NORMALIZING MAPPINGS OF AN ANALYTIC GENERIC CR MANIFOLD WITH ZERO LEVI FORM

Won K. Park

Abstract. It is well-known that an analytic generic CR submanifold $M$ of codimension $m$ in $\mathbb{C}^{n+m}$ is locally transformed by a biholomorphic mapping to a plane $\mathbb{C}^n \times \mathbb{R}^m \subset \mathbb{C}^n \times \mathbb{C}^m$ whenever the Levi form $L$ on $M$ vanishes identically. We obtain such a normalizing biholomorphic mapping of $M$ in terms of the defining function of $M$. Then it is verified without Frobenius theorem that $M$ is locally foliated into complex manifolds of dimension $n$.

0. Introduction

Let $\rho_1, \cdots, \rho_m$ be real-valued functions near the origin in $\mathbb{C}^{n+m}$ such that

$$\rho_1|_0 = \cdots = \rho_m|_0 = 0$$

and

$$\partial \rho_1 \wedge \cdots \wedge \partial \rho_m|_0 \neq 0.$$ 

Suppose that a generic CR submanifold $M$ of codimension $m$ in a sufficiently small domain $\Omega \ni 0$ is defined by the real-valued functions $\rho_1, \cdots, \rho_m$ as follows

$$\rho_1 = \cdots = \rho_m = 0.$$ 

Then there is a natural differential system $D$ on $M$ defined by

$$d\rho_1 = \cdots = d\rho_m = d^c \rho_1 = \cdots = d^c \rho_m = 0$$

where $d^c$ is the imaginary part of $\partial$. The differential system $D$ is indeed a subbundle of real dimension $2n$ in $TM$. Further, the complex structure of $\mathbb{C}^{n+m}$ induces a bundle automorphism $I$ on $D$ satisfying the following conditions

2000 Mathematics Subject Classification: Primary 32H99.
Key words and phrases: CR submanifold, Levi form, biholomorphic mapping.
(1) \[ I^2 U = -U \]

(2) \[ [U, V] - [IU, IV], \quad [IU, V] + [U, IV] \in \Gamma D \]

(3) \[ [U, V] - [IU, IV] + I([IU, V] + [U, IV]) = 0 \]

for all \( U, V \in \Gamma D \). By (1), we have the following decomposition
\[ D \otimes \mathbb{C} = H \oplus \overline{H}, \]

where
\[ IW = iW \quad \text{for} \quad W \in \Gamma H. \]

Then (2) and (3) are equivalent to
\[ [W, Z] \in \Gamma H \quad \text{for} \quad W, Z \in \Gamma H. \]

Then the Levi form \( L \) of the generic CR submanifold \( M \) is defined by the intrinsic objects \( (M, D, I) \) as the composition of the following sequence
\[ D \otimes D \xrightarrow{b_1} TM \xrightarrow{b_2} TM/D, \]

where \( b_1 \) is the Lie bracket with the operation \( I \) as follows
\[ b_1(U, V) = [U, IV] \]

and \( b_2 \) is the natural projection. Clearly, the Levi form \( L \) is also an intrinsic object of \( M \). With (1) and (2), we obtain the following properties of the Levi form \( L \)
\[
\begin{align*}
L(fU, V) &= L(U, fV) = fL(U, V) \\
L(U, V) &= L(V, U) \\
L(IU, IV) &= L(U, V)
\end{align*}
\]

for \( f \in \Gamma(M, \mathbb{R}) \) and \( U, V \in \Gamma D \). Hence we obtain
\[ L(W, Z) = L(\overline{W}, \overline{Z}) = 0 \]

for \( W, Z \in \Gamma H. \) Thus the Levi form \( L \) is completely determined by the value \( L(W, Z) \).

Note that the operation \( I \) is an automorphism on \( D \). Thus the Levi form \( L \) is faithfully represented by a two-form \( l \) obtained by composing the following sequence
\[ A^2 D \xrightarrow{b_1} TM \xrightarrow{b_2} TM/D \rightarrow TM/D \otimes (TM/D)^* \rightarrow M \times \mathbb{R}, \]
where $b^*_1$ is the Lie bracket. Since the generic CR submanifold $M$ is defined by the real-valued functions $\rho_1, \ldots, \rho_m$ satisfying the condition $\partial \rho_1 \wedge \cdots \wedge \partial \rho_m \neq 0$, the one-forms $d^c \rho_1, \ldots, d^c \rho_m$ make a basis of $(TM/D)^*$. Then we define a two-form $l = (l_1, \ldots, l_m)$ as follows
\[
\begin{align*}
l_1(U, V) &= -d^c \rho_1([U, V]) = 2dd^c \rho_1(U, V) = 2i\partial \bar{\partial} \rho_1(U, V) \\
&\quad \vdots \\
l_m(U, V) &= -d^c \rho_m([U, V]) = 2dd^c \rho_m(U, V) = 2i\partial \bar{\partial} \rho_m(U, V)
\end{align*}
\]
for $U, V \in \Gamma D$. Note that the differential system $D$ on $M$ is defined by the one-forms
\[d\rho_1, \ldots, d\rho_m, d^c \rho_1, \ldots, d^c \rho_m.\]
Thus the Levi form $L$ is essentially equivalent to the information of the two-form
\[l = 2dd^c \rho = 2i\partial \bar{\partial} \rho\]
up to
\[\text{mod } d\rho_1, \ldots, d\rho_m, d^c \rho_1, \ldots, d^c \rho_m.\]

Then the zero Levi form is represented by the following condition
\[l \equiv 0 \mod d\rho_1, \ldots, d\rho_m, d^c \rho_1, \ldots, d^c \rho_m.\]
Since we have
\[\phi^* \circ \partial = \partial \circ \phi^*, \quad \phi^* \circ \bar{\partial} = \bar{\partial} \circ \phi^*\]
for any biholomorphic mapping $\phi$, the zero Levi form leaves invariant under a biholomorphic mapping $\phi$ as follows
\[
\begin{align*}
2i\partial \bar{\partial} \phi^* \rho &= 2i\phi^* \partial \bar{\partial} \rho \\
&\equiv 0 \mod d\phi^* \rho_1, \ldots, d\phi^* \rho_m, d^c \phi^* \rho_1, \ldots, d^c \phi^* \rho_m.
\end{align*}
\]
It is well-known that a generic CR submanifold $M$ with zero Levi form is locally foliated into complex manifolds (cf. [1]). Further, an analytic generic CR submanifold $M$ with zero Levi form is locally biholomorphic to a plane $\mathbb{C}^n \times \mathbb{R}^m \subset \mathbb{C}^n \times \mathbb{C}^m$. We shall obtain a biholomorphic mapping in terms of the defining functions $\rho_1, \ldots, \rho_m$ which transforms $M$ to a plane $\mathbb{C}^n \times \mathbb{R}^m$. Thus it is verified that $M$ is locally analytically foliated into complex manifolds of complex dimension $n$, which has been obtained within our knowledge under the assumption of Frobenius theorem for the existence of a foliation and Newlander-Nirenberg theorem/Levi-Civita theorem for its leaf to be a complex manifold (cf. [1]).
1. Straightening a totally real surface \( \Gamma \)

Let \( M \) be an analytic generic CR submanifold in \( \Omega \subset \mathbb{C}^{n+m} \) near the origin defined by

\[
\rho_1 = \cdots = \rho_m = 0,
\]

where

\[
\partial \rho_1 \wedge \cdots \wedge \partial \rho_m \neq 0.
\]

Then we may take a coordinate \((z, w) \in \mathbb{C}^n \times \mathbb{C}^m\), if necessary, after a suitable linear change of coordinates such that

\[
\rho = -v + F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0,
\]

where \(\rho = (\rho_1, \cdots, \rho_m)\), \(u = \Re w\) and \(v = \Im w\). Thus \( M \) is defined near the origin by the following equation

\[
v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0.
\]

Let \( \Gamma \) be an analytic real surface of dimension \( m \) on \( M \), which is transversal to the complex tangent hyperplane at the origin 0. Then the equation of \( \Gamma \) is given near the origin as follows

\[
\Gamma \left\{ \begin{array}{l}
z = p(\mu) \\
u = q(\mu)
\end{array} \right.,
\]

where

\[
p(0) = q(0) = 0, \quad \det q'(0) \neq 0.
\]

By the condition \( F|_0 = dF|_0 = 0 \), we can take the \( \mathbb{R}^m \)-valued parameter \( \mu \) such that

\[
q'(0) = Id_{m \times m}, \quad \Re q(\mu) = \mu,
\]

where \( Id_{m \times m} \) is the identity matrix and \( \Re q(\mu) \) is the real part of \( q(\mu) \). Hence the real surface \( \Gamma \) on \( M \) determines a unique function \( p(\mu) \) and \( \Gamma \) is uniquely described by the function \( p(\mu) \) via the following equation

\[
\Gamma \left\{ \begin{array}{l}
z = p(\mu) \\
u = \mu \\
v = F(p(\mu), \bar{p}(\mu), \mu).
\end{array} \right.
\]

Assume that the generic CR submanifold \( M \) and the surface \( \Gamma \) on \( M \) are both analytic so that the functions \( F(z, \bar{z}, u) \) and \( p(\mu) \) are both analytic. Then there is a unique holomorphic function \( g(z, w) \), which is implicitly defined by the equations
(5) \[ g(z, w) - g(0, w) = -2iF(p(w), \bar{p}(w), w) + 2iF\left(z + p(w), \bar{p}(w), w + \frac{1}{2}\left\{g(z, w) - g(0, w)\right\}\right), \]
\[ g(0, w) = iF(p(w), \bar{p}(w), w). \]

The holomorphic function \( g(z, w) \) is well defined because of the condition
\[ F|_0 = dF|_0 = 0, \]
which implies
\[ g|_0 = \frac{\partial g}{\partial z}|_0 = \frac{\partial g}{\partial w}|_0 = 0. \]

Then we consider a holomorphic mapping near the origin as follows
\[ z = z^* + p(w^*), \]
\[ w = w^* + g(z^*, w^*). \]

By (6), the mapping (7) is biholomorphic near the origin for any analytic function \( p(w) \). We claim that the generic CR submanifold \( M \) is transformed to a generic CR submanifold \( M' \) of the form
\[ v = \sum_{s,t=1}^{\infty} F^*_{st}(z, \bar{z}, u) \]
and the surface \( \Gamma \) on \( M \) via the equation (4) is mapped on the \( u \)-plane, \( z = v = 0 \), under the biholomorphic mapping (7).

Suppose that the generic CR submanifold \( M' \) is defined by
\[ v^* = F^*(z^*, \bar{z}^*, u^*). \]
The mapping (7) yields the following equality
\[ F(z, \bar{z}, u) = F^*(z^*, \bar{z}^*, u^*) + \frac{1}{2i}\left\{g(z^*, u^* + iv^*) - \bar{g}(\bar{z}^*, u^* - iv^*)\right\}, \]
where
\[ z = z^* + p(u^* + iv^*), \]
\[ \bar{z} = \bar{z}^* + \bar{p}(u^* - iv^*), \]
\[ u = u^* + \frac{1}{2}\left\{g(z^*, u^* + iv^*) + \bar{g}(\bar{z}^*, u^* - iv^*)\right\}. \]
Since $F$ and $F^*$ are both real-analytic, we can consider $z^*, \bar{z}^*$ and $u^*$ as independent variables. Hence the condition of $F^*(z^*, 0, u^*) = v^* = 0$ is equivalent to the following equality

\begin{equation}
(8) \quad g(z, u) - \bar{g}(0, u) = 2iF\left(z + p(u), \bar{p}(u), u + \frac{1}{2}\left\{g(z, u) - \bar{g}(0, u)\right\}\right).
\end{equation}

We obtain an equality by taking $z = 0$

\begin{equation}
\nonumber g(0, u) - \bar{g}(0, u) = 2iF\left(p(u), \bar{p}(u), u + \frac{1}{2}\left\{g(0, u) - \bar{g}(0, u)\right\}\right),
\end{equation}

which implies that

\[ g(0, u) + \bar{g}(0, u) = 0 \]

if and only if

\[ g(0, u) = iF(p(u), \bar{p}(u), u). \]

Hence (8) reduces to

\begin{equation}
\nonumber g(z, u) - g(0, u) = -2iF\left(p(u), \bar{p}(u), u\right)
+ 2iF\left(z + p(u), \bar{p}(u), u + \frac{1}{2}\left\{g(z, u) - g(0, u)\right\}\right).
\end{equation}

Thus the equality (8) is satisfied by the function $g(z, u)$ defined in the mapping (5). By putting

\[ z^* = \bar{z}^* = v^* = 0 \]

in (7), we obtain

\begin{align*}
z & = p(u^*), \\
u & = u^*, \\
v & = F(p(u^*), \bar{p}(u^*), u^*).
\end{align*}

Thus the surface $\Gamma$ on $M$ in (4) is mapped on the $u$-plane by the biholomorphic mapping (7).
From the equation (5), we obtain the holomorphic function $g(z, w)$ up to order 2 inclusive of the variable $z$ as follows

$$
g(z, w) = iF(p(w), \bar{p}(w), w) + 2i(Id - iF')^{-1} \left\{ \sum_{\alpha=1}^{n} z^\alpha \left( \frac{\partial F}{\partial z^\alpha} \right)(p(w), \bar{p}(w), w) + \sum_{\alpha, \beta=1}^{n} \frac{z^\alpha z^\beta}{2} \left( \frac{\partial^2 F}{\partial z^\alpha \partial z^\beta} \right)(p(w), \bar{p}(w), w) \right\}
$$

$$
-2(Id - iF')^{-1} \left\{ \sum_{\alpha=1}^{n} z^\alpha \left( \frac{\partial F}{\partial z^\alpha} \right)(p(w), \bar{p}(w), w) \right\}
\times (Id - iF')^{-1} \left\{ \sum_{\alpha=1}^{n} z^\alpha \left( \frac{\partial F}{\partial z^\alpha} \right)(p(w), \bar{p}(w), w) \right\}
$$

$$
-2i(Id - iF')^{-1} F''
\times \left( (Id - iF')^{-1} \left\{ \sum_{\alpha=1}^{n} z^\alpha \left( \frac{\partial F}{\partial z^\alpha} \right)(p(w), \bar{p}(w), w) \right\} \right)^2 + \sum_{|l|=3} O(z^l),
$$

where

$$
(F')_{ab} = \left( \frac{\partial F^a}{\partial u^b} \right)(p(w), \bar{p}(w), w),
$$

$$
\left( \frac{\partial F'}{\partial z^\alpha} \right)_{ab} = \left( \frac{\partial^2 F^a}{\partial z^\alpha \partial u^b} \right)(p(w), \bar{p}(w), w),
$$

$$
(F'')_{abc} = \frac{1}{2} \left( \frac{\partial^2 F^a}{\partial u^b \partial u^c} \right)(p(w), \bar{p}(w), w).
$$

We shall examine the dependence of the function $F^*_i(z, \bar{z}, u)$ of the lowest type $(1, 1)$ on the function $p(u)$ and its derivatives.

**Lemma 1.** Let $M'$ be the generic CR submanifold obtained from $M$ by the mapping (7) and defined by

$$
v = F^*(z, \bar{z}, u) = \sum_{s, t=1}^{\infty} F^*_{st}(z, \bar{z}, u).
$$
Then the function \( F'_{11}(z, \bar{z}, u) \) depends on \( p(u) \) and \( p'(u) \) as follows

\[
F'_{11}(z, \bar{z}, u) = \left\{ \begin{array}{c}
\text{Id} - i(\text{Id} + iF') \sum_{\alpha=1}^{n} \left( \frac{\partial F}{\partial z^\alpha} \right) p^\alpha \\
+i(\text{Id} - iF') \sum_{\alpha=1}^{n} \left( \frac{\partial F}{\partial z^\alpha} \right) \bar{p}^\alpha + (F')^2 \end{array} \right\}^{-1}
\times \left\{ \begin{array}{c}
\sum_{\alpha, \beta=1}^{n} \left( \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \right) z^\alpha \bar{z}^\beta \\
-i \sum_{\alpha, \beta=1}^{n} \left( \frac{\partial F'}{\partial z^\alpha} \right) (\text{Id} + iF')^{-1} z^\alpha \bar{z}^\beta \\
+i \sum_{\alpha, \beta=1}^{n} \left( \frac{\partial F'}{\partial z^\alpha} \right) (\text{Id} - iF')^{-1} \left( \frac{\partial F}{\partial \bar{z}^\beta} \right) z^\alpha \bar{z}^\beta \\
-2 \sum_{\alpha, \beta=1}^{n} F''(\text{Id} - iF')^{-1} \left( \frac{\partial F}{\partial z^\alpha} \right) z^\alpha (\text{Id} + iF')^{-1} \left( \frac{\partial F}{\partial \bar{z}^\beta} \right) \bar{z}^\beta
\end{array} \right\},
\]

where

\[
\left( \frac{\partial F}{\partial z^\alpha} \right)_a = \left( \frac{\partial F^a}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u),
\left( \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \right)_a = \left( \frac{\partial^2 F^a}{\partial z^\alpha \partial \bar{z}^\beta} \right) (p(u), \bar{p}(u), u),
\left\{ \left( \frac{\partial F}{\partial z^\alpha} \right) p^\alpha \right\}_{ab} = \left( \frac{\partial F^a}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u) \left( \frac{\partial p^a}{\partial u^b} \right) (u),
\left( \frac{\partial F'}{\partial z^\alpha} \right)_{ab} = \left( \frac{\partial^2 F^a}{\partial z^\alpha \partial u^b} \right) (p(u), \bar{p}(u), u),
(F')_{ab} = \left( \frac{\partial F^a}{\partial u^b} \right) (p(u), \bar{p}(u), u),
(F'')_{abc} = \frac{1}{2} \left( \frac{\partial^2 F^a}{\partial u^b \partial u^c} \right) (p(u), \bar{p}(u), u).
\]
Proof. The generic CR submanifold \( M' \) is defined by the following equation

\[
\begin{align*}
v &= F(z + p(u + iv), \bar{z} + \bar{p}(u - iv), \\
&\quad u + \frac{1}{2}\{g(z, u + iv) + \bar{g}(\bar{z}, u - iv)\} \\
&\quad - \frac{1}{2i}\{g(z, u + iv) - \bar{g}(\bar{z}, u - iv)\} \\
&= A(z, \bar{z}, u) + B(z, \bar{z}, u)v + O(|v|^2),
\end{align*}
\]

where

\[
A(z, \bar{z}, u) = F\left(z + p(u), \bar{z} + \bar{p}(u), u + \frac{1}{2}\{g(z, u) + \bar{g}(\bar{z}, u)\}\right) \\
- \frac{1}{2i}\{g(z, u) - \bar{g}(\bar{z}, u)\}
\]

\[
B(z, \bar{z}, u) = i \sum_{\alpha=1}^{n} \left( \frac{\partial F'}{\partial z^\alpha} \right) \left( z + p(u), \bar{z} + \bar{p}(u), u + \frac{1}{2}\{g(z, u) + \bar{g}(\bar{z}, u)\}\right) p^{\alpha'}(u) \\
- i \sum_{\alpha=1}^{n} \left( \frac{\partial F'}{\partial \bar{z}^\alpha} \right) \left( z + p(u), \bar{z} + \bar{p}(u), u + \frac{1}{2}\{g(z, u) + \bar{g}(\bar{z}, u)\}\right) \bar{p}^{\alpha'}(u) \\
- F'\left(z + p(u), \bar{z} + \bar{p}(u), u + \frac{1}{2}\{g(z, u) + \bar{g}(\bar{z}, u)\}\right) \\
\times \frac{1}{2i}\{g'(z, u) - \bar{g}'(\bar{z}, u)\} \\
- \frac{1}{2}\{g'(z, u) + \bar{g}'(\bar{z}, u)\}.
\]

With the function \( g(z, w) \) in (5), we can put

\[
A(z, \bar{z}, u) = \sum_{s, t \geq 1} A_{st}(z, \bar{z}, u),
\]

\[
B(z, \bar{z}, u) = \sum_{s, t \geq 0} B_{st}(z, \bar{z}, u).
\]

By using the expansion (9) of \( g(z, w) \), we obtain

\[
v = \left\{I - B_{00}(z, \bar{z}, u)\right\}^{-1} A_{11}(z, \bar{z}, u) + O(|z|^3),
\]

\[
\left\{ A_{11}(z, \bar{z}, u) \right\}_a = \sum_{\alpha, \beta = 1}^n z^\alpha \bar{z}^\beta \left( \frac{\partial^2 F^a}{\partial z^\alpha \partial \bar{z}^\beta} \right) (p(u), \bar{p}(u), u) \\
+ \sum_{\alpha, \beta = 1}^n \sum_{b = 1}^m \frac{z^\alpha \bar{z}^\beta}{2} \left( \frac{\partial^2 F^a}{\partial z^\alpha \partial u^b} \right) (p(u), \bar{p}(u), u) \left( \frac{\partial g^b}{\partial z^\beta} \right) (0, u) \\
+ \sum_{\alpha, \beta = 1}^n \sum_{b = 1}^m \frac{z^\alpha \bar{z}^\beta}{2} \left( \frac{\partial^2 F^a}{\partial z^\beta \partial u^b} \right) (p(u), \bar{p}(u), u) \left( \frac{\partial g^b}{\partial z^\alpha} \right) (0, u) \\
+ \sum_{\alpha, \beta = 1}^n \sum_{b, c = 1}^m \frac{z^\alpha \bar{z}^\beta}{4} \left( \frac{\partial^2 F^a}{\partial u^b \partial u^c} \right) (p(u), \bar{p}(u), u) \\
\quad \times \left( \frac{\partial g^b}{\partial z^\alpha} \right) (0, u) \left( \frac{\partial \bar{g}^c}{\partial z^\beta} \right) (0, u)
\]

\[
\left\{ B_{00}(z, \bar{z}, u) \right\}_{ab} = i \sum_{\alpha = 1}^n \left( \frac{\partial F^a}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u) \left( \frac{\partial p^\alpha}{\partial u^b} \right) (u) \\
- i \sum_{\alpha = 1}^n \left( \frac{\partial F^a}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u) \left( \frac{\partial \bar{p}^\alpha}{\partial u^b} \right) (u) \\
- \frac{1}{2i} \sum_{c = 1}^m \left( \frac{\partial F^a}{\partial u^c} \right) (p(u), \bar{p}(u), u) \\
\quad \times \left\{ \left( \frac{\partial g^c}{\partial u^b} \right) (0, u) - \left( \frac{\partial \bar{g}^c}{\partial u^b} \right) (0, u) \right\}.
\]

From the expansion (9), we obtain
\[
g(0, u) = iF(p(u), \bar{p}(u), u),
\]
\[
\left( \frac{\partial g^b}{\partial z^\alpha} \right) (0, u) = \left\{ 2i(Id - iF')^{-1} \left( \frac{\partial F}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u) \right\}_b,
\]
\[
\left( \frac{\partial g^b}{\partial u^c} \right) (0, u) = \left\{ i \sum_{\alpha} \left( \frac{\partial F}{\partial z^\alpha} \right) p^\alpha(u) + i \sum_{\alpha} \left( \frac{\partial F}{\partial z^\alpha} \right) \bar{p}^\alpha(u) + iF' \right\}_{bc}.
\]
where
\[
\left\{ \left( \frac{\partial F}{\partial z^a} \right) p^{\alpha'}(u) \right\}_{ab} = \left( \frac{\partial F^a}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u) \left( \frac{\partial p^\alpha}{\partial u^b} \right) (u),
\]
\[
(F')_{ab} = \left( \frac{\partial F^a}{\partial u^b} \right) (p(u), \bar{p}(u), u).
\]
This completes the proof. \(\square\)

Note that the functions \(F^a_s(z, \bar{z}, u)\) in Lemma 1 are functionals of the function \(p(u)\), i.e., functions of the function \(p(u)\) and its derivatives. The highest order of the derivatives of the function \(p(u)\) in \(F^a_s(z, \bar{z}, u)\) depends on the type \((s, t)\) of \(F^a_s(z, \bar{z}, u)\).

2. Zero Levi form

We shall study a generic CR submanifold \(M\) with Levi form \(L\) vanishing identically on \(M\).

**Lemma 2.** Suppose that a generic CR submanifold \(M\) is defined near the origin by
\[ v = F(z, \bar{z}, u) = \sum_{s+t \geq 2} F^a_{st}(z, \bar{z}, u). \]

Then the \(u\)-plane, \(z = v = 0\), is on \(M\) and
\[ 2dd^c \rho \big|_{z=v=0} = 2i \sum_{\alpha, \beta = 1}^{n} \left( \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \right) (0, 0, 0, u) dz^\alpha \wedge d\bar{z}^\beta, \]
where
\[ \rho = -v + F(z, \bar{z}, u). \]

**Proof.** By the definition of \(d^c\), we have
\[ 2id^c \rho = \sum_{\alpha = 1}^{n} \left( \frac{\partial \rho}{\partial z^\alpha} dz^\alpha + \frac{\partial \rho}{\partial \bar{z}^\alpha} d\bar{z}^\alpha \right) + \sum_{\alpha = 1}^{m} \left( \frac{\partial \rho}{\partial w^\alpha} dw^\alpha + \frac{\partial \rho}{\partial \bar{w}^\alpha} d\bar{w}^\alpha \right). \]
Thus we obtain
\[
2i dd^c \rho = -2 \left( \sum_{a,b=1}^n \frac{\partial^2 \rho}{\partial z^a \partial \bar{z}^b} dz^a \wedge d\bar{z}^b + \sum_{a=1}^m \frac{\partial^2 \rho}{\partial w^a \partial \bar{w}^b} dw^a \wedge d\bar{w}^b \right)
\]
\[
-2 \sum_{a=1}^n \sum_{a=1}^m \left( \frac{\partial^2 F}{\partial z^a \partial w^a} dz^a \wedge dw^a - \frac{\partial^2 F}{\partial \bar{z}^a \partial \bar{w}^a} d\bar{z}^a \wedge d\bar{w}^a \right)
\]
\[
= -2 \sum_{a,b=1}^n \frac{\partial^2 F}{\partial z^a \partial \bar{z}^b} dz^a \wedge d\bar{z}^b - \frac{1}{2} \sum_{a=1}^m \frac{\partial^2 F}{\partial w^a \partial \bar{u}^b} dw^a \wedge d\bar{u}^b
\]
\[
- \sum_{a=1}^n \sum_{a=1}^m \left( \frac{\partial^2 F}{\partial z^a \partial u^a} dz^a \wedge dw^a - \frac{\partial^2 F}{\partial \bar{z}^a \partial \bar{u}^a} d\bar{z}^a \wedge d\bar{u}^a \right).
\]

Note that the generic CR submanifold $M$ contains the $u$-plane, $z = v = 0$, since
\[
F(0,0,u) = 0.
\]

Further, the condition
\[
F_{10}(z, \bar{z}, u) = F_{01}(z, \bar{z}, u) = 0
\]
gives the following equality on the $u$-plane
\[
2i dd^c \rho|_{z=v=0} = -2 \sum_{a,b=1}^n \left( \frac{\partial^2 F}{\partial z^a \partial \bar{z}^b} \right)(0,0,u) dz^a \wedge d\bar{z}^b.
\]

This completes the proof. $\square$

Note that the differential system defined by
\[
d\rho_1 = \cdots = d\rho_m = d^c \rho_1 = \cdots = d^c \rho_m = 0
\]
along the $u$-plane on $M$ in Lemma 2 is given by the complex tangent planes of the variable $z$ in $\mathbb{C}^n \times \mathbb{C}^m$. Thus the Levi form $L$ on $M$ in Lemma 2 is faithfully represented on the $u$-plane by the two-form in (10).

**Lemma 3.** Suppose that an analytic generic CR submanifold $M$ is defined near the origin by
\[
v = F(z, \bar{z}, u) = \sum_{s,t \geq 1} F_{st}(z, \bar{z}, u)
\]
and the Levi form \( L \) on \( M \) vanishes identically. Then

\[
F(z, \bar{z}, u) = 0,
\]

i.e., \( M \) is a plane \( \mathbb{C}^n \times \mathbb{R}^m \) defined by \( v = 0 \).

**Proof.** Let \( M' \) be the generic CR submanifold obtained from \( M \) by the biholomorphic mapping as in (7)

\[
z = z^* + p(w^*),
\]

\[
w = w^* + g(z^*, w^*)
\]

for a given function \( p(u) \). Then \( M' \) is given near the origin by

\[
v = \sum_{s,t \geq 1} F^*_s(z, \bar{z}, u),
\]

where

\[
F^*_s(z, \bar{z}, u) = \left( \text{Id} - B_{00}(0, 0, u) \right)^{-1} A_{11}(z, \bar{z}, u).
\]

Since the generic CR submanifold \( M \) is defined by

\[
v = F(z, \bar{z}, u) = \sum_{s,t \geq 1} F_{st}(z, \bar{z}, u),
\]

we have the following equalities

\[
\sum_{\alpha=1}^n z^\alpha \left( \frac{\partial F}{\partial z^\alpha} \right)(z, \bar{z}, u) = \sum_{s,t \geq 1} s F_{st}(z, \bar{z}, u),
\]

\[
\sum_{\alpha=1}^n \bar{z}^\alpha \left( \frac{\partial F}{\partial \bar{z}^\alpha} \right)(z, \bar{z}, u) = \sum_{s,t \geq 1} t F_{st}(z, \bar{z}, u),
\]

\[
\sum_{\alpha, \beta=1}^n z^\alpha \bar{z}^\beta \left( \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \right)(z, \bar{z}, u) = \sum_{s,t \geq 1} st F_{st}(z, \bar{z}, u).
\]
Then from Lemma 1, we obtain
\[
\sum_{\alpha,\beta=1}^{n} P^{\alpha \beta} \left( \frac{\partial^{2} A_{11}}{\partial z^{\alpha} \partial \bar{z}^{\beta}} \right) (0,0,u) = \sum_{s,t \geq 1} s t F_{st}(p, \bar{p}, u) \\
- \iota \left\{ \sum_{s,t \geq 1} s F_{st}'(p, \bar{p}, u) \right\} (I d + i F')^{-1} \left\{ \sum_{s,t \geq 1} t F_{st}(p, \bar{p}, u) \right\} \\
+ \iota \left\{ \sum_{s,t \geq 1} t F_{st}'(p, \bar{p}, u) \right\} (I d - i F')^{-1} \left\{ \sum_{s,t \geq 1} s F_{st}(p, \bar{p}, u) \right\} \\
- 2 F''(I d - i F')^{-1} \left\{ \sum_{s,t \geq 1} s F_{st}(p, \bar{p}, u) \right\} \\
\times (1 + i F')^{-1} \left\{ \sum_{s,t \geq 1} t F_{st}(p, \bar{p}, u) \right\},
\]
\[
(11)
\]
where
\[
\left\{ F_{st}'(p, \bar{p}, u) \right\}_{ab} = \left( \frac{\partial F_{st}^a}{\partial u^b} \right) (p, \bar{p}, u).
\]

Note that the Levi form \( L' \) on \( M' \) vanishes identically whenever the Levi form \( L \) on \( M \) vanishes identically. By Lemma 2, the function \( F_{11}^a(z, \bar{z}, u) \) vanishes identically for any function \( p(u) \). Thus the equality (11) yields the following identity
\[
\sum_{s,t \geq 1} s t F_{st}(z, \bar{z}, u) \\
= \iota \left\{ \sum_{s,t \geq 1} s F_{st}'(z, \bar{z}, u) \right\} (I d + i F')^{-1} \left\{ \sum_{s,t \geq 1} t F_{st}(z, \bar{z}, u) \right\} \\
- \iota \left\{ \sum_{s,t \geq 1} t F_{st}'(z, \bar{z}, u) \right\} (I d - i F')^{-1} \left\{ \sum_{s,t \geq 1} s F_{st}(z, \bar{z}, u) \right\} \\
+ 2 F''(I d - i F')^{-1} \left\{ \sum_{s,t \geq 1} s F_{st}(z, \bar{z}, u) \right\} \\
\times (I d + i F')^{-1} \left\{ \sum_{s,t \geq 1} t F_{st}(z, \bar{z}, u) \right\},
\]
\[
(12)
\]
In the identity (12), we expand the right hand side with respect to \( z \) and \( \bar{z} \). Then we observe that the function
\[
\sum_{s,t \geq 1, s+t=k} s t F_{st}(z, \bar{z}, u)
\]
is represented by a linear combination of products of the following functions

\[ F_{st}(z, \bar{z}, u) \quad \text{for} \quad s + t \leq k - 2, \]
\[ F'_{st}(z, \bar{z}, u) \quad \text{for} \quad s + t \leq k - 2, \]
\[ F''_{st}(z, \bar{z}, u) \quad \text{for} \quad s + t \leq k - 4, \]

where

\[ \{ F'_{st}(z, \bar{z}, u) \}_{ab} = \left( \frac{\partial F_{st}^a}{\partial u^b} \right) (z, \bar{z}, u), \]
\[ \{ F''_{st}(z, \bar{z}, u) \}_{abc} = \frac{1}{2} \left( \frac{\partial^2 F_{st}^a}{\partial u^b \partial u^c} \right) (z, \bar{z}, u). \]

We easily see that

\[ \sum_{s, t \geq 1, s + t = 2, 3} sF_{st}(z, \bar{z}, u) = 0 \]

so that

\[ F_{st}(z, \bar{z}, u) = 0 \quad \text{for} \quad s + t = 2, 3. \]

As inductive hypothesis, we suppose that

\[ F_{st}(z, \bar{z}, u) = 0 \]

for \( s + t = k \geq 4 \). Then we obtain

\[ \sum_{s, t \geq 1, s + t \leq k + 2} sF_{st}(z, \bar{z}, u) = 0 \]

so that

\[ F_{st}(z, \bar{z}, u) = 0 \quad \text{for} \quad s + t \leq k + 2. \]

Therefore we conclude that \( F(z, \bar{z}, u) = 0 \). This completes the proof. \( \square \)

Hence we have proved the following theorem

**Theorem 4.** Let \( M \) be an analytic generic CR submanifold of codimension \( m \) with zero Levi form defined by

\[ v = F(z, \bar{z}, u), \quad F| = dF| = 0. \]

Then \( M \) is locally transformed to a plane \( \mathbb{C}^n \times \mathbb{R}^m \) defined by

\[ v = 0 \]
by the following biholomorphic mapping

\begin{align}
  z &= z^*, \\
  w &= w^* + g(z^*, w^*),
\end{align}

where the function \( g(z, w) \) is implicitly defined by

\begin{equation}
  g(z, w) = -iF(0, 0, w) + 2iF(z, 0, w - \frac{i}{2}F(0, 0, w) + \frac{i}{2}g(z, w)).
\end{equation}

Let \( \phi \) be a biholomorphic mapping near the origin, which transforms the generic CR submanifold \( M \) in Theorem 4 to the plane \( v = 0 \). Then the mapping \( \phi \) is factorized to the mapping (13) and an element of the pseudo-group of the local biholomorphic automorphisms of the plane \( v = 0 \) such that

\begin{align}
  z^* &= f(z, w), \\
  w^* &= q(w)
\end{align}

where

\[ \text{det}(f_z|_0) \neq 0, \quad \Re q(u) = 0 \quad \text{and} \quad \text{det} q'(0) \neq 0. \]

Note that the biholomorphic mapping (13) is a local trivialization of a family of complex manifolds of complex dimension \( n \) parametrized by a subset of \( \mathbb{R}^m \). Thus the analytic generic CR submanifold \( M \) with zero Levi form is locally foliated into complex manifolds. Further, the leaves of the complex foliation on \( M \) are locally given by the complex submanifold near the origin as follows

\[ w = \tau + g(z, \tau) \]

for \( \tau \in \mathbb{R}^m \), where the function \( g(z, \tau) \) is defined by the equation (14).

**Corollary 5.** Let \( M \) be an analytic generic CR submanifold of CR dimension \( n \) with zero Levi form in a complex manifold. Then there is an open neighborhood \( U \) of each point of \( M \) such that \( M \cap U \) is an analytic foliation of complex manifolds of complex dimension \( n \).

This corollary is a well-known special case of a general result (cf. [1]). The significance of this article is that we do not require Frobenius theorem and Newlander-Nirenberg theorem/Levi-Civita theorem in the proof (cf. [1]).
References


Department of Mathematics  
University of Seoul, Korea  
*E-mail:* wonkpark@mindmath.uos.ac.kr