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CRITICAL POINTS AND CONFORMALLY FLAT METRICS

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ABSTRACT. It has been conjectured that, on a compact 3–dimensional manifold, a critical point of the total scalar curvature functional restricted to the space of constant scalar curvature metrics of volume 1 is Einstein. In this paper we find a sufficient condition that a critical point is Einstein. This condition is equivalent for a critical point to be conformally flat. Its relationship with the Fisher-Marsden conjecture is also discussed.

I. Introduction

The total scalar curvature functional $\mathcal{S}$ of a compact 3–dimensional manifold (M^3, g) is defined by $\mathcal{S}(g) = \int_{M^3} s_g dv_g$, where s_g denotes the scalar curvature of the metric g and dv_g is the volume form. Let $\mathcal{M}_1$ be all smooth Riemannian structures on M^3 of volume 1. It is well known that a critical point of the scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{M}_1$ is Einstein. When we restrict the domain of the scalar curvature functional $\mathcal{S}$ to the space of constant scalar curvature metrics of volume 1, the equation for a critical point is given by

$$z_g = D_g df - (\Delta_g f)g - f r_g \tag{1}$$

where z_g is the traceless Ricci tensor, r_g the Ricci tensor, f a function on M with vanishing mean value, and $\Delta_g f = -\frac{s_g}{2} f$ [2].

J. Lafontaine showed that if a solution to the equation (1) is conformally flat, such a metric is Einstein [5]. M. Obata proved that an Einstein

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critical point with nonzero f is isometric to a standard 3−sphere [7]. In [3], it was shown that if the minimum value of f is -1, g is Einstein. It has been conjectured that the only solutions to these equations are Einstein metrics, with either $f \equiv 0$ or f a first eigenfunction of the Laplacian on (S^3, g_0) (Conjecture A).

In the present paper, we find a sufficient condition that a critical point is Einstein, and discuss its relationship with the Fisher-Marsden Conjecture. This condition will be expressed in terms of z_g only. In fact, we prove in this paper that this condition is equivalent to a statement that a solution metric g is conformally flat. In virtue of previously mentioned results, this g is isometric to a standard 3−sphere. In a forthcoming paper, using the technique developed in this paper(especially the equation (15)), we will show that a solution metric g is Einstein in a more general setting.

Now our main result can be stated as follows:

THEOREM 1. *Suppose that z_g of a critical point g has eigenvalues $2z_1 = 2z_2 = -z_3$ on $f^{-1}(c)$ for regular values c's, where z_3 is the eigenvalue corresponding to the normal vector field ν_c, and z_1 and z_2 to the tangent vector fields to $f^{-1}(c)$. Then g is conformally flat. In particular, (M^3, g) is isometric to a standard 3−sphere.*

REMARK 1. Note that if we set $s_g = 0$, the equation (1) is reduced to the so-called *static Einstein vacuum equations* given by

$$\begin{aligned} hr_g &= D_g dh \\ \Delta_g h &= 0 \end{aligned}$$

where $h = 1 + f$. Thus we may think of the equations (1) as a compact manifold version of the static equations. Note that static equations are nontrivial only for noncompact manifolds. If a 3−dimensional solution to the static equation has eigenvalues $2z_1 = 2z_2 = -z_3$ everywhere, such a solution metric is called *degenerate*. Degenerate static metrics are classified by Levi-Civita [4]. The induced metrics on the hypersurfaces of the Schwarzschild metric and of the so-called $B1$−metric are among them. Theorem 1 shows that "degenerate" critical points are uniquely characterized as standard 3−spheres.

REMARK 2. There is a conjecture that any metric $g \in \mathcal{M}_1$ realizing the Sigma constant $\sigma(M^n)$ is Einstein for $n \geq 3$ (Conjecture B), cf.

[1]. The Sigma constant $\sigma(M^n)$ is defined by a minimax procedure. In particular, a metric g realizing $\sigma(M^n)$ is a critical point of S restricted to the space of constant scalar curvature metrics. Thus the resolution of Conjecture A implies Conjecture B. If $\sigma(M) \leq 0$, it is known that such a metric g is Einstein. The case of $\sigma(M) > 0$ is more complicated and not much known. We have the following partial answer to the conjecture B as an immediate consequence of Theorem 1.

COROLLARY 1. *A metric $g \in \mathcal{M}_1$ realizing $\sigma(M^n)$ and satisfying the assumption of* Theorem 1 *is* Einstein.

II. Conformally flat metrics

This section is devoted to the proof of Theorem 1. For the proof, we consider a non-negative function $\mathsf{D}_c = ||z||^2 - \frac{3}{2}z(\nu_c, \nu_c)^2$ on $f^{-1}(c)$ for regular values c's. It is easy to see that the condition that z_g has eigenvalues $2z_1 = 2z_2 = -z_3$ on $f^{-1}(c)$ for regular values c's is equivalent to saying that $\mathsf{D}_c = 0$ on each $f^{-1}(c)$. Throughout present section, we represent tensors in a given coordinate system $\{e_a\}_{a=1,2,3}$ with $e_3 = \nu$. Let (g, f) be a solution of (1) on a compact manifold M^3.

DEFINITION 1. Let H be the *Weyl-Schouten tensor field* defined by $H = r_g - \frac{s_g}{4} g$. We denote by $d^D H$ the differential operator from $C^\infty(S^2(M))$ into $\Lambda^2 M \otimes T^* M$ defined by

$$d^D H(x, y, z) \equiv D_x H(y, z) - D_y H(x, z)$$

or, in a given coordinate system,

$$d^D H_{abc} \equiv R_{abc} = r_{ab;c} - r_{ac;b} + \frac{1}{4}(g_{ac} s_{;b} - g_{ab} s_{;c}) \tag{2}$$

where we denote by ";" the covariant differentiation.

For dimension 3, it is well known that a metric is conformally flat if and only if $d^D H$ derived from the Weyl-Schouten tensor H vanishes identically. The equation in the following lemma has the same flavor as Robinson equalities, c.f. [8].

LEMMA 1. *On M^3, the traceless* Ricci *tensor z_g and the tensor $d^D H$ are related by*

$$h^2 W||z_g||^2 = \frac{h^4}{8}||d^D H||^2 + \frac{3}{8}|dW + \frac{sf}{3}df|^2 \tag{3}$$

with $W = |df|^2$.

Proof. The equation (1) can be rewritten as

$$(1+f)z_g = D_g df + \frac{s_g f}{6} g. \tag{4}$$

A representation of (4) in a given coordinate system $\{e_a\}_{a=1,2,3}$ is

$$hr_{ab} = f_{;ab} + \frac{2+3f}{6} s g_{ab} \tag{5}$$

with $h = 1+f$. Substitution of (5) into (2) gives

$$h^2 R_{abc} = h(f_{;abc} - f_{;acb}) - (f_{;ab}f_{;c} - f_{;ac}f_{;b}) + \frac{s}{6}(f_{;c}g_{ab} - f_{;b}g_{ac}). \tag{6}$$

In virtue of the Ricci identity $f_{;abc} - f_{;acb} = R_{bcla}f^{;l}$, the first term of the right-hand side of (6) is $hR_{bcla}f^{;l}$. Since, in dimension 3, R_{ijkl} are given by

$$R_{ijkl} = -\frac{s}{2}(g_{ik}g_{jl} - g_{il}g_{jk}) + (r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il})$$

we have

$$hR_{bcla}f^{;l} = \frac{s}{2}h(g_{ba}f_{;c} - g_{ca}f_{;b}) + h(r_{bl}g_{ca}f^{;l} - r_{cl}g_{ba}f^{;l}) + h(r_{ca}f_{;b} - r_{ba}f_{;b}). \tag{7}$$

Substituting the following equations

$$\begin{aligned} hr_{bl}g_{ca}f^{;l} &= f_{;bl}f^{;l}g_{ca} + \frac{2+3f}{6}sf_{;b}g_{ca} = \frac{W_{;b}}{2}g_{ca} + \frac{2+3f}{6}sf_{;b}g_{ca} \\ hr_{ca}f_{;b} &= f_{;ca}f_{;b} + \frac{2+3f}{6}sg_{ca}f_{;b} \end{aligned}$$

together with their permutations of indices into (7), (6) becomes

$$h^2 R_{abc} = \frac{sf}{2}(g_{ca}f_{;b} - g_{ba}f_{;c}) + 2(f_{;ac}f_{;b} - f_{;ab}f_{;c}) + \frac{1}{2}(W_{;b}g_{ca} - W_{;c}g_{ba}). \tag{8}$$

In virtue of (8) we have

(9)

$$h^4|d^D H|^2 = h^4 R_{abc}R^{abc} = -s^2 f^2 W - 2sf\langle df, dW\rangle - 3|dW|^2 + 8|D_g df|^2 W.$$

Let $\mathsf{L} \equiv dW + \frac{sf}{3}df$. Then

$$3|\mathsf{L}|^2 = 3|dW|^2 + 2sf\langle df, dW\rangle + \frac{s^2f^2}{3}. \tag{10}$$

On the other hand, (4) gives

$$h^2||z_g||^2 = ||D_g df||^2 - \frac{s^2f^2}{12}. \tag{11}$$

Substituting (10) and (11) into (9), we finally have

$$h^4||d^DH||^2 = -3|\mathsf{L}|^2 + 8Wh^2||z_g||^2 \tag{12}$$

which proves our assertion. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Since the relation

$$dW + \frac{sf}{3}df = 2D_{df}df + \frac{sf}{6}df = 2hz(df, \cdot)$$

holds in virtue of (4), we may rewrite $|\mathsf{L}|^2$ as

$$|\mathsf{L}|^2 = |dW + \frac{sf}{3}df|^2 = 4h^2\sum_{i=1}^{3} z(df, e_i)^2. \tag{13}$$

Hence, in virtue of (13), the relation (3) in Lemma 1 becomes

$$W||z_g||^2 = \frac{h^2}{8}||d^DH||^2 + \frac{3}{2}\sum_{i=1}^{3} z(df, e_i)^2 \tag{14}$$

or

$$W\mathsf{D}_c = \frac{h^2}{8}||d^DH||^2 + \frac{3}{2}W\sum_{i=1}^{2} z(\nu_c, e_i)^2. \tag{15}$$

The equation (15) says that the tensor d^DH and D_c are related by

$$\frac{h^2}{8}||d^DH||^2 \leq |df|^2\mathsf{D}_c. \tag{16}$$

Our assumption that $\mathsf{D}_c = ||z_g||^2 - \frac{3}{2}z(\nu_c, \nu_c)^2 = 0$ on $f^{-1}(c)$ for regular values c's implies that the left-hand side of (16) vanishes. Hence, by continuity of d^DH, we may conclude that d^DH vanishes identically on all of M^3, since the set of critical points has measure zero. This implies that the metric g is conformally flat. A conformally flat metric g

should be isometric to a standard 3−sphere by [5] and [7], or by a brute computation. This completes the proof of Theorem 1.

REMARK 3 [A geometric meaning of μ]. In [3], we proved that if $r_g > \frac{1}{\mu}\frac{s}{3}$ with $\mu = \min_{M^3} h$, there is no embedded compact oriented stable minimal surface Σ in M^3. It is also mentioned that if $r_g \geq \frac{1}{\mu}\frac{s}{3}$, M^3 might have such a surface Σ in $h^{-1}(\mu)$. Our result in this article, especially (16), implies that $d^D H \equiv 0$ on $h^{-1}(\mu)$. In other words, the metric g is conformally flat on $h^{-1}(\mu)$.

III. Remarks on Fisher-Marsden conjecture

The conjecture of Fisher-Marsden states that a standard n−sphere is the only solution to the equation

$$0 = D_\gamma df - (\Delta_\gamma f)\gamma - f r_\gamma \tag{17}$$

on a compact manifold (N^n, γ). The solution metric γ satisfying (17) shares similar properties with the critical point g satisfying (1). For example, we have

$$\int_{N^3} f||z_\gamma||^2 = 0$$

(compare with Lemma 1 of [3]). If we consider $\tilde{d}^D H$ and $\tilde{D}_c$ for γ, the calculation similar in the proof of Theorem 1 shows that

$$\frac{f^2}{8}||\tilde{d}^D H||^2 \leq |df|^2 \tilde{D}_c. \tag{18}$$

Therefore, if $\tilde{D}_c$ vanishes on N^3, $d^{\tilde{D}} H \equiv 0$ identically. In other words, the metric γ is conformally flat. However, this γ may not be isometric to S^3, since there exist conformally flat metrics satisfying (17) and not being isometric to standard 3−spheres. Among them are finite quotient of $S^1 \times S^2$ or a warped product of S^1 and S^2, c.f. [5]. Therefore the last statement of Theorem 1 does not hold for the solution of (17).

The failure of the Fisher-Marsden conjecture is related to the existence of embedded compact oriented stable minimal surfaces. Note that each element in $H_2(N^3, \mathbb{Z})$ can be represented by sums of such surfaces, c.f. [6]. It is shown that if $r_g > \frac{1}{\mu}\frac{s}{3}$, $H_2(M^3, \mathbb{Z}) = 0$ [3]. In particular such a Σ does not exist in M^3 as mentioned in Remark 3. For a solution metric γ of (17), there is no known obstruction of the existence of such a surface Σ in N^3 under generic curvature condition. Thus it may be that $H_2(N^3, Z) \neq 0$. However, there is no embedded compact oriented stable minimal surface Σ in N^3 which is not totally geodesic. It is due to the following structure result for Σ:

THEOREM 2. *An embedded compact oriented stable minimal hypersurface Σ in N^n for $n \geq 3$ is totally geodesic.*

Proof. The Laplacian Δ_γ and the intrinsic Laplacian Δ' on the minimal surface Σ are related by

$$\Delta f = \Delta' f + Ddf(\nu, \nu) \tag{19}$$

where ν is a normal vector field on Σ. From (17), we also have

$$Ddf(\nu, \nu) = f r_\gamma(\nu, \nu) + \Delta f. \tag{20}$$

Since substitution of (20) into (19) gives

$$\Delta' f + f r_\gamma(\nu, \nu) = 0 \tag{21}$$

it follows that

$$\int_\Sigma f^2 r(\nu, \nu) = \int_\Sigma |\nabla f|_\gamma^2. \tag{22}$$

On the other hand, the stability condition may be written as

$$\int_\Sigma f^2(r(\nu, \nu) + ||II||^2) \leq \int_\Sigma |\nabla f|_\gamma^2 \tag{23}$$

where $||II||^2$ is the length of the second fundamental form of Σ and $|\cdot|_\gamma$ is a norm from the induced metric on Σ. On the Hence, substitution of (22) into (23) gives

$$\int_\Sigma f^2 ||II||^2 = 0.$$

Therefore, $f^2||II||^2 \equiv 0$ or $II = 0$ on Σ possibly except at any point p in a set $\Omega = \{x \in N^n | f(x) = 0\}$. Suppose that there is a neighborhood U_p of p such that $U_p \subset \Omega$. It is easy to see that U_p is totally geodesic, since $Ddf = 0$ on Ω with $|df|$ constant. Thus $II_p = 0$. If such a neighborhood does not exist, it is clear that $II_p = 0$ by continuity of II. □

The above proof is a slight modification of the proof of Main Theorem in [3]. It should be noted that Theorem 2 is comparable with Theorem 1 of [9]. As an immediate consequence of Theorem 2, being solutions of (17), in a finite quotient of $(S^1, dt^2) \times (S^{n-1}, g_0)$ or a finite quotient of the warped product $(S^1 \times S^{n-1}, dt^2 + \phi^2(t) g_0)$ where (S^{n-1}, g_0) is the standard sphere and ϕ is a periodic solution of

$$s_\gamma \phi^2 = s_{g_0} - (n-1)((n-2)\phi'^2 + 2\phi\phi'') \qquad \text{if} \quad n \neq 4$$
$$s_\gamma \phi^2 + 3(\phi^2)'' = s_{g_0} \qquad \text{if} \quad n = 4$$

every oriented stable minimal hypersurface should be totally geodesic. In S^n, it is well known that there are no compact stable minimal submanifolds in it.

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