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ON THE MAXIMAL OPERATORS GENERATED BY QUASIRADIAL FOURIER MULTIPLIERS AND ITS APPLICATIONS TO P.D.E.'S

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ABSTRACT. Let $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a distance function which is homogeneous with respect to a dilation group $\{t^P\}_{t>0}$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we consider the maximal operators generated by quasiradial Fourier multiplier $\mathfrak{m} \circ \varrho$ which is defined by

$$\mathcal{M}_{\mathfrak{m}\circ\varrho} f(x) = \sup_{t>0} \left| \mathcal{F}^{-1}\left[\mathfrak{m} \circ (\varrho/t) \widehat{f}\right](x) \right|$$

where $\mathfrak{m}$ is a function given on $\mathbb{R}_+$. Suppose that the sphere $\Sigma_\varrho \doteqdot \{\xi \in \mathbb{R}^n \,|\, \varrho(\xi) = 1\}$ satisfies a certain finite type condition and that $\mathfrak{m}$ vanishes at infinity and satisfies $\int_0^\infty s^\delta |\mathfrak{m}^{(\delta+1)}(s)|\, ds \le C$ for $\delta > (n-1)\left|1/p - 1/2\right|$, $1 \le p \le \infty$. Then we prove that $\mathcal{M}_{\mathfrak{m}\circ\varrho}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$; moreover, it is of weak type $(1, 1)$.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, denote the inner product in $\mathbb{R}^n$ by $\langle x, \xi \rangle = \sum_{i=1}^n x_i \xi_i$ for $x, \xi \in \mathbb{R}^n$, and denote by $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$. Let $\mathfrak{S}(\mathbb{R}^n)$ be the space of all infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we denote the Fourier transform of f by

$$\mathcal{F}[f](x) = \hat{f}(x) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(\xi)\, d\xi.$$

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Then the inverse Fourier transform of f is given by

$$\mathcal{F}^{-1}[f](\xi) = \check{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(x)\,dx.$$

Let P be a real $n\times n$ matrix whose eigenvalues α_i satisfy $\mathrm{Re}(\alpha_i) > 0$; set $a = \min_{1\le i\le n} \mathrm{Re}(\alpha_i)$ and $A = \max_{1\le i\le n} \mathrm{Re}(\alpha_i)$, and denote by ν the trace of P. Then we consider the dilation group $\{t^P\}_{t>0}$ in $\mathbb{R}^n$ generated by the infinitesimal generator P, where $t^P = \exp(P\log t)$ for $t > 0$. We introduce quasi-homogeneous distance functions ϱ associated with the dilation group $\{t^P\}_{t>0}$; that is, ϱ is a continuous function on $\mathbb{R}^n$ with positive values satisfying $\varrho(t^P\xi) = t\varrho(\xi)$ for all $\xi\in\mathbb{R}^n$. One can refer to [2] and [10] for its fundamental properties. In what follows we shall denote the adjoint of P by P^* and we shall always assume that $\varrho\in C^\infty(\mathbb{R}^n_0)$.

If $\mathfrak{m}(s)$ is defined on $(0,\infty)$ we consider its extension to $\mathbb{R}^n_0$ via $\mathfrak{m}\circ\varrho(\xi)$, which is called a quasiradial function, and its corresponding maximal operator defined by

$$\mathcal{M}_{\mathfrak{m}\circ\varrho} f(x) = \sup_{t>0} \left|\mathcal{F}^{-1}[\mathfrak{m}\circ(\varrho/t)\hat{f}\,](x)\right|,\;\; f\in\mathfrak{S}(\mathbb{R}^n).$$

In order to state our result on the maximal operator associated with quasiradial Fourier multipliers, we introduce the notion of fractional derivatives and certain surface condition on dilations of the unit sphere

$$\Sigma_\varrho = \{\xi\in\mathbb{R}^n \,|\, \varrho(\xi) = 1\}.$$

For $\omega > 0$, $0 < \gamma < 1$, and a locally integrable function h on $\mathbb{R}$ vanishing identically on $(-\infty, 0)$, one can define (see Gasper and Trebels [3]) the fractional integrals by

$$\mathcal{I}^\gamma_\omega(h)(s) = \frac{1}{\Gamma(\gamma)}\chi_{(-\infty,\omega)}(s)\int_s^\omega (\tau - s)^{\gamma-1} h(\tau)\,d\tau$$

where $\chi_{(-\infty,\omega)}(s)$ is the characteristic function on $(-\infty,\omega)$ and $\Gamma(\gamma)$ is the Gamma function, and define fractional derivatives of order $\delta > 0, \delta = [\delta] + \gamma$ with $[\delta]$ being the largest integer less than or equal to δ, by

$$h^{(\gamma)}(s) = \lim_{\omega\to\infty} \frac{d}{ds}[-\mathcal{I}^{1-\gamma}_\omega(h)(s)],\;\; 0<\gamma<1,$$

and

$$h^{(\delta)}(s) = \left(\frac{d}{ds}\right)^{[\delta]} h^{(\gamma)}(s)$$

whenever the right-hand sides exist; in fact, if $\gamma \in \mathbb{N}$ then $h^{(\gamma)}$ means the classical derivative. In the following we assume that $\mathcal{I}_\omega^{1-\gamma}(h)$ is locally absolutely continuous for each $\omega > 0$ as well as $h^{(\gamma)}$ and $h^{(\delta)}$. Note that a heuristic calculation gives $\mathcal{F}[h^{(\delta)}](\sigma) = (-1)^{[\delta]}(-i\sigma)^\delta \hat{h}(\sigma)$, $\sigma \in \mathbb{R}$.

Let Σ be a smooth convex hypersurface of $\mathbb{R}^n$, $n \geq 2$, with which every tangent line makes finite order of contact; that is, Σ is called to be of finite type. Let $\mathcal{E}(\Sigma)$ be the set of points of Σ at which the Gaussian curvature κ vanishes, and let $\mathcal{N}(\Sigma) = \{n(\xi) | \xi \in \mathcal{E}(\Sigma)\}$ where $n(\xi)$ denotes the outer unit normal to Σ at $\xi \in \Sigma$. For $x \in \mathbb{R}^n$, denote by $\mathcal{B}(\xi(x), r)$ the spherical cap near $\xi(x) \in \Sigma$ cut off from Σ by a plane parallel to $T_{\xi(x)}(\Sigma)$ (the affine tangent plane to Σ at $\xi(x)$) at distance $r > 0$ from it; that is,

$$\mathcal{B}\big(\xi(x), r\big) = \big\{\xi \in \Sigma |\, d(\xi, T_{\xi(x)}(\Sigma)) < r\big\},$$

where $\xi(x)$ is the point of Σ whose outer unit normal is in the direction x. For $t > 0$, let $\Sigma_\varrho^t = \{\xi \in \mathbb{R}^n |\, \varrho(\xi) = t\}$ and let $\mathcal{B}_t(\xi_t(x), r) = \{\xi \in \Sigma_\varrho^t |\, d(\xi, T_{\xi_t(x)}(\Sigma_\varrho^t)) < r\}$, where $\xi_t(x)$ is the point of Σ_ϱ^t whose outer unit normal is in the direction x. We say that Σ is of finite type k, if $k \geq 2$ is the maximal order of contact on Σ.

DEFINITION 1.1. Let Σ_ϱ be a smooth convex hypersurface of finite type and $\{t^P\}_{t>0}$ be a dilation group as in the above. Then we say that (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$ if there exists some $\epsilon > 0$ such that $\Omega_\epsilon \in L^1(S^{n-1})$ where

$$\Omega_\epsilon(\theta) = \sup_{(r,t)\in\mathbb{R}_+\times[1,1+\epsilon]} \sigma_t[\mathcal{B}_t(\xi_t(r\theta), 1/r)](1+r)^{\frac{n-1}{2}}, \tag{1.1}$$

$d\sigma_t$ is the surface measure on Σ_ϱ^t, and the polar coordinates on $\mathbb{R}^n$ is denoted by $x = r\theta$ for $r = |x|$ and $\theta = x/|x| \in S^{n-1}$.

REMARK. We shall now give several examples that exemplify the above definition.

(i) Let t^P be a general dilation matrix as in the above and suppose that Σ_ϱ has nonvanishing Gaussian curvature. Then it is obvious that (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$.

(ii) Let us consider a general dilation matrix t^P as in the above, $\mathcal{E}_t = \mathcal{E}(\Sigma_\varrho^t)$, and $\mathcal{N}_t = \mathcal{N}(\Sigma_\varrho^t)$, and for $\epsilon > 0$ set $\mathcal{N}^\epsilon = \cup_{1\le t\le 1+\epsilon}\mathcal{N}_t$. Let Σ_ϱ be a smooth convex hypersurface of finite type $k \geq 2$ and suppose that there is an $\epsilon_0 > 0$ such that $\mathcal{N}^{\epsilon_0}$ is a m-dimensional submanifold of $\mathbb{R}^n$ which is on S^{n-1}, where $m < [k(n-1)]/[2(k-1)]$. Then we see (refer to [7]) that (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$.

(iii) Let $t^P = \mathrm{diag}\big(t^{1/\lambda_1}, t^{1/\lambda_2}, \cdots, t^{1/\lambda_n}\big)$ where each λ_i is even integer, and consider its associated quasi-homogeneous distance functions defined by $\varrho(\xi) = \sum_{i=1}^n |\xi_i|^{\lambda_i}$. Then it is known in [8] that (Σ_ϱ, t^P) is also uniformly spherically integrable near $t = 1$.

THEOREM 1.2. *Let (Σ_ϱ, t^P) be uniformly spherically integrable near $t = 1$. Let $\mathfrak{m}$ be a measurable function on $\mathbb{R}_+$ vanishing at infinity and satisfying*

$$\|[\mathfrak{m}]\|_{\delta,1} \doteq \int_0^\infty s^\delta \left|\mathfrak{m}^{(\delta+1)}(s)\right| ds < \infty \tag{1.2}$$

for $\delta > (n-1)|1/p - 1/2|$, $1 \leq p \leq \infty$. Then $\mathcal{M}_{\mathfrak{m}\circ\varrho}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$; moreover, it is of weak type $(1,1)$.

REMARK. It is well-known in Dappa and Trebels [2] that under less smoothness condition on ϱ, i.e., $\varrho \in C^{[n/2+1]}(\mathbb{R}_0^n)$, if $\mathfrak{m}$ satisfies (1.2) for $\delta > (n-1)(1/2-1/p)$ then $\mathcal{M}_{\mathfrak{m}\circ\varrho}$ is bounded on $L^p(\mathbb{R}^n)$ for $2 \leq p \leq \infty$. Thus we concentrate on the proof of the above theorem on the range $1 \leq p \leq 2$.

Next we write out a routine result for almost everywhere convergence of averages $\mathcal{F}^{-1}[\mathfrak{m} \circ (\varrho/t)\hat{f}\,]$ given by quasiradial Fourier multiplier $\mathfrak{m} \circ \varrho$, which is controlled by the weak type $(1,1)$-estimate and $L^p(\mathbb{R}^n)$-estimate of $\mathfrak{M}_\varrho^\delta$.

COROLLARY 1.3. *Suppose that (Σ_ϱ, t^P) is uniformly spherically integrable near $t=1$. Let $\mathfrak{m}$ be a measurable function on $\mathbb{R}_+$ vanishing at infinity and satisfying (1.2) for $\delta > (n-1)|1/p-1/2|$, $1 \le p \le \infty$. If $f \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, then we have that*

$$\lim_{t\to\infty} \mathcal{F}^{-1}\left[\mathfrak{m}\circ(\varrho/t)\hat{f}\,\right](x) = \left(\int_0^\infty s^\delta\, \mathfrak{m}^{(\delta+1)}(s)\,ds\right) f(x) \quad a.e.\,.$$

2. Proof of the main theorem

In this section, we prove $L^2(\mathbb{R}^n)$-boundedness and weak type $(1,1)$-estimate for the maximal operator $\mathfrak{M}_{\mathfrak{m}\circ\varrho}$ generated by quasiradial Fourier multiplier $\mathfrak{m}\circ\varrho$. An application of the complex interpolation theorem for analytic family of operators along the lines of Stein and Weiss [11] leads us to complete the proof of the main theorem. We shall see that there is an interrelation between the maximal operator $\mathcal{M}_{\mathfrak{m}\circ\varrho}$ and quasiradial Bochner-Riesz means. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we consider quasiradial Bochner-Riesz means $\mathcal{R}^\delta_{\varrho,t}f$ of index $\delta > 0$ defined by

$$\widehat{\mathcal{R}^\delta_{\varrho,t}f}(\xi) = \left(1-\varrho(\xi)/t\right)^\delta_+\hat{f}(\xi)$$

and the associated maximal operator

$$\mathfrak{M}^\delta_\varrho f(x) = \sup_{t>0}\left|\mathcal{R}^\delta_{\varrho,t}f(x)\right|.$$

Let $\mu \ge 0$ and $\delta > 0$. Then it follows from Gasper and Trebels [3] that

$$\mathfrak{m}^{(\mu)}(s) = \frac{1}{\Gamma(\delta+1-\mu)}\int_0^\infty (t-s)_+^{\delta-\mu}\mathfrak{m}^{(\delta+1)}(t)dt,$$

for $0 \le \mu < \delta$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, by simple calculation we now have that

$$\begin{aligned}&\left|\mathcal{F}^{-1}\left[\mathfrak{m}\circ(\varrho/t)\hat{f}\right](x)\right|\\ &= \frac{1}{(2\pi)^n}\frac{1}{\Gamma(\delta+1)}\left|\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\int_0^\infty \left(1-\varrho(\xi)/ts\right)^\delta_+ s^\delta\mathfrak{m}^{(\delta+1)}(s)\,ds\hat{f}(\xi)d\xi\right|\\ &\le \frac{1}{\Gamma(\delta+1)}\mathfrak{M}^\delta_\varrho f(x)\int_0^\infty s^\delta\left|\mathfrak{m}^{(\delta+1)}(s)\right|ds,\end{aligned}$$

and so we have that

$$\mathcal{M}_{\mathfrak{m}\circ\varrho}f(x) \le \frac{\|[\mathfrak{m}]\|_{\delta,1}}{\Gamma(\delta+1)}\mathfrak{M}^{\delta}_{\varrho}f(x). \tag{2.1}$$

Thus the proof of the main theorem relies heavily upon the mapping properties of the maximal quasiradial Bochner-Riesz operator $\mathfrak{M}^{\delta}_{\varrho}$. In order to obtain weak type $(1,1)$-estimate for $\mathfrak{M}^{\delta}_{\varrho}$, we recall the lemma [6] about asymptotics of quasiradial Bochner-Riesz kernel.

We now introduce polar coordinates with respect to a t^P-homogeneous distance function $\varrho \in C^{\infty}(\mathbb{R}^n)$, which is given by the following diffeomorphism

$$\mathbb{R}_+ \times \Sigma_\varrho \to \mathbb{R}^n_0,\ (\varrho,\zeta) \mapsto t^P\zeta = \xi, \varrho > 0, \zeta \in \Sigma_\varrho.$$

It follows from this that the transformation rule of the Lebesgue measure $d\xi$ is given by

$$d\xi = \varrho^{\nu}\langle P\zeta, n(\zeta)\rangle d\varrho\, d\sigma(\zeta),$$

where $d\sigma$ denotes the surface measure on Σ_ϱ and $n(\zeta)$ is the outer unit normal vector to Σ_ϱ at ζ.

Fix some $\zeta_0 \in \Sigma_\varrho$. Then the unit sphere Σ_ϱ can be parametrized near ζ_0 by a map $w \mapsto \mathfrak{P}(w), w \in \mathbb{R}^{n-1}, |w| < 1$ such that $\mathfrak{P}(0) = \zeta_0$. Then there is a neighborhood $\mathcal{U}_0$ of ζ_0 with compact closure, a neighborhood $\mathcal{V}_0$ of the origin in $\mathbb{R}^{n-1}$, and an interval $I_0 = (1-\epsilon_0, 1+\epsilon_0)$ so that the map

$$\mathcal{Q} : I_0 \times \mathcal{V}_0 \to \mathcal{U}_0, (\varrho, w) \mapsto \mathcal{Q}(\varrho, w) = \varrho^P\mathfrak{P}(w)$$

is a diffeomorphism with $\mathcal{Q}(1,0) = \zeta_0$. The Jacobian of $\mathcal{Q}$ is given by

$$\mathfrak{J}(\varrho, w) = \varrho^{\nu-1}\langle P\mathfrak{P}(w), n(\mathfrak{P}(w))\rangle\, \mathfrak{R}(w),$$

where $\mathfrak{R}(w)$ is positive and

$$[\mathfrak{R}(w)]^2 = \det\left(\left[\frac{d\mathfrak{P}}{dw}\right]^*\left[\frac{d\mathfrak{P}}{dw}\right]\right).$$

LEMMA 2.1. *Let $\zeta_0, \mathcal{U}_0$, and $\mathfrak{P}$ be as above. Then there is $\varepsilon_0 > 0$, a neighborhood $\mathcal{U}_1$ of ζ_0 whose closure is contained in $\mathcal{U}_0$, a neighborhood $\mathcal{V}_1$ of the origin in $\mathbb{R}^{n-1}$, and an interval $I_1 = (1-\epsilon_1, 1+\epsilon_1)$ with $\epsilon_1 \ll 1$ such that $\mathcal{Q}$ maps $I_1 \times \mathcal{V}_1$ to $\mathcal{U}_1$, and such that for all $\chi \in C_c^\infty(\mathcal{U}_1)$ and for all $t \in I_1$*

$$\left|\mathcal{F}^{-1}[(1-\varrho/t)_+^\delta \chi](x)\right| \leq C_N(1+|x|)^{-N} \quad \text{for} \quad |\langle x/|x|, n(\zeta_0)\rangle| \leq 1-\varepsilon_0$$

and

$$\mathcal{F}^{-1}[(1-\varrho/t)_+^\delta \chi](x) = \sum_{\ell=0}^{N-1} \mathfrak{p}_\ell(x,t) + \mathcal{O}(|x|^{-N}) \text{ for } |\langle x/|x|, n(\zeta_0)\rangle| \geq 1-\varepsilon_0$$

where

$$\mathfrak{p}_\ell(x,t) = |x|^{-\delta-\ell-1} \int_{\Sigma_\varrho} e^{i|x|\langle t^{P^*}\frac{x}{|x|}, \zeta\rangle} \mathfrak{q}_\ell(\zeta, t, x/|x|)\, d\mu(\zeta)$$

and $\mathfrak{q}_\ell \in C_c^\infty(\mathfrak{P}(\mathcal{V}_0) \times I_1 \times S^{n-1})$ for $\ell = 0, 1, \cdots, N-1$. In particular, $\mathfrak{q}_0(\zeta, t, x/|x|) = C(\delta)\chi(t^P\zeta)t^\nu\langle P\zeta, n(\zeta)\rangle \left[\langle P^* t^{P^*}\frac{x}{|x|}, \zeta\rangle\right]^{-\delta-1}$ where $C(\delta) = \Gamma(\delta+1)\, e^{-i\pi(\delta+1)/2}$.

As you can observe in the above lemma, the decay estimate for the Bochner-Riesz kernel is closely related with the Fourier transform of the unit sphere Σ_ϱ. So it is very natural to compare the relative sizes of the spherical caps on dilations of Σ_ϱ with respect to a dilation group $\{t^P\}_{t>0}$ which is introduced in section 1. We now recall the following lemma [7] for this natural need.

LEMMA 2.2. *There is an $\epsilon_0 > 0$ such that*
(i) $\mathcal{B}_t(\xi_t(t^{-P^*}x), r/2) \subset t^P[\mathcal{B}_1(\xi_1(x), r)] \subset \mathcal{B}_t(\xi_t(t^{-P^*}x), 2r)$,
(ii) $\sigma_1[\mathcal{B}_1(\xi_1(x), r)]/2 \leq \sigma_t\left[t^P(\mathcal{B}_1(\xi_1(x), r))\right] \leq 2\,\sigma_1[\mathcal{B}_1(\xi_1(x), r)]$,
and
(iii) $\sigma_t[\mathcal{B}_t(\xi_t(x), r)]/2 \leq \sigma_1\left[t^{-P}(\mathcal{B}_t(\xi_t(x), r))\right] \leq 2\,\sigma_t[\mathcal{B}_t(\xi_t(x), r)]$
for any $t \in [1-\epsilon_0, 1+\epsilon_0]$ and for any $x \in \mathbb{R}_0^n$

We also see that the proof of the doubling property in [1] leads us to obtain the stronger estimate for the surface measure $d\sigma$ on Σ_ϱ as in the following lemma.

LEMMA 2.3. *Let Σ_ϱ be a smooth convex hypersurface of finite type. Then there is a constant $C > 0$ so that $\sigma[\mathcal{B}(\xi(x), \mathcal{A}r)] \leq C\,\mathcal{A}^{(n-1)/2}\sigma[\mathcal{B}(\xi(x), r)]$ for any $x \in \mathbb{R}^n_0$, for any $r > 0$, and for $\mathcal{A} \geq 1$.*

We learned from (1.2) and (2.1) that the study of the maximal operator $\mathcal{M}_{\mathfrak{m}\circ\varrho}$ generated by quasiradial Fourier multipliers $\mathfrak{m}\circ\varrho$ relies heavily upon that of the maximal quasiradial Bochner-Riesz operator $\mathfrak{M}^\delta_\varrho$. So we prove weak type $(1,1)$-estimate and $L^p(\mathbb{R}^n)$-boundedness of $\mathfrak{M}^\delta_\varrho$ for $1 < p \leq 2$ in the following lemma.

LEMMA 2.4. *Let Σ_ϱ be a smooth convex hypersurface of finite type and $\{t^P\}_{t>0}$ be a dilation group as in the above. If (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$, then $\mathfrak{M}^\delta_\varrho$ is bounded on $L^p(\mathbb{R}^n)$ for $\delta > (n-1)(1/p - 1/2)$, $1 < p \leq 2$; moreover, it is of weak type $(1,1)$ for $\delta > (n-1)/2$.*

Proof. Let $\mathcal{K}(x) = \mathcal{F}^{-1}[(1-\varrho)^\delta_+](x)$ and $\mathcal{K}_t(x) = \mathcal{F}^{-1}[(1-\varrho/t)^\delta_+](x)$ for $t > 0$. Then we have $\mathcal{K}_t(x) = t^\nu \mathcal{K}(t^{P^*}x)$. Choose $\epsilon = \epsilon_0 > 0$ with $\epsilon \ll a$ satisfying (1.1) and such that for any $t \in [1-\epsilon_0, 1+\epsilon_0]$

$$\inf_{x\in\mathbb{R}^n_0} \frac{|t^{P^*}x|}{|x|} \geq 1/2 \quad \text{and} \quad \sup_{x\in\mathbb{R}^n_0} \frac{|t^{P^*}x|}{|x|} \leq 2. \tag{2.2}$$

Since $\widehat{\mathcal{K}} \in L^\infty(\mathbb{R}^n)$ is compactly supported, in order to get weak type $(1,1)$-estimate for $\mathfrak{M}^\delta_\varrho$, it suffices by the results in [7] to show up that

$$\int_{\mathbb{R}^n} \sup_{1\leq t\leq 1+\epsilon} |\mathcal{K}_t(x)|\, dx < \infty$$

and

$$\mathcal{J} \coloneqq \sup_{y\in\mathbb{R}^n_0} \sum_{\{h|(1+\epsilon)^h r(y)>1\}} \int_{r(x)\geq(1+\epsilon)^h r(y)} \sup_{1\leq t\leq 1+\epsilon} |\mathcal{K}_t(x)|\, dx < \infty.$$

By Lemma 2.1 and Theorem A [1], we obtain that

$$|\mathcal{K}(x)| \leq \frac{C}{(1+|x|)^{\delta+1}} \sigma_1[\mathcal{B}_1(\xi_1(x), 1/|x|)].$$

So we have that for any $t \in [1-\epsilon, 1+\epsilon]$,

$$|\mathcal{K}_t(x)| \leq \frac{C}{(1+|x|)^{\delta+1}} \sigma_1[\mathcal{B}_1(\xi_1(t^{P^*}x), 1/|t^{P^*}x|)]. \tag{2.3}$$

It also follows from (i) of Lemma 2.2 that

$$\mathcal{B}_1(\xi_1(t^{P^*}x), 1/|t^{P^*}x|) \subset t^{-P}[\mathcal{B}_t(\xi_t(x), 2/|t^{P^*}x|)].$$

By (2.2), we get that for any $t \in [1-\epsilon, 1+\epsilon]$,

$$1 \leq \frac{2}{\sup_{x\in\mathbb{R}^n_0} \frac{|t^{P^*}x|}{|x|}} = \mathcal{A}_0 \leq 2 \sup_{x\in\mathbb{R}^n_0} \frac{|x|}{|t^{P^*}x|} = \frac{2}{\inf_{x\in\mathbb{R}^n_0} \frac{|t^{P^*}x|}{|x|}} \leq 4,$$

where $\mathcal{A}_0 = 2\inf_{x\in\mathbb{R}^n_0} \frac{|x|}{|t^{P^*}x|}$. Combining this with Lemma 2.3,

$$\sigma_t\big[\mathcal{B}_t(\xi_t(x), 2/|t^{P^*}x|)\big] \leq 2C\mathcal{A}_0^{\frac{n-1}{2}} \sigma_t\big[\mathcal{B}_t(\xi_t(x), 1/|x|)\big].$$

Thus by (i) and (iii) of Lemma 2.2, we have that

(2.4)

$$\begin{aligned}\sigma_1\big[\mathcal{B}_1(\xi_1(t^{P^*}x), 1/|t^{P^*}x|)\big] &\leq \sigma_1\big[t^{-P}(\mathcal{B}_t(\xi_t(x), 2/|t^{P^*}x|))\big] \\ &\leq 2\sigma_t\big[\mathcal{B}_t(\xi_t(x), 2/|t^{P^*}x|)\big] \\ &\leq 4C\mathcal{A}_0^{\frac{n-1}{2}} \sigma_t\big[\mathcal{B}_t(\xi_t(x), 1/|x|)\big].\end{aligned}$$

From (2.3) and (2.4), we get that for all $t \in [1-\epsilon, 1+\epsilon]$,

$$|\mathcal{K}_t(x)| \leq \frac{C}{(1+|x|)^{\delta+1}} \sigma_t\big[\mathcal{B}_t(\xi_t(x), 1/|x|)\big].$$

Since (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$, using $\Omega_\epsilon(\theta)$ defined in (1,1) we have that

$$\begin{aligned}&\sup_{1\leq t\leq 1+\epsilon} |\mathcal{K}_t(x)| \\ &\leq \frac{C}{(1+|x|)^{\delta+\frac{n+1}{2}}} \sup_{(|x|,t)\in\mathbb{R}_+\times[1,1+\epsilon]} \sigma_t[\mathcal{B}_t(\xi_t(x), 1/|x|)](1+|x|)^{\frac{n-1}{2}} \\ &\leq \frac{C}{(1+|x|)^{\delta+\frac{n+1}{2}}} \Omega_\epsilon(x/|x|).\end{aligned}$$

If $\delta > (n-1)/2$, then since $\Omega_\epsilon \in L^1(S^{n-1})$ we have that

$$\begin{aligned}
&\left\| \sup_{1\le t\le 1+\epsilon} |\mathcal{K}_t(\cdot)| \right\|_{L^1(\mathbb{R}^n)} \\
&\le C \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{\delta+\frac{n+1}{2}}}\,\Omega_\epsilon(x/|x|)\,dx \\
&= C\,\|\Omega_\epsilon\|_{L^1(S^{n-1})} \int_0^\infty \frac{r^{n-1}dr}{(1+r)^{\delta+\frac{n+1}{2}}} \\
&< \infty.
\end{aligned}$$

Finally if $\delta > (n-1)/2$, then we have that

$$\begin{aligned}
\mathcal{J} &\le C \sup_{y\in\mathbb{R}^n_0} \sum_{\{h|(1+\epsilon)^h r(y)>1\}} \int_{|x|\ge((1+\epsilon)^h r(y))^{a-\epsilon}} \frac{1}{(1+|x|)^{\delta+\frac{n+1}{2}}}\,\Omega_\epsilon(x/|x|)\,dx \\
&= C\,\|\Omega_\epsilon\|_{L^1(S^{n-1})} \sup_{y\in\mathbb{R}^n_0} \sum_{\{h|(1+\epsilon)^h r(y)>1\}} \int_{((1+\epsilon)^h r(y))^{a-\epsilon}}^\infty \frac{r^{n-1}\,dr}{(1+r)^{\delta+\frac{n+1}{2}}} \\
&\le C \sup_{y\in\mathbb{R}^n_0} \sum_{\{h|(1+\epsilon)^h r(y)>1\}} \frac{1}{((1+\epsilon)^h r(y))^{(a-\epsilon)(\delta-\frac{n-1}{2})}} < \infty.
\end{aligned}$$

Thus we conclude that $\mathfrak{M}^\delta_\varrho$ is of weak type $(1,1)$ for $\delta > (n-1)/2$.

Finally we utilize the standard linearization technique of the maximal operator in order to employ the Marcinkiewicz interpolation theorem. It now suffices to consider the analytic family of linear operators given by $z \mapsto \mathcal{R}^z_{\varrho,t(x)} f(x)$ where $x \mapsto t(x)$ is an arbitrary measurable function on $\mathbb{R}^n$ with values in $\mathbb{R}_+$. Combining this with the above weak type $(1,1)$-estimate for $\mathfrak{M}^\delta_\varrho$, we get $L^p(\mathbb{R}^n)$- boubdedness of $\mathfrak{M}^\delta_\varrho$ near $p=1$. By the complex interpolation method, we interpolate this estimate with $L^2(\mathbb{R}^n)$-boundedness for $\mathfrak{M}^\delta_\varrho$ on $\delta > 0$. Therefore we complete the proof. □

3. Applications to P.D.E.'s

In this section, we discuss about applications of Theorem 1.2 to several partial differential equations.

(a) It is well-known that the solution $\mathcal{U}$ of the Schrödinger initial value problem

$$\Delta\mathcal{U} = i\frac{\partial}{\partial t}\mathcal{U}, t > 0, \ \ \mathcal{U}(x,0) = f(x), f \in \mathfrak{S}(\mathbb{R}^n)$$

is given by $\mathcal{U}(x,t) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{it|\xi|^2}\hat{f}(\xi)\,d\xi$, and can be extended to all of $L^p(\mathbb{R}^n)$ only if $p = 2$ (see [5]). Sjöstrand [9] studied $L^p(\mathbb{R}^n)$-boundedness of the α-th Riesz means of $\mathcal{U}(x,t)$ defined by

$$\mathcal{R}_t^\alpha(x) = \frac{\alpha+1}{t^{\alpha+1}}\int_0^t (t-s)^\alpha \mathcal{U}(x,s)\,ds.$$

Then we have that

$$\begin{aligned}
\widehat{\mathcal{R}_t^\alpha}(\xi) &= \frac{\alpha+1}{t^{\alpha+1}}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\int_0^t (t-s)^\alpha \mathcal{U}(x,s)\,ds\,dx \\
&= \frac{\alpha+1}{t^{\alpha+1}}\int_0^t (t-s)^\alpha\left(\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\mathcal{U}(x,s)\,dx\right)ds \\
&= \left(\frac{\alpha+1}{t^{\alpha+1}}\int_0^t (t-s)^\alpha e^{is|\xi|^2}\,ds\right)\hat{f}(\xi).
\end{aligned}$$

So the associated Fourier multiplier is

$$\begin{aligned}
\mathfrak{m}_\alpha(t|\xi|^2) &= \frac{\alpha+1}{t^{\alpha+1}}\int_0^t (t-s)^\alpha e^{is|\xi|^2}\,ds \\
&= \frac{\alpha+1}{t^{\alpha+1}}\int_0^{t|\xi|^2} (t|\xi|^2-s)^\alpha e^{is}\,ds.
\end{aligned}$$

Now we consider the generalized Schrödinger operator defined by

$$\mathcal{S}_{\varrho,t}[f](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{it\varrho(\xi)}\hat{f}(\xi)\,d\xi$$

for $t > 0$ and $x \in \mathbb{R}^n$, where (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$, and α-th Riesz means of $\mathcal{S}_{\varrho,t}[f](x)$ defined by

$$\mathcal{R}_{\varrho,t}^\alpha(x) = \frac{\alpha+1}{t^{\alpha+1}}\int_0^t (t-s)^\alpha \mathcal{S}_{\varrho,t}[f](x)\,ds.$$

Similarly to the above, the associated Fourier multiplier is

$$\mathfrak{m}_\alpha(t\varrho(\xi)) = \frac{\alpha+1}{t^{\alpha+1}} \int_0^{t\varrho(\xi)} (t\varrho(\xi) - s)^\alpha e^{is}\, ds.$$

Note that $\mathfrak{m}_\alpha(t) = \frac{\alpha+1}{t^{\alpha+1}} \int_0^t (t-s)^\alpha e^{is}\, ds = (\alpha+1)\int_0^1 (1-s)^\alpha e^{ist}\, ds$. We see that the major contribution of the asymptotic decay for the function $\mathfrak{m}_\alpha^{(\delta+1)}(t)$ comes from some small intervals near $t = 0$ and $t = 1$. Using a function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(s) = 1$ for $s \geq 3/4$ and $\phi(s) = 0$ for $s \leq 1/2$, we split the function $\mathfrak{m}_\alpha^{(\delta+1)}(t)$ as follows;

$$\begin{aligned}\mathfrak{m}_\alpha^{(\delta+1)}(t) &= C \int_0^1 (1-s)^\alpha s^{\delta+1} e^{ist}\, ds \\ &= v_\alpha(t) + w_\alpha(t),\end{aligned}$$

where $v_\alpha(t) = C \int_0^1 (1-s)^\alpha s^{\delta+1} e^{ist} \phi(s)$ and $w_\alpha(t) = C \int_0^1 (1-s)^\alpha s^{\delta+1}\, e^{ist} (1-\phi(s))\, ds$. It then follows from an usual asymptotic formula in A. Erdélyi (Asymptotic Expansion, Dover, 1956) that $v_\alpha(t) \sim 1/(1+t)^{\alpha+1}$ and $w_\alpha(t) \sim 1/(1+t)^{\delta+2}$. So we have that for $\alpha > \delta > (n-1)|1/p - 1/2|$,

$$\int_0^1 s^\delta \left|\mathfrak{m}_\alpha^{(\delta+1)}(s)\right| ds \leq C \int_0^\infty s^\delta \left[(1+s)^{-(\alpha+1)} + (1+s)^{-(\delta+2)}\right] ds < \infty.$$

Thus by Theorem 1.2 we conclude that if $\alpha > (n-1)|1/p - 1/2|$, then $\mathcal{M}_{\mathfrak{m}_\alpha \circ \varrho}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$; moreover, it is of weak type $(1,1)$ for $\alpha > (n-1)/2$.

(b) We now consider the heat initial value problem

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{U} = 0, t > 0, \;\; \mathcal{U}(x,0) = f(x), f \in \mathfrak{S}(\mathbb{R}^n).$$

Then it is well-known that the solution $\mathcal{U}$ is given by

$$\mathcal{U}(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t|\xi|^2} \hat{f}(\xi)\, d\xi.$$

Suppose that (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$. Then we define the generalized heat operator by

$$\mathcal{H}_{\varrho,t}[f](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t\varrho(\xi)} \hat{f}(\xi)\, d\xi$$

for $t > 0$ and $x \in \mathbb{R}^n$. We easily see that the associated Fourier multiplier is $\mathfrak{m}(t\varrho(\xi))$ where $\mathfrak{m}(s) = e^{-s}$. Since $\mathfrak{m}(s)$ trivially satisfies (1.2) for all $\delta > 0$, by Theorem 1.2 we conclude that the associated maximal operator $\mathcal{M}_{\mathfrak{m}\circ\varrho}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$; moreover, it is of weak type $(1, 1)$.

(c) Suppose that (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$. Since $\mathfrak{m}(s) = (1 - e^{-s})/s$ obviously satisfies (1.2) for all $\delta > 0$, by Theorem 1.2 we have that

$$\left\| \sup_{t>0} \left| \mathcal{F}^{-1}\left[(t\varrho)^{-1}(1 - e^{-t\varrho})\hat{f}\,\right](\cdot)\right| \right\|_{L^p(\mathbb{R}^n)} \le C\,\|f\|_{L^p(\mathbb{R}^n)}, \ 1 < p \le \infty$$

and

$$\left| \left\{ x \in \mathbb{R}^n \,|\, \sup_{t>0} |\mathcal{F}^{-1}\left[(t\varrho)^{-1}(1 - e^{t\varrho})\hat{f}\,\right](x)| > \lambda \right\} \right| \le \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}, \ \lambda > 0,$$

which may be useful to prove almost everywhere convergence theorems of Voronovskaya type for the generalized Weierstrass means (see [4]).

(d) Suppose that (Σ_ϱ, t^P) is uniformly spherically integrable near $t = 1$ and let $\mathfrak{m}_\delta(s) = (1 - s)_+^\delta$. Then we get that $\mathfrak{m}_\delta^{(\lambda)}(s) = C(1 - s)_+^{(\delta-\lambda)}$ for $\lambda < \delta + 1$. By Theorem 1.2, we have that $\mathfrak{M}_\varrho^\delta$ is bounded on $L^p(\mathbb{R}^n)$ for $\delta > (n-1)|1/p - 1/2|$, $1 < p \le \infty$, and also it is of weak type $(1, 1)$ for $\delta > (n-1)/2$. This result coincides with that of Lemma 2.4.

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