x\bar{y}z)[1+\mathrm{Re}(x\bar{y}z)-(|x|^2+|y|^2+|z|^2)]+2(|yz|^2+|xz|^2)$. Since $|xz|^2 \geq |xyz|^2 \geq [\mathrm{Re}(x\bar{y}z)]^2$, and $|yz|^2 \geq |xyz|^2 \geq [\mathrm{Re}(x\bar{y}z)]^2$,we have $2(|xz|^2+|yz|^2) \geq 4[\mathrm{Re}(x\bar{y}z)]^2$. Hence, per $A-$ per $(A \circ \bar{A}) = |x|^2+|y|^2+|z|^2-[|x|^4+|y|^4+|z|^4]+2\mathrm{Re}(x\bar{y}z)-2|xyz|^2 \geq 4\mathrm{Re}(x\bar{y}z)[1+2\mathrm{Re}(x\bar{y}z)-(|x|^2+|y|^2+|z|^2)] = 4\mathrm{Re}(x\bar{y}z)\det A \geq 0$.

Case 2. $\mathrm{Re}(x\bar{y}z) \leq 0$.

Since $\det A \geq 0$, we have $1 \geq 1+2\mathrm{Re}(x\bar{y}z) \geq |x|^2+|y|^2+|z|^2$.

If $|x| \leq \frac{1}{2}$, $|y| \leq \frac{1}{2}$, and $|z| \leq \frac{1}{2}$, then $|x|^4+|y|^4+|z|^4 \leq \frac{1}{4}(|x|^2+|y|^2+|z|^2)$. We now apply the geometric-arithmetic mean inequality to obtain $|xyz|^2 \leq \left(\frac{|x|^2+|y|^2+|z|^2}{3}\right)^3 = \frac{1}{27}(|x|^2+|y|^2+|z|^2)^3 \leq \frac{1}{27}(|x|^2+|y|^2+|z|^2)^2$, since $1 \geq |x|^2+|y|^2+|z|^2$, and hence, $|xyz| \leq \left(\frac{|x|^2+|y|^2+|z|^2}{3}\right)^{\frac{3}{2}} \leq \frac{1}{\sqrt{27}}(|x|^2+|y|^2+|z|^2)$. But then $-\mathrm{Re}(x\bar{y}z) = |\mathrm{Re}(x\bar{y}z)| \leq |xyz| \leq \frac{1}{\sqrt{27}}(|x|^2+|y|^2+|z|^2)$. Hence, $2\mathrm{Re}(x\bar{y}z)-2|xyz|^2-(|x|^4+|y|^4+|z|^4) \geq -\left(\frac{2}{\sqrt{27}}+\frac{2}{27}+\frac{1}{4}\right)(|x|^2+|y|^2+|z|^2)$. It follows that per $A-$per $(A\circ\bar{A}) \geq \left(1-\frac{2}{\sqrt{27}}-\frac{2}{27}-\frac{1}{4}\right)(|x|^2+|y|^2+|z|^2) \geq 0$.

Thus we may assume that one of $|x|$, $|y|$, or $|z|$ is greater than $\frac{1}{2}$ and without loss of generality we assume that $|y| > \frac{1}{2}$. Since $|x|^2+|y|^2+|z|^2 \leq 1$, we have $|x|^2+|y|^2+|z|^2-(|x|^4+|y|^4+|z|^4) \geq (|x|^2+|y|^2+|z|^2)^2-(|x|^4+|y|^4+|z|^4) = 2(|xy|^2+|yz|^2+|xz|^2)$. But $2|xz|^2 \geq 2|xyz|^2$ and $|xy^2z|^2 = |x^2y^2||y^2z^2| \leq \left(\frac{|xy|^2+|yz|^2}{2}\right)^2$, so that $2|xy^2z| \leq (|xy|^2+|yz|^2)$. Since $|y| > \frac{1}{2}$, we have $|xyz| < 2|xy^2z| \leq (|xy|^2+|yz|^2)$. Here, $-\mathrm{Re}(x\bar{y}z) = |\mathrm{Re}(x\bar{y}z)| \leq |xyz| \leq (|xy|^2+|yz|^2)$, and hence, $|x|^2+|y|^2+|z|^2-(|x|^4+|y|^4+|z|^4) \geq -\mathrm{Re}(x\bar{y}z)+2|xyz|^2$. Hence per $A-$per $(A\circ\bar{A}) = 2\mathrm{Re}(x\bar{y}z)+|x|^2+|y|^2+|z|^2-2|xyz|^2-|x|^4-|y|^4-|z|^4 \geq 0$. □

We now show that for some infinite classes of matrices, inequality (2), and hence inequality (3), holds.

THEOREM 3.2. *Let A be an $n\times n$ positive semidefinite real symmetric matrix of order n such that $a_{i,j} \geq 0$ for all $i,j = 1,\cdots,n$. If B is any positive semidefinite* Hermitian *matrix then* per $(A\circ B) \leq$ per $A\prod_{i=1}^{n} b_{i,i}$.

Proof. Suppose A and B are positive definite Hermitian matrices and $a_{i,j} \geq 0$ for all $i, j \leq n$. Since B is a positive definite Hermitian matrix, we have that $b_{i,i} > 0$ for all i. Let $D = (B \circ I)^{\frac{1}{2}}$ that is, D is the diagonal matrix whose diagonal entries are the positive square roots of the diagonal entries of B. Let $R = D^{-1}BD^{-1}$, so that $r_{i,j} = \frac{b_{i,j}}{(b_{i,i})^{\frac{1}{2}}(b_{j,j})^{\frac{1}{2}}}$. Then, R is a correlation matrix and $|r_{i,j}| \leq 1$. Now,

$$\begin{aligned}
\text{per}\,(A \circ B) &= \text{per}\,(A \circ DRD) \\
&= \text{per}\,(D(A \circ R)D) \\
&= \text{per}\,D^2 \text{per}\,(A \circ R) \\
&= \text{per}\,D^2 \left|\text{per}\,(A \circ R)\right| \\
&= \text{per}\,D^2 \left|\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} r_{i,\sigma(i)}\right| \\
&\leq \text{per}\,D^2 \sum_{\sigma \in S_n} \prod_{i=1}^{n} \left| a_{i,\sigma(i)} \right| \left| r_{i,\sigma(i)} \right| \\
&\leq \text{per}\,D^2 \sum_{\sigma \in S_n} \prod_{i=1}^{n} \left|a_{i,\sigma(i)}\right| \\
&= \text{per}\,D^2 \left|\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}\right| \\
&= \text{per}\,D^2 \left|\text{per}\,A\right| \\
&= \text{per}\,D^2 \text{per}\,A \\
&= \text{per}\,A \prod_{i=1}^{n} b_{i,i}.
\end{aligned}$$

Since the set of positive semidefinite Hermitian matrices is the closure of the set of positive definite Hermitian matrices, the lemma follows. □

COROLLARY 3.2.1. *If A is an $n \times n$ entrywise nonnegative correlation matrix then* per $(A \circ \bar{A}) \leq$ per A.

COROLLARY 3.2.2. *If A is a totally positive $n \times n$ semidefinite matrix and B is any $n \times n$ positive semidefinite* Hermitian *matrix then* per $(A \circ B) \leq$ per $A \prod_{i=1}^{n} b_{i,i}$.

Proof. Since a totally positive matrix has all its minors positive, each entry is positive. □

COROLLARY 3.2.3. *If A is an $n \times n$ tridiagonal positive semidefinite* Hermitian *matrix and B is any $n \times n$ positive semidefinite* Hermitian *matrix then* per $(A \circ B) \leq$ per $A \prod_{i=1}^{n} b_{i,i}$.

Proof. Since every tridiagonal semidefinite Hermitian matrix is diagonally congruent to a real entrywise nonnegative correlation matrix, the corollary follows from Theorem 3.2. □

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