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COMPUTATION OF NUMERICAL INVARIANTS *col*(-), *row*(-) FOR A RING $k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$

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Abstract. We find the values of numerical invariants $col(R)$ and $row(R)$ for $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$, where k is a field and $e \geq 4$. We also show that $col(R) = crs(R)$ and $row(R) = drs(R)$, but they are strictly less than the reduction number of R plus 1.

I. Introduction

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a Noetherian local ring, and all modules are unitary.

Recently, it was proved in [3] that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings. From these restrictions, some previously known results in commutative ring theory are slightly improved; for examples, Herzog's extension of Kunz's result to a characterization of modules of finite projective and injective dimensions in characteristic $p > 0$ ([3, Corollary 2.8]), and Eisenbud's and Dutta's results on the existence of free summands in syzygy modules ([3, Proposition 2.2]).

Also, using these restrictions, some new numerical invariants of local rings were introduced in [3]. They are $col(A)$ and $row(A)$ associated

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with the columns and rows, respectively, of the maps in infinite minimal resolutions. This paper deals with these invariants. In [4], two more invariants were defined, say $crs(A)$ and $drs(A)$, which are associated with the cyclic modules determined by regular sequences and their Matlis duals, respectively. More precisely, we have the following definitions.

DEFINITION 1.1. In defining $row(A)$ and $drs(A)$ below, we assume that A is Cohen-Macaulay. We denote by $\varphi_i(M)$ the i-th map in a minimal resolution of a finitely generated A-module M. We also use the usual notation $Soc(M) = \mathrm{Hom}_A(A/\mathfrak{m}, M)$ to denote the socle of M.

i) $col(A) = \inf\ \{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each column of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i > 1 +$ depth A $\}$.

$row(A) = \inf\ \{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each row of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i >$ depth A $\}$.

ii) $crs(A) = \inf\ \{t \geq 1 : Soc(A/(\mathbf{x})) \not\subset \mathfrak{m}^t(A/(\mathbf{x}))$ for some maximal regular sequence $\mathbf{x} = x_1, \cdots, x_n\}$.

$drs(A) = \inf\ \{t \geq 1 : Soc((A/(\mathbf{x}))^\vee) \not\subset \mathfrak{m}^t((A/(\mathbf{x}))^\vee)$ for some system of parameters $\mathbf{x} = x_1, \cdots, x_d\}$.

When A is regular local, we interpret the above definition as $col(A) = row(A) = 1$. These invariants are related by the following inequalities.

PROPOSITION 1.2. ([4, Proposition 1.3]) *Let* $(A, \mathfrak{m})$ *be a* Noetherian *local ring. Then*
i) $1 \leq col(A) \leq crs(A) < \infty$.

ii) *If* A *is* Cohen-Macaulay, *then* $1 \leq row(A) \leq drs(A) < \infty$.

iii) *A is a regular local ring if and only if any, equivalently all, of the invariants in* i) *and* ii) *is* 1.

PROPOSITION 1.3. ([4, Proposition 4.1]) *Let* $(A, \mathfrak{m})$ *be a* Gorenstein *local ring. Then*
i) $col(A) = row(A)$,

ii) $crs(A) = drs(A)$.

In [4], the following was conjectured.

CONJECTURE 1.4. *If $A/\mathfrak{m}$ is infinite, then two invariants in* i) *and* ii) *above agree, that is, $col(A) = crs(A)$, and if A is* Cohen-Macaulay *then $row(A) = drs(A)$.*

In a few cases, the conjecture is in the affirmative. For instances, if a non-regular Cohen-Macaulay local ring A has a minimal multiplicity, i.e., $\text{mult}A = 1 + \text{edim}A - \text{dim}A$, then these invariants are all equal to 2 (see Corollary 3.7 in [4]), and if A is hypersurface, these invariants are the same as the multiplicity of A (see Theorem 4.3 in [4]). It was also shown in [4] that if $\text{dim}A = 0$, then the conjecture holds. Nevertheless, the conjecture is still open even in the case of $\text{dim}A = 1$. In this article, we take a particular ring $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$ for $e \geq 4$, which is often used as an example ring in many papers (e.g. see [6], or [7]). It is known that R is a Cohen-Macaulay local ring of dimension 1, but its associated graded ring is not Cohen-Macaulay.

Like other invariants, it is almost impossible to find the exact values of the invariants defined above for arbitrary local rings. In the next section, we compute the values of $col(R)$ and $row(R)$, and also show that the conjecture holds for the ring R.

2. Invariants $col(R)$ and $row(R)$

Throughout this section we set $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$, where k is a field and $e \geq 4$, and $\mathfrak{m}$ its maximal ideal. Since R is a one-dimensional domain, R is Cohen-Macaulay. It is known that R is not Gorenstein. (See Remark 2.7.(2)) We find the generators of each power of $\mathfrak{m}$ in the next lemma, whose proof is quite elementary. We first note that $\mathfrak{m} = (t^e, t^{e+1}, t^{(e-1)e-1})$.

LEMMA 2.1. *For every positive integer $\ell \geq (e-2)e$, each t^ℓ belongs to $\mathfrak{m}$. Moreover, if $r \geq 2$,*

$$\mathfrak{m}^r = (t^{re},\ t^{re+1},\ \cdots,\ t^{re+r}).$$

Proof. Since t^e and $t^{e+1} \in \mathfrak{m}$, we know $t^{2e}, t^{2e+1}, t^{2e+2} \in \mathfrak{m}$. Inductively, we can check that $t^{je+k} \in \mathfrak{m}$ for any positive integer j and $0 \leq k \leq j$. The first part follows since $(e-1)e-1 = (e-2)e+e-2+1$, $(e-2)e+e-2+1+1 = (e-1)e$, and $t^{(e-1)e-1} \in \mathfrak{m}$.

For the second part, we use induction on r. For $r=2$, it suffices to show that

$$t^{(e-1)e-1}t^e,\ t^{(e-1)e-1}t^{e+1},\ (t^{(e-1)e-1})^2 \in (t^{2e},\ t^{2e+1},\ t^{2e+2}).$$

Since

$$\begin{aligned} t^{(e-1)e-1}t^e &= t^{(e-3)(e+1)}t^{2e+2}, \\ t^{(e-1)e-1}t^{e+1} &= t^{(e-2)e}t^{2e}, \\ (t^{(e-1)e-1})^2 &= t^{2(e-2)e-2}t^{2e}, \end{aligned}$$

and $t^{(e-3)(e+1)}, t^{(e-2)e}, t^{2(e-2)e-2} \in \mathfrak{m}$ by the first part, the case of $r=2$ is complete.

Suppose $\mathfrak{m}^j = (t^{je}, t^{je+1}, \cdots, t^{je+j})$ for j with $2 \leq j < r$. We now show that

$$\mathfrak{m}^r = (t^{re}, t^{re+1}, \cdots, t^{re+r}).$$

It is clear that $(t^{re}, t^{re+1}, \cdots, t^{re+r}) \subseteq \mathfrak{m}^r$ since $t^{re+k} = t^{(r-k)e}t^{k(e+1)} \in \mathfrak{m}^r$ for $0 \leq k \leq r$. To show the other inclusion, it is sufficient, by induction hypothesis, to show that for $0 \leq i \leq r-1$,

$$t^{(e-1)e-1}t^{(r-1)e+i} \in (t^{re}, t^{re+1}, \cdots, t^{re+r}).$$

If $i=0$, then

$$t^{(e-1)e-1}t^{(r-1)e} = t^{(e-3)(e+1)}t^{re+2} \in (t^{re}, t^{re+1}, \cdots, t^{re+r}).$$

If $1 \leq i \leq r-1$, then

$$t^{(e-1)e-1}t^{(r-1)e+i} = t^{(e-2)e}t^{re-1+i} \in (t^{re}, t^{re+1}, \cdots, t^{re+r}).$$

This completes the proof. $\square$

REMARK 2.2. In fact, it is easily obtained that for $r \geq 2$

$$\mathfrak{m}^r = (t^{re}, t^{re+1}, \cdots, t^{re+min\{r,e-1\}}).$$

Indeed, if $e \leq k \leq r$, then since $k = k_1 e + k_2$ for some $k_1, 0 \leq k_2 < e$, we have

$$t^{re+k} = t^{re+k_1 e+k_2} = (t^e)^{k_1} t^{re+k_2} \in (t^{re}, t^{re+1}, \cdots, t^{re+min\{r,e-1\}}).$$

Thus for $r \geq 2$, the minimal number of generators of $\mathfrak{m}^r$, $\mu(\mathfrak{m}^r)$, is $min\{r, e-1\} + 1$.

We say that a system of parameter $\mathbf{x} = x_1, \cdots, x_d$ of a local ring $(A, \mathfrak{m})$ is a minimal reduction of the maximal ideal $\mathfrak{m}$ if $(\mathbf{x})\mathfrak{m}^r = \mathfrak{m}^{r+1}$ for some positive integer r. It is known in [5] that such a minimal reduction always exists, provided that a residue field $A/\mathfrak{m}$ is infinite. The reduction number of A, denoted by $red(A)$, is the smallest integer r for which there is a minimal reduction $\mathbf{x}$ with $(\mathbf{x})\mathfrak{m}^r = \mathfrak{m}^{r+1}$.

The following lemma is a well known fact, but we include a proof for completeness.

LEMMA 2.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Suppose $(\mathbf{x})\mathfrak{m}^r = \mathfrak{m}^{r+1}$ for some positive integer r and a system of parameter $\mathbf{x} = x_1, \cdots, x_d$ of A. Then $x_i \in \mathfrak{m} - \mathfrak{m}^2$ for each $i = 1, \cdots, d$.*

Proof. If all x_i are in $\mathfrak{m}^2$, then $\mathfrak{m}^{r+1} = (\mathbf{x})\mathfrak{m}^r \subseteq \mathfrak{m}^{r+2}$, which implies $\mathfrak{m} = 0$ by Nakayama lemma. Suppose $x_1, \cdots, x_j \in \mathfrak{m} - \mathfrak{m}^2$ and $x_{j+1}, \cdots, x_d \in \mathfrak{m}^2$. Now, let $N := (x_1, \cdots, x_j)\mathfrak{m}^{r-1}$ be a submodule of $M := (x_1, \cdots, x_d)\mathfrak{m}^{r-1}$. Then it is easy to show that $\mathfrak{m}M + N = M$ since $x_{j+1}, \cdots, x_d \in \mathfrak{m}^2$ and $\mathfrak{m}M = \mathfrak{m}^{r+1}$ by assumption. Therefore, by Nakayama lemma, we have $M = N$, i.e., $(x_1, \cdots, x_d)\mathfrak{m}^{r-1} = (x_1, \cdots, x_j)\mathfrak{m}^{r-1}$, which implies $j = d$ since $x_1, \cdots, x_d$ is a system of parameter of A. Hence all x_i are in $\mathfrak{m} - \mathfrak{m}^2$. □

In [1], a generalized Loewy length of a Noetherian local ring $(A, \mathfrak{m})$ is defined as

$$\ell.\ell.(A) = \min\{\ell : \mathfrak{m}^\ell \subseteq (x_1, \cdots, x_d) \text{ for some system of parameter } x_1, \cdots, x_d \text{ of } A\}.$$

By definitions, it is clear that $\ell.\ell.(A) \leq red(A) + 1$. If A is Cohen-Macaulay, then one of our invariants, $drs(A)$, is the same as $\ell.\ell.(A)$ (see [4]). Thus we have the following inequalities

$$(*) \qquad row(A) \leq drs(A) = \ell.\ell.(A) \leq red(A) + 1.$$

In the following proposition, we show that $red(R) = e - 1$ and $\ell.\ell.(R) \leq e - 1$ (in fact, the equality holds (see Theorem 2.6 below)). Thus the last inequality in $(*)$ could be strict.

PROPOSITION 2.4. *(1) $(t^e)\mathfrak{m}^{e-1} = \mathfrak{m}^e$ and $(x)\mathfrak{m}^{e-2} \neq \mathfrak{m}^{e-1}$ for any $x \in \mathfrak{m}$. In particular, $red(R) = e - 1$.*
(2) $\mathfrak{m}^{e-1} \subseteq (t^e)$. In particular, $\ell.\ell.(R) \leq e - 1$.

Proof. (1) By Lemma 2.1 and Remark 2.2, it is easy to check the first part. For the second part, it suffices to show that $(x)\mathfrak{m}^{e-2} \neq \mathfrak{m}^{e-1}$ only for $x \in \mathfrak{m} - \mathfrak{m}^2$ by the above lemma. Let $x = at^e + bt^{e+1} + ct^{(e-1)e-1}$. Then at least one of a, b, and c must be a unit. If a is a unit, then $t^{(e-1)e+e-1}$ is not in $(x)\mathfrak{m}^{e-2}$ but in $\mathfrak{m}^{e-1}$ since $\mathfrak{m}^{e-2} = (t^{(e-2)e}, \cdots, t^{(e-2)e+e-2})$, $\mathfrak{m}^{e-1} = (t^{(e-1)e}, \cdots, t^{(e-1)e+e-1})$, $t^e t^{(e-1)e-1} = t^{(e-1)e+e-1}$, and $t^{(e-1)e-1} \notin \mathfrak{m}^{e-2}$. Similarly, if b or c is a unit, then $t^{(e-1)e} \in \mathfrak{m}^{e-1} - (x)\mathfrak{m}^{e-2}$. Hence $(x)\mathfrak{m}^{e-2} \neq \mathfrak{m}^{e-1}$ for any $x \in \mathfrak{m}$, and so we have $red(R) = e - 1$ by the definition of $red(R)$.

(2) We note that $\mathfrak{m}^{e-1} = (t^{(e-1)e}, t^{(e-1)e+1}, \cdots, t^{(e-1)e+(e-1)})$ by Lemma 2.1. Since for $0 \leq j < e - 1$

$$t^{(e-1)e+j} = t^{(e-(j+1))e+je+j} = t^{je+j}(t^e)^{e-(j+1)} \in (t^e)$$

and

$$t^{(e-1)e+e-1} = t^{e+(e-1)e-1} = t^{(e-1)e-1}t^e \in (t^e),$$

we have $\mathfrak{m}^{e-1} \subseteq (t^e)$.

In particular, $\ell.\ell.(R) \leq e - 1$ by the definition of $\ell.\ell.(R)$. □

REMARK 2.5. It is known that if A is a Cohen-Macaulay local ring of dimension 1, then $red(A)$ is independent, i.e., $red(A)$ takes the same value for any minimal reduction of A (see [2]). Thus for Proposition 2.4 (1), it suffices to show that $(t^e)\mathfrak{m}^{e-1} = \mathfrak{m}^e$, and $(t^e)\mathfrak{m}^{e-2} \neq \mathfrak{m}^{e-1}$.

Now we prove that the conjecture in the introduction holds for R. We recall the definition of the invariant $row(A)$ of a Cohen-Macaulay local ring A $row(A)$ is the least number of $t \geq 1$ such that for each finitely generated A-module M of infinite projective dimension, each row of i-th map in a minimal resolution of M contains an element outside $\mathfrak{m}^t$, for all $i >$ depth A.

THEOREM 2.6. *Let* $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$, *where* k *is a field and* $e \geq 4$. *Then*
(1) $col(R) = 2 = crs(R)$,
(2) $row(R) = e - 1 = drs(R)$.
In particular, Conjecture 1.4 *holds for* R.

Proof. (1) Since $col(R) \leq crs(R)$ and R is not regular, it is enough to show that $crs(R) \leq 2$. We will show that $\overline{t^{(e-1)e-1}} \in Soc(R/(t^e)) - \mathfrak{m}^2(R/(t^e))$, which will force $crs(R) \leq 2$ by definition of $crs(R)$. Since $\mathfrak{m} = (t^e,\ t^{e+1},\ t^{(e-1)e-1})$,

$$t^{(e-1)e-1}t^{e+1} = (t^e)^e \in (t^e) \quad \text{and} \quad t^{(e-1)e-1}t^{(e-1)e-1} = (t^{2e+1})^{e-2}t^e \in (t^e),$$

we know that $\overline{t^{(e-1)e-1}} \in \mathrm{Soc}\big(R/(t^e)\big)$. It is easy to check that $\overline{t^{(e-1)e-1}} \notin \mathfrak{m}^2(R/(t^e))$. Indeed if $\overline{t^{(e-1)e-1}} \in \mathfrak{m}^2\big(R/(t^e)\big)$, then $t^{(e-1)e-1} + xt^e \in \mathfrak{m}^2$ for some $x \in R$, which contradicts that $\{\ t^{(e-1)e-1},\ t^e\ \}$ is a part of minimal generators of $\mathfrak{m}$. Thus we have $1 < col(R) \leq crs(R) \leq 2$, which implies that

$$col(R) = 2 = crs(R).$$

(2) Since

$$row(R) \leq drs(R) = \ell.\ell(R) \leq e - 1 \quad \text{by Proposition 2.4,}$$

it suffices to show that $row(R) \geq e - 1$. Let us consider the minimal resolution $(F_\bullet, \varphi_\bullet)$ of $R/(t^{(e-2)e}, -t^{(e-2)e+e-2})$

$$\cdots \longrightarrow R^{n_3} \xrightarrow{\varphi_3} R^{n_2} \xrightarrow{\varphi_2} R^2 \xrightarrow{\varphi_1} R \longrightarrow R/(t^{(e-2)e}, -t^{(e-2)e+e-2}) \longrightarrow 0.$$

We note that $t^{(e-2)e}$, $t^{(e-2)e+e-2} \in \mathfrak{m}^{e-2}$. Since $\varphi_1 = \begin{bmatrix} t^{(e-2)e} \\ -t^{(e-2)e+e-2} \end{bmatrix}$, we know $(t^{(e-2)e+e-2}, t^{(e-2)e}) \in \operatorname{Ker}\varphi_1$. If $\left(t^{(e-2)e-e-2}, t^{(e-2)e}\right)$ is a part of minimal generators of $\operatorname{Ker}\varphi_1$, then $row(R) \geq e-1$ by the definition of $row(R)$ since $\left(t^{(e-2)e}, t^{(e-2)e+e-2}\right)$ is one of the rows in the map φ_2, and $t^{(e-2)e}$, $t^{(e-2)e+e-2} \in \mathfrak{m}^{e-2}$. Thus we claim that $\left(t^{(e-2)e+e-2}, t^{(e-2)e}\right)$ is a part of minimal generators of $\operatorname{Ker}\varphi_1$. We need to show that $\left(t^{(e-2)e+e-2}, t^{(e-2)e}\right) \notin \mathfrak{m}\operatorname{Ker}\varphi_1$. Suppose on the contrary that $\left(t^{(e-2)e+e-2}, t^{(e-2)e}\right) = \sum r_i(x_i, y_i)$, where $r_i \in \mathfrak{m}$ and $\{(x_i, y_i)\}$ is a set of minimal generators of $\operatorname{Ker}\varphi_1$. Since $\varphi_1(x_i, y_i) = 0$, i.e., $x_i t^{(e-2)e} - y_i t^{(e-2)e+e-2} = 0$, $x_i = t^{e-2} y_i$ for each i in $k[\![t]\!]$. Thus we may assume that $\left(t^{(e-2)e+e-2}, t^{(e-2)e}\right) = \sum(\pm)t^{p_i}(t^{k_i}, t^{\ell_i})$, where t^{p_i}, t^{k_i}, and $t^{\ell_i} \in \mathfrak{m}$, and $k_i - \ell_i = e-2$ for each i. Then, for some i_0

$$(*) \qquad p_{i_0} + k_{i_0} = (e-2)e + e - 2 \qquad \text{and} \qquad p_{i_0} + \ell_{i_0} = (e-2)e.$$

To simplify notation, let $p = p_{i_0}, k = k_{i_0}$, and $\ell = \ell_{i_0}$. Since t^k, $t^\ell \in \mathfrak{m}$, we may assume that $k = k_1 e + k_2$ and $\ell = \ell_1 e + \ell_2$ by Lemma 2.1 and Remark 2.2 for some positive integers $k_1 \geq k_2, \ell_1 \geq \ell_2$. (We have these inequalities from $(*)$. The case $k_1 < k_2$ happens only for $t^{(e-1)e-1}$. See Remark 2.7 below). Note that $\ell_1 \leq k_1 \leq e-2$ since $p > 0$ and $k - \ell = e-2$. If $k_1 = \ell_1$, then we must have $k_1 = \ell_1 = e-2$, $k_2 = e-2$, and $\ell_2 = 0$ to satisfy the conditions $k_1 \geq k_2, \ell_1 \geq \ell_2$ and $k_2 - \ell_2 = e-2$. This implies that $p = 0$ by $(*)$, i.e., $t^p \notin \mathfrak{m}$, which is a contradiction. Thus we may assume that by Lemma 2.1,

$$(**) \qquad \ell_2 \leq \ell_1 < k_1 \leq e-2, \quad \text{and} \quad k_2 \leq k_1.$$

If $k_1 = k_2$, then $k - \ell = k_1 e + k_2 - (\ell_1 e + \ell_2) = (k_1 - \ell_1)e + k_1 - \ell_2 > e-2$ by $(**)$, which contradicts the fact that $k - \ell = e-2$.

Suppose $k_1 > k_2$. Since $p + k = (e-2)e + e - 2$, we have

$$\begin{aligned} p &= (e-2)e + e - 2 - k \\ &= (e-2)e + e - 2 - (k_1 e + k_2) \\ &= (e - 2 - k_1)e + e - 2 - k_2. \end{aligned}$$

Since $k_1 > k_2$,

$$(e-2-k_1)e+e-2-k_1 \;<\; p=(e-2-k_1)e+e-2-k_2$$
$$<\; (e-2-k_1+1)e.$$

This implies that $t^p \notin \mathfrak{m}$ by Lemma 2.1 (e.g. $2e+3 \notin \mathfrak{m}$), which is a contradiction. Hence $\left(t^{(e-2)e+e-2}, t^{(e-2)e}\right)$ is a part of minimal generators of $\operatorname{Ker}\varphi_1$, and $row(R) \geq e-1$. □

REMARK 2.7. (1) In case of resolving $R/\left(t^{(e-1)e}, -t^{(e-1)e+(e-1)}\right)$ for $t^{(e-1)e}, t^{(e-1)e+(e-1)} \in \mathfrak{m}^{e-1}$, $\left(t^{(e-1)e+(e-1)}, t^{(e-1)e}\right)$ is not a part of minimal generators of $\operatorname{Ker}\varphi_1$ since $\left(t^{(e-1)e+(e-1)}, t^{(e-1)e}\right) = t^e\left(t^{(e-1)e-1}, t^{(e-2)e}\right)$, where $\left(t^{(e-1)e-1}, t^{(e-2)e}\right) \in \operatorname{Ker}\varphi_1$. We note that $t^{(e-1)e-1} = t^{(e-2)e+(e-2)+1}$, and $k_1 = e-2 \ngtr (e-2)+1 = k_2$.

(2) If A is a Gorenstein local ring, it is known that $col(A) = row(A)$, and $crs(A) = drs(A)$ (see Proposition 1.3). Thus by Theorem 2.6, $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$ is not Gorenstein.

References

[1] S. Ding, *Auslander's δ-invariants of Gorenstein local rings,* Proc. Amer. Math. Soc. 122 (1994), 649–656.

[2] S. Huckaba, *Reduction numbers for ideals of analytic spread one,* J. Alg. 108 (1987), 503–512.

[3] J. Koh and K. Lee, *Some restrictions on the maps in minimal resolutions,* J. Alg. 202 (1998), 671–689.

[4] ———, *New invariants of Noetherian local rings,* J. Alg. (to appear).

[5] D.G. Northcott and D. Rees, *Reductions of ideals in local rings,* Proc. Camb. Phil. Soc. 50 (1954), 145–158.

[6] J. Sally, *On the associated graded ring of a local Cohen-Macaulay ring,* J. Math. Kyoto Univ. 17 (1977), 19–21.

[7] G. Valla, *On form rings which are Cohen-Macaulay,* J. Alg. 58 (1979), 247–250.

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