J. Korean Math. Soc. **37** (2000), No. 4, pp. 545–563

PLANCHEREL AND PALEY-WIENER THEOREMS FOR AN INDEX INTEGRAL TRANSFORM

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Abstract. An integral transform with the Bessel function $J_\nu(z)$ in the kernel is considered. The transform is related to a singular Sturm-Liouville problem on a half line. This relation yields a Plancherel's theorem for the transform. A Paley-Wiener-type theorem for the transform is also derived.

1. Introduction

In this paper, we will derive and study the following pair of transforms

$$(1)\quad G(\tau) = \int_0^1 \Im\left\{\left[\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right] J_{-i\tau}(t)\right\} g(t)\,\frac{dt}{t},$$

$$\alpha \in \left[\frac{-\pi}{4},\, 0\right] \cup \left[\frac{3\pi}{4},\, \pi\right],$$

$$(2)\quad g(t) = \int_0^\infty \frac{2\tau\, \Im\left\{\left[\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right] J_{-i\tau}(t)\right\} G(\tau)}{\sinh \pi\tau\, \left|\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right|^2}\, d\tau,$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν[1], and $\Im z$ denotes the imaginary part of z. An extensive table of integral transforms involving the Bessel functions in the kernels is collected in [6]. Since the integration in (2) is with respect to the order of the Bessel function, such a pair of integral transforms is called index transform. Details about many other index transforms can be found in [15].

Received July 12, 1999.

2000 Mathematics Subject Classification: Primary 44A15, 34B24; Secondary 33C10.

Key words and phrases: Plancherel theorem, Paley-Wiener theorem, singular Sturm-Liouville problem, index transform, Bessel function.

In Section 2, we will show that the pair of transforms (1)-(2) arises from a singular Sturm-Liouville problem on a half line. As a consequence, a Plancherel's theorem and a Parseval's formula for the pair of transforms (1)-(2) will be established.

In Section 3, we will characterize function $g(t)$ as the transform (2) of a function $G(\tau)$ with a compact support. The classical Paley-Wiener theorem [5] for the Fourier transform gives a characterization of the space of square integrable functions with compact support in terms of its image under the Fourier transform by showing that $f \in L_2(R)$ has a compact support if and only if its Fourier transform $\hat{f}$ can be continued analytically to the whole complex plane as an entire function of exponential type whose restriction to the real axis belongs to $L_2(R)$. Notwithstanding the strength of its statement, the proof of the theorem does not lend itself very naturally to other integral transforms. Alternative approaches using real analysis techniques have been developed to characterize the images of spaces of the form $L_2[I, d\rho]$, for some measure $d\rho$ and a finite interval I under various integral transforms, such as the Mellin [12], Hankel [11], Y[10], and Airy transforms [13]. Recently in [14], a unified approach to derive Paley-Wiener-type theorems for a large class of integral transforms that includes not only most of the above transforms, but also any new ones, has been developed. This class of integral transforms arises from two types of singular Sturm-Liouville problems: singular on a half line and singular on the whole line. The latter is more complicated, but includes more interesting examples. The approach can be briefly described as follows [9].

Let L be a differential operator with a continuous spectrum Ω_1 and $\phi(x,\lambda)$ be an eigenfunction with corresponding eigenvalue λ: $L\phi = -\lambda\phi$. In addition, suppose that $T : L_2(\Omega_1; d\rho_1) \to L_2(\Omega_2; d\rho_2)$

$$f(x) = (TF)(x) = \int_{\Omega_1} F(\lambda)\phi(x,\lambda)\, d\rho_1(\lambda)$$

is a unitary transformation

$$\int_{\Omega_2} |f(x)|^2\, d\rho_2(x) = \int_{\Omega_1} |F(\lambda)|^2\, d\rho_1(\lambda).$$

In that case, if $\lambda^n F(\lambda) \in L_2(\Omega_1;\, d\rho_1)$, we have

$$L^n f(x) = \int_{\Omega_1} (-\lambda)^n F(\lambda)\phi(x,\lambda)\, d\rho_1(\lambda),$$

and

$$\int_{\Omega_2} |L^n f(x)|^2 \, d\rho_2(x) = \int_{\Omega_1} |\lambda^n F(\lambda)|^2 \, d\rho_1(\lambda). \tag{3}$$

Raising both sides of (3) to the power $1/(2n)$ and taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \|L^n f(x)\|_{L_2(\Omega_2;\, d\rho_2)}^{1/n} = \sup_{\lambda \in \operatorname{supp} F} |\lambda|, \tag{4}$$

where $\operatorname{supp} F$ denotes the support of F, the smallest closed set, outside which the function F vanishes almost everywhere.

From (4) it is obvious that if

$$\lim_{n\to\infty} \|L^n f(x)\|_{L_2(\Omega_2;\, d\rho_2)}^{1/n} < \infty, \tag{5}$$

then F has a compact support. Hence, formula (4) plays a decisive role in studying integral transforms of functions with compact supports. It can be shown [9, 14] that under some "extra conditions" on f inequality (5) gives the necessary and sufficient condition for a function f to be a T-transform of a function $F \in L_2(\Omega_1;\, d\rho_1)$ with compact support. Formula (4) has been first discovered in [2] for one-dimensional Fourier transform with $L = \frac{d}{dx}$, and independently in [3, 8] for multidimensional Fourier transform with L being the Laplacian or any polynomial differential operator. In this paper, such approach will be applied to obtain a Paley-Wiener-type theorem for the transform (2).

2. A related singular Sturm-Liouville problem

In this section, we will show that the pair of transforms (1)-(2) can be interpreted as an eigenfunction expansion associated with a singular Sturm-Liouville problem on a half line. As a consequence, a Plancherel theorem for the pair of transforms (1)-(2) is established.

Let us consider the general singular Sturm-Liouville problem on the half line

$$Ly := \frac{d^2y}{dx^2} - q(x)y = -\lambda y, \quad 0 \le x < \infty \tag{6}$$

with

$$\begin{aligned} y(0)\cos\alpha + y'(0)\sin\alpha &= 0, \quad 0 \le \alpha < 2\pi, \\ |y(\infty)| &< \infty, \end{aligned} \tag{7}$$

and $q(x)$ is assumed to be continuous on $R^+ = [0, \infty)$, and $q(x) \in L_1(R^+)$. Under this assumption, the spectrum of the operator L consists of the continuous part R^+ and a possible finite discrete part on $(-\infty, 0)$ [4, 7].

Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be the solutions of equation (6) satisfying the initial conditions

$$\phi(0, \lambda) = \sin \alpha, \qquad \phi'(0, \lambda) = -\cos \alpha, \tag{8}$$
$$\theta(0, \lambda) = \cos \alpha, \qquad \theta'(0, \lambda) = \sin \alpha.$$

Throughout the paper $\phi'(x, \lambda)$ will mean $\frac{\partial}{\partial x}\phi(x, \lambda)$.

It is known [7] that any non-real λ, there exists a function $m(\lambda)$, analytic in the upper and lower half planes that are not necessarily analytic continuation of each other so that

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda) \tag{9}$$

as a function of x, is in $L_2(R^+)$ for non-real λ. Moreover,

$$-\frac{1}{\pi} \lim_{\delta \to 0} \int_0^\lambda \Im\, m(u + i\delta)\, du = \rho(\lambda), \tag{10}$$

where $\rho(\lambda)$ is a non-decreasing function. We assume that the discrete part of the spectrum is empty, that is the case, when $m(\lambda)$ is real-valued on the negative axis and has no poles there. in this case, it is known fact [7] that if $f(x) \in L_2(R^+)$, then

$$F(\lambda) = \int_0^\infty f(x)\phi(x, \lambda)\, dx \tag{11}$$

belongs to $L_2(R^+, d\rho)$ and

$$f(x) = \int_0^\infty F(\lambda)\phi(x, \lambda)\, d\rho(\lambda) \tag{12}$$

with

$$\|f\|_{L_2(R^+)} = \|F\|_{L_2(R^+, d\rho)}. \tag{13}$$

Here the integral $\int_0^\infty$ with respect to x is interpreted as $\lim_{N\to\infty} \int_0^N$ with convergence in the metric of $L_2(R^+, d\rho)$, while the integral $\int_0^\infty$ with respect to λ is interpreted as $\lim_{N\to\infty} \int_0^N$ with convergence in the metric of $L_2(R^+)$. Conversely, if $F(\lambda) \in L_2(R^+, d\rho)$, then $f(x)$, given by (12), belongs to $L_2(R^+)$ and formula (11) holds.

Now we are ready to consider the following particular singular Sturm-Liouville problem

$$
\begin{aligned}
(14) \qquad Ly = y'' + e^{-2x} y &= -\lambda y, \quad 0 \le x < \infty, \\
y(0)\cos\alpha + y'(0)\sin\alpha &= 0, \quad 0 \le \alpha < 2\pi, \quad |y(\infty)| < \infty.
\end{aligned}
$$

Since $q(x) = -e^{-2x} \in L_1(R^+)$, the continuous part of the spectrum of the singular Sturm-Liouville problem (14) is R^+. A later computation will show that under some restriction on α the discrete part of the spectrum is empty.

Making the change of variable $t = e^{-x}$ transforms equation (14) into a Bessel differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(1 + \frac{\lambda}{t^2}\right) y = 0$$

whose general solution is [1]

$$y(t) = aJ_{i\sqrt{\lambda}}(t) + bJ_{-i\sqrt{\lambda}}(t).$$

Hence, the general solution of equation (14) is

$$y(x) = aJ_{i\sqrt{\lambda}}\left(e^{-x}\right) + bJ_{-i\sqrt{\lambda}}\left(e^{-x}\right).$$

The solution $\phi(x,\lambda)$ satisfies the initial conditions

$$\phi(0,\lambda) = \sin\alpha, \quad \phi'(0,\lambda) = -\cos\alpha.$$

Thus

$$
\begin{aligned}
aJ_{i\sqrt{\lambda}}(1) + bJ_{-i\sqrt{\lambda}}(1) &= \sin\alpha, \\
aJ'_{i\sqrt{\lambda}}(1) + bJ'_{-i\sqrt{\lambda}}(1) &= \cos\alpha.
\end{aligned}
$$

Using the Wronskian equality for the Bessel functions [1]

$$(15) \qquad W(J_\nu(x), J_{-\nu}(x)) := J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x) = -\frac{2\sin(\pi\nu)}{\pi x}$$

to solve this system of equations, we get

$$
\begin{aligned}
a &= \frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\left(\sin\alpha J'_{-i\sqrt{\lambda}}(1) - \cos\alpha J_{-i\sqrt{\lambda}}(1)\right), \\
b &= \frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\left(\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right).
\end{aligned}
$$

Thus,

$$\begin{aligned}(16)\qquad &\phi(x,\lambda)\\ &= \frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big[\Big(\sin\alpha J'_{-i\sqrt{\lambda}}(1)-\cos\alpha J_{-i\sqrt{\lambda}}(1)\Big)J_{i\sqrt{\lambda}}(e^{-x})\\ &+\Big(\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\Big)J_{-i\sqrt{\lambda}}(e^{-x})\Big].\end{aligned}$$

If $\lambda>0$, then $J_{-i\sqrt{\lambda}}(x)=\overline{J_{i\sqrt{\lambda}}(x)}$, $J'_{-i\sqrt{\lambda}}(x)=\overline{J'_{i\sqrt{\lambda}}(x)}$ and formula (16) can be simplified further:

$$\begin{aligned}(17)\qquad &\phi(x,\lambda)\\ &= \frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big[\Big(\sin\alpha\overline{J'_{i\sqrt{\lambda}}(1)}-\cos\alpha\overline{J_{i\sqrt{\lambda}}(1)}\Big)\overline{J_{-i\sqrt{\lambda}}(e^{-x})}\\ &-\Big(\sin\alpha J'_{i\sqrt{\lambda}}(1)-\cos\alpha J_{i\sqrt{\lambda}}(1)\Big)J_{-i\sqrt{\lambda}}(e^{-x})\Big]\\ &= \frac{\pi}{\sinh(\pi\sqrt{\lambda})}\Im\Big\{\Big[\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\Big]J_{-i\sqrt{\lambda}}(e^{-x})\Big\}\end{aligned}$$

Similarly, the solution $\theta(x,\lambda)$ satisfies the initial conditions

$$\theta(0,\lambda)=\cos\alpha,\quad \theta'(0,\lambda)=\sin\alpha,$$

that yields

$$\begin{aligned} aJ_{i\sqrt{\lambda}}(1)+bJ_{-i\sqrt{\lambda}}(1) &= \cos\alpha,\\ aJ'_{i\sqrt{\lambda}}(1)+bJ'_{-i\sqrt{\lambda}}(1) &= -\sin\alpha.\end{aligned}$$

Solving this system we get

$$a=\frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big(\cos\alpha J'_{-i\sqrt{\lambda}}(1)+\sin\alpha J_{-i\sqrt{\lambda}}(1)\Big),$$

$$b=-\frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big(\cos\alpha J'_{i\sqrt{\lambda}}(1)+\sin\alpha J_{i\sqrt{\lambda}}(1)\Big).$$

Therefore,

$$\begin{aligned}(18)\qquad &\theta(x,\lambda)\\ &= \frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big[\Big(\cos\alpha J'_{-i\sqrt{\lambda}}(1)+\sin\alpha J_{-i\sqrt{\lambda}}(1)\Big)J_{i\sqrt{\lambda}}(e^{-x})\\ &-\Big(\cos\alpha J'_{i\sqrt{\lambda}}(1)+\sin\alpha J_{i\sqrt{\lambda}}(1)\Big)J_{-i\sqrt{\lambda}}(e^{-x})\Big].\end{aligned}$$

We have

$$
\begin{aligned}
&\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)\\
&=\frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big[\big(\cos\alpha+\sin\alpha\, m(\lambda)\big)J'_{-i\sqrt{\lambda}}(1)\\
&\qquad\qquad+\big(\sin\alpha-\cos\alpha\, m(\lambda)\big)J_{-i\sqrt{\lambda}}(1)\Big]J_{i\sqrt{\lambda}}(e^{-x})\\
&\quad-\frac{\pi i}{2\sinh(\pi\sqrt{\lambda})}\Big[\big(\cos\alpha+\sin\alpha\, m(\lambda)\big)J'_{i\sqrt{\lambda}}(1)\\
&\qquad\qquad+\big(\sin\alpha-\cos\alpha\, m(\lambda)\big)J_{i\sqrt{\lambda}}(1)\Big]J_{-i\sqrt{\lambda}}(e^{-x}).
\end{aligned}
\tag{19}
$$

Using the asymptotic formula for the Bessel function

$$
J_\nu(z)=\frac{z^\nu}{2^\nu\Gamma(\nu+1)}\big(1+O\left(z^2\right)\big),\quad z\to 0,
\tag{20}
$$

we get

$$
J_{i\sqrt{\lambda}}(e^{-x})=\frac{e^{-i\sqrt{\lambda}x}}{2^{i\sqrt{\lambda}}\Gamma(i\sqrt{\lambda}+1)}(1+O(e^{-2x})),\quad x\to\infty,
$$

$$
J_{-i\sqrt{\lambda}}(e^{-x})=\frac{e^{i\sqrt{\lambda}x}}{2^{-i\sqrt{\lambda}}\Gamma(-i\sqrt{\lambda}+1)}\left(1+O\left(e^{-2x}\right)\right),\quad x\to\infty.
$$

Hence, if $\Im\lambda>0$, then $J_{-i\sqrt{\lambda}}(e^{-x})\in L_2(R^+)$, but $J_{i\sqrt{\lambda}}(e^{-x})\notin L_2(R^+)$. Consequently, for $\Im\lambda>0$ the function $\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$ belongs to $L_2(R^+)$ if and only if the coefficient of $J_{i\sqrt{\lambda}}(e^{-x})$ in the formula (19) vanishes:

$$
\big(\cos\alpha+\sin\alpha\, m(\lambda)\big)J'_{-i\sqrt{\lambda}}(1)+\big(\sin\alpha-\cos\alpha\, m(\lambda)\big)J_{-i\sqrt{\lambda}}(1)=0.
$$

Solving this equation we obtain

$$
m(\lambda)=\frac{\cos\alpha J'_{-i\sqrt{\lambda}}(1)+\sin\alpha J_{-i\sqrt{\lambda}}(1)}{\cos\alpha J_{-i\sqrt{\lambda}}(1)-\sin\alpha J'_{-i\sqrt{\lambda}}(1)}.
\tag{21}
$$

Since $\lim_{\delta\to 0+}m(\lambda+i\delta)=m(\lambda)$ for $\lambda\in R,\ \lambda\neq 0$, it follows from (10) that

$$
d\rho(\lambda)=-\frac{1}{\pi}\Im\, m(\lambda)\,d\lambda,\quad \lambda\in R,\quad \lambda\neq 0.
$$

If $\lambda<0$, then $J_{-i\sqrt{\lambda}}(1)=J_{\sqrt{|\lambda|}}(1)$ and $J'_{-i\sqrt{\lambda}}(1)=J'_{\sqrt{|\lambda|}}(1)$, therefore, $\Im\, m(\lambda)=0$. Consequently, the negative part of the spectrum is discrete.

From the integral representation [6]

$$J_\nu(x) = \frac{x^\nu}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(x\sin t)(\cos t)^{2\nu}\, dt, \quad \nu > -\frac{1}{2},$$

it is clear that $J_\nu(1)$ is positive when ν is positive. Moreover,

$$\begin{aligned} J_{\nu+1}(1) &= \frac{1}{\sqrt{\pi}2^{\nu}\Gamma\left(\nu+\frac{3}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(\sin t)(\cos t)^{2\nu+2}\, dt \\ &\leq \frac{1}{2\nu+1}\frac{1}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(\sin t)(\cos t)^{2\nu}\, dt = \frac{J_\nu(1)}{2\nu+1}. \end{aligned}$$

Hence, applying the relation [6]

$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

we get

$$\frac{J_\nu'(1)}{J_\nu(1)} = \nu - \frac{J_{\nu+1}(1)}{J_\nu(1)} \geq \nu - \frac{1}{2\nu+1} > -1,$$

if ν is positive. Thus, the function

$$\cos\alpha J_{\sqrt{|\lambda|}}(1) - \sin\alpha J'_{\sqrt{|\lambda|}}(1)$$

has no zeros if $-\frac{\pi}{4} \leq \alpha \leq 0$ or $\frac{3\pi}{4} \leq \alpha \leq \pi$. Consequently, $m(\lambda)$ has no poles on the negative axis, and therefore, the discrete negative spectrum is empty.

Thus, we can assume that $\lambda > 0$. We have

$$\begin{aligned} m(\lambda) &= \frac{\left(\cos\alpha J'_{-i\sqrt{\lambda}}(1) + \sin\alpha J_{-i\sqrt{\lambda}}(1)\right)\left(\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right)}{\left(\cos\alpha J_{-i\sqrt{\lambda}}(1) - \sin\alpha J'_{-i\sqrt{\lambda}}(1)\right)\left(\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right)} \\ &= \frac{\cos^2\alpha\, J_{i\sqrt{\lambda}}(1)\, J'_{-i\sqrt{\lambda}}(1) - \sin^2\alpha\, J_{-i\sqrt{\lambda}}(1)\, J'_{i\sqrt{\lambda}}(1)}{\left|\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2} \\ &\quad + \frac{\sin\alpha\, \cos\alpha\left(\left|J_{i\sqrt{\lambda}}(1)\right|^2 - \left|J'_{i\sqrt{\lambda}}(1)\right|^2\right)}{\left|\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}. \end{aligned}$$

Therefore, since

$$\frac{\sin\alpha\,\cos\alpha\left(\left|J_{i\sqrt{\lambda}}(1)\right|^2-\left|J'_{i\sqrt{\lambda}}(1)\right|^2\right)}{\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}$$

has no imaginary part, we have

$$\begin{aligned}
\Im\, m(\lambda) &= \Im\,\frac{\cos^2\alpha\, J_{i\sqrt{\lambda}}(1)\,J'_{-i\sqrt{\lambda}}(1)-\sin^2\alpha\, J_{-i\sqrt{\lambda}}(1)\,J'_{i\sqrt{\lambda}}(1)}{\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}\\
&= \frac{\cos^2\alpha\,\left[J_{i\sqrt{\lambda}}(1)\,J'_{-i\sqrt{\lambda}}(1)-\overline{J_{i\sqrt{\lambda}}(1)\,J'_{-i\sqrt{\lambda}}(1)}\right]}{2i\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}\\
&\quad -\frac{\sin^2\alpha\,\left[J_{-i\sqrt{\lambda}}(1)\,J'_{i\sqrt{\lambda}}(1)-\overline{J_{-i\sqrt{\lambda}}(1)\,J'_{i\sqrt{\lambda}}(1)}\right]}{2i\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}\\
&= \frac{J_{i\sqrt{\lambda}}(1)\,J'_{-i\sqrt{\lambda}}(1)-J_{-i\sqrt{\lambda}}(1)\,J'_{i\sqrt{\lambda}}(1)}{2i\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}\\
&= -\frac{\sinh\pi\sqrt{\lambda}}{\pi\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(22)\qquad d\rho(\lambda) &= -\frac{1}{\pi}\,\Im\, m(\lambda)\,d\lambda\\
&= \frac{\sinh\pi\sqrt{\lambda}}{\pi^2\left|\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2}\,d\lambda,\quad \lambda>0.
\end{aligned}$$

We arrive at the following pair of integral transforms

$$(23)\quad F(\lambda)=\frac{\pi}{\sinh\pi\sqrt{\lambda}}\int_0^\infty \Im\left\{\left[\cos\alpha J_{i\sqrt{\lambda}}(1)-\sin\alpha J'_{i\sqrt{\lambda}}(1)\right]J_{-i\sqrt{\lambda}}(e^{-x})\right\}f(x)\,dx,$$

$$(24)\quad f(x)$$
$$= \int_0^\infty \frac{\pi}{\sinh \pi\sqrt{\lambda}} \Im \left\{ \left[\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right] J_{-i\sqrt{\lambda}}\left(e^{-x}\right)\right\}$$
$$\frac{\sinh \pi\sqrt{\lambda}}{\pi^2 \left|\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2} F(\lambda)\, d\lambda$$
$$= \int_0^\infty \frac{\Im\left\{\left[\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right] J_{-i\sqrt{\lambda}}\left(e^{-x}\right)\right\} F(\lambda)}{\pi\left|\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2} d\lambda,$$

with the Parseval equality

$$(25) \int_0^\infty |f(x)|^2\, dx = \int_0^\infty \frac{\sinh \pi\sqrt{\lambda}}{\pi^2\left|\cos\alpha J_{i\sqrt{\lambda}}(1) - \sin\alpha J'_{i\sqrt{\lambda}}(1)\right|^2} |F(\lambda)|^2\, d\lambda.$$

Making the change of variables

$$(26)\quad e^{-x} = t, \quad \sqrt{\lambda} = \tau, \quad f(x) = g(t), \quad \frac{\sinh \pi\sqrt{\lambda}}{\pi} F(\lambda) = G(\tau),$$

we finally arrive at the pair of transforms (1)-(2):

$$(27)\quad G(\tau) = \int_0^1 \Im\left\{\left[\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right] J_{-i\tau}(t)\right\} g(t)\, \frac{dt}{t},$$

$$(28)\quad g(t) = \int_0^\infty \frac{2\tau\, \Im\left\{\left[\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right] J_{-i\tau}(t)\right\} G(\tau)}{\sinh \pi\tau\ \left|\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right|^2} d\tau,$$

and the Parseval equality takes the form

$$(29)\quad \int_0^1 |g(t)|^2\, \frac{dt}{t} = \int_0^\infty \frac{2\tau\, |G(\tau)|^2}{\sin h\pi\tau\ \left|\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right|^2} d\tau.$$

Thus we obtain

THEOREM 1.(Plancherel's Theorem) *Let* $\alpha \in \left[-\frac{\pi}{4}, 0\right] \cup \left[\frac{3\pi}{4}, \pi\right]$. *The integral transform* (27) *is a homeomorphism from the space* $L_2([0,1], t^{-1}dt)$ *onto the space* $L_2\left(R^+, \frac{2\tau}{\sinh \pi\tau \left|\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right|^2}\, d\tau\right)$, *with the inverse having the form* (28). *Moreover, the Parseval equality* (29) *holds.*

The integral $\int_0^1$ in (27) with respect to t is interpreted as $\lim_{\epsilon\to 0+}\int_\epsilon^1$ with convergence in the metric of the space $L_2\Big(R^+, \frac{2\tau}{\sinh\pi\tau\left|\cos\alpha J_{i\tau}(1)-\sin\alpha J'_{i\tau}(1)\right|^2}\, d\tau\Big)$ and the integral $\int_0^\infty$ in (28) with respect to τ is interpreted as $\lim_{N\to\infty}\int_0^N$ with convergence in the metric of the space $L_2\big([0,1],t^{-1}\,dt\big)$.

As an example take $\alpha = 0$. We get the pair of transforms

$$G(\tau) = \int_0^1 \Im\left\{J_{i\tau}(1)J_{-i\tau}(t)\right\}g(t)\,\frac{dt}{t}, \tag{30}$$

$$g(t) = \int_0^\infty \frac{2\tau\,\Im\left\{J_{i\tau}(1)J_{-i\tau}(t)\right\}G(\tau)}{\sinh\pi\tau\;|J_{i\tau}(1)|^2}d\tau, \tag{31}$$

with the Parseval equality

$$\int_0^1 |g(t)|^2\,\frac{dt}{t} = \int_0^\infty \frac{2\tau\,|G(\tau)|^2}{\sinh\pi\tau\;|J_{i\tau}(1)|^2}\,d\tau. \tag{32}$$

3. A Paley-Wiener-type theorem

Throughout this section we assume $\alpha \in \left[-\frac{\pi}{4},0\right]\cup\left[\frac{3\pi}{4},\pi\right]$. Let

$$\Phi(t,\tau) = \Im\left\{\left[\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)\right]J_{-i\tau}(t)\right\}. \tag{33}$$

Then

$$D\Phi := \left(t^2\frac{\partial^2}{\partial t^2} + t\frac{\partial}{\partial t} + t^2\right)\Phi(t,\tau) = -\tau^2\Phi(t,\tau) \tag{34}$$

and

$$\Phi(1,\tau) = \frac{1}{\pi}\sin\alpha\,\sinh\pi\tau, \quad \Phi'(1,\tau) = -\frac{1}{\pi}\cos\alpha\,\sinh\pi\tau. \tag{35}$$

Hence,

$$\cos\alpha\,\Phi(1,\tau) + \sin\alpha\,\Phi'(1,\tau) = 0. \tag{36}$$

Moreover, from the asymptotics of $J_{-i\tau}(t)$ (see formula (20)) it is clear that $\Phi(t,\tau)$ and $t\Phi'(t,\tau)$ are uniformly bounded with respect to τ as $t\to 0+$. Denote

$$d\xi(\tau) = \frac{2\tau}{\sinh \pi\tau \; |\cos\alpha J_{i\tau}(1) - \sin\alpha J'_{i\tau}(1)|^2}\, d\tau. \tag{37}$$

The pair of transforms (1)-(2) takes a simpler form

$$G(\tau) = \int_0^1 \Phi(t,\tau)\, g(t)\, \frac{dt}{t}, \tag{38}$$

$$g(t) = \int_0^\infty \Phi(t,\tau)\, G(\tau)\, d\xi(\tau), \tag{39}$$

and the Parseval equality (29) becomes

$$\int_0^1 |g(t)|^2 \frac{dt}{t} = \int_0^\infty |G(\tau)|^2\, d\xi(\tau). \tag{40}$$

We will study now the transform (39) called the Φ-transform. The transform (24) is named the ϕ- transform.

Lemma 1. *Let $G(\tau)$ be such that $\tau^n G(\tau) \in L_2(R^+, d\xi)$ for all $n = 0, 1, 2, \ldots$. A function $g(t)$ is the Φ-transform (39) of G if and only if*

i) *$g(t)$ is infinitely differentiable on $(0,1)$, and $(D^n g)(t) \in L_2\big([0,1], t^{-1}, dt\big)$ for all $n = 0,\ 1,\ 2,\ \ldots$*

ii) *$\lim_{t\to 0+}(D^n g)(t) = \lim_{t\to 0+} t\,\frac{d}{dt}(D^n g)(t) = 0$ for all $n = 0,\ 1,\ 2,\ \ldots$*

iii) *$\lim_{t\to 1-}\left\{\cos\alpha\,(D^n g)(t) + \sin\alpha\,\frac{d}{dt}(D^n g)(t)\right\} = 0$ for all $n = 0,\ 1,\ 2,\ \ldots$*

Proof. Necessity: Let $\tau^n G(\tau) \in L_2(R^+, d\xi)$ for any n. Then $\lambda^n F(\lambda) \in L_2(R^+, d\rho)$ for any n, where F and G are related by the formula (26).

i) Let $f(x)$ and $h(x)$ be the ϕ-transformations (24) of $F(\lambda) \in L_2(R^+, d\rho)$ and $\lambda F(\lambda) \in L_2(R^+, d\rho)$, respectively. Then both f and h belong to $L_2(R^+)$.

The Green's function $G(x,y,\mu)$ of the problem (14) is defined for μ non-real as [4, 7]

$$G(x,y,\mu) = \begin{cases} \psi(x,\mu)\phi(y,\mu), & y \le x \\ \phi(x,\mu)\psi(y,\mu), & x < y \end{cases}, \tag{41}$$

with ϕ and ψ being defined as in (8) and (9), respectively. If $k \in L_2(R^+)$, then by $R_\mu k$ we denote the resolvent function of the boundary value problem (14)

$$
\begin{aligned}
(R_\mu k)(x) &= \int_0^\infty G(x,y,\mu)k(y)\,dy \\
&= \psi(x,\mu)\int_0^x \phi(y,\mu)k(y)\,dy + \phi(x,\mu)\int_x^\infty \psi(y,\mu)k(y)\,dy, \qquad (42)
\end{aligned}
$$

which is easily seen to be in $L_2(R^+)$ for μ non-real. Moreover, $R_\mu k$ is twice differentiable and $(L+\mu)R_\mu k = k$ [4, 7], where the operator L is defined in (14). The resolvent function $R_\mu k$ has the following integral representation [4]

$$
\int_0^\infty G(x,y,\mu)\,k(y)\,dy = \int_0^\infty \phi(x,\lambda)\frac{K(\lambda)}{\mu-\lambda}\,d\rho(\lambda)\,, \quad \mu \text{ non-real}, \qquad (43)
$$

where $K(\lambda)$ is the transform (23) of $k(x)$.

Since $f, h \in L_2(R^+)$, the integral representations of the form (43) of the resolvent functions $R_\mu f$ and $R_\mu h$ yield

$$
\begin{aligned}
f(x) &= \int_0^\infty (\mu-\lambda)\phi(x,\lambda)\frac{F(\lambda)}{\mu-\lambda}\,d\rho(\lambda) \\
&= \mu\int_0^\infty \phi(x,\lambda)\frac{F(\lambda)}{\mu-\lambda}\,d\rho(\lambda) - \int_0^\infty \phi(x,\lambda)\frac{\lambda F(\lambda)}{\mu-\lambda}\,d\rho(\lambda) \\
&= R_\mu\,(\mu f - h)\,(x),
\end{aligned}
$$

where μ is a non-real number. Because $\mu f - h \in L_2(R^+)$, $R_\mu\,(\mu f-h)$ is twice differentiable, so is f. Hence, $g(t) = f(-\ln t)$ is also twice differentiable. Moreover,

$$
(L+\mu)f = (L+\mu)R_\mu\,(\mu f - h) = \mu f - h.
$$

Hence, $Lf = -h$. Making the change of variables $e^{-x} = t$ and $\sqrt{\lambda} = \tau$, we get $Lf = Dg$ and

$$
Dg(t) = -\int_0^\infty \tau^2 G(\tau)\Phi(t,\tau)\,d\xi(\tau). \qquad (44)
$$

By induction one can show that

$$
(D^n g)(t) = \int_0^\infty (-\tau^2)^n G(\tau)\Phi(t,\tau)\,d\xi(\tau), \qquad (45)
$$

for any n. It means g is infinitely differentiable. Because $\tau^{2n}G(\tau) \in L_2(R^+, d\xi)$ it follows from the Plancherel theorem that $(D^n g)(t) \in L_2$ $([0,1], t^{-1}dt)$ for any n.

ii) Since $h \in L_2(R^+)$ and $Lf = -h$, then $Lf \in L_2(R^+)$. Because $q(x) = -e^{-2x}$ is bounded, $qf \in L_2(R^+)$, and therefore, $f'' = Lf + qf \in L_2(R^+)$. Hence, zero-extensions of f and f'' to the negative axis that we denote again by f and f'' belong to $L_2(R)$. Now the fact that $f, f'' \in L_2(R)$ yields $f' \in L_2(R)$, that means, the restriction of f' on R^+ belongs to $L_2(R^+)$. In fact, if we denote the Fourier transform of $f(x)$ by $\hat{f}(\omega)$, then the Fourier transform of f' and f'' are $i\omega\hat{f}(\omega)$ and $(i\omega)^2\hat{f}(\omega)$, respectively. To show that $f' \in L_2(R)$ it suffices to show that $\omega\hat{f}(\omega) \in L_2(R)$. But this follows from the Cauchy- Schwarz inequality

$$\int_{-\infty}^{\infty} \omega^2 \left|\hat{f}(\omega)\right|^2 d\omega \le \left(\int_{-\infty}^{\infty} \omega^4 \left|\hat{f}(\omega)\right|^2 d\omega\right)\left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega\right) < \infty.$$

The last inequality holds since $\hat{f}(\omega)$ and $\omega^2\hat{f}(\omega)$ are in $L_2(R)$.

Now we have

$$2\int_0^x f(x)f'(x)\,dx = f^2(x) - f^2(0).$$

But since $f(x)f'(x) \in L_1(R^+)$, the limit of the right-hand side exists as $x \to \infty$. Consequently, $\lim_{x\to\infty} f^2(x)$ exists. But $f \in L_2(R^+)$, then the limit must be zero:

$$\lim_{x\to\infty} f(x) = 0.$$

Similarly, from the relation

$$2\int_0^x f'(x)f''(x)\,dx = f'^2(x) - f'^2(0)$$

and $f' \in L_2(R^+)$ it follows

$$\lim_{x\to\infty} f'(x) = 0.$$

Making the change $g(t) = f(-\ln t)$ completes the proof of ii) for $n = 0$. The general case can be proved in a similar way.

iii) Since both $(D^n g)(t)$ and $\frac{d}{dt}(D^n g)(t)$ are continuous at $t = 1$, by taking the limit of (45) as $t \to 1-$, we have

$$\lim_{t\to 1-} (D^n g)(t) = (D^n g)(1) = \int_0^\infty (-\tau^2)^n\, G(\tau)\Phi(1,\tau)\, d\xi(\tau) = A_n \sin\alpha,$$

where

$$A_n = \frac{1}{\pi}\int_0^\infty (-\tau^2)^n \sinh \pi\tau\, G(\tau)\, d\xi(\tau),$$

and

$$\lim_{t\to 1-} \frac{d}{dt}(D^n g)(t) = \frac{d}{dt}(D^n g)(1)$$
$$= \int_0^\infty (-\tau^2)^n\, G(\tau)\Phi'(1,\tau)\, d\xi(\tau) = -A_n \cos\alpha.$$

Hence,

$$\lim_{t\to 1-} \left\{ \cos\alpha\, (D^n g)(t) + \sin\alpha\, \frac{d}{dt}(D^n g)(t) \right\} = 0.$$

Sufficiency: Let g satisfy conditions i)-iii) of the lemma. We need only to show that

$$(-\tau^2)^n G(\tau) = \int_0^1 (D^n g)(t)\Phi(t,\tau)\, \frac{dt}{t}. \tag{46}$$

Then, because $(D^n g)(t) \in L_2([0,1]; t^{-1}dt)$, the Plancherel theorem implies $(-\tau^2)^n\, G(\tau) \in L_2(R^+, d\xi)$. Hence, $\tau^n\, G(\tau) \in L_2(R^+, d\xi)$ for any n.

We use induction on n. Clearly, formula (46) holds for $n = 0$. Let it hold for n. Then, in view of (34) integration by parts gives

$$\begin{aligned}
(-\tau^2)^{n+1} G(\tau) &= \int_0^1 (D^n g)(t)(-\tau^2)\Phi(t,\tau)\, \frac{dt}{t} \\
&= \int_0^1 (D^n g)(t) D\Phi(t,\tau)\, \frac{dt}{t} \\
&= \int_0^1 (D^n g)(t) \left(t\, \frac{d^2}{dt^2} + \frac{d}{dt} + t \right) \Phi(t,\tau) dt \\
&= \left[t(D^n g)(t)\Phi'(t,\tau) - t\frac{d}{dt}(D^n g)(t)\Phi(t,\tau) \right] \Bigg|_0^1 \\
&\quad + \int_0^1 (D^{n+1} g)(t)\Phi(t,\tau)\, \frac{dt}{t}.
\end{aligned}$$

But from (35) and property iii) it is easy to see that

$$\lim_{t\to 1-} t \left[(D^n g)(t)\, \Phi'(t,\tau) - \frac{d}{dt}(D^n g)(t)\, \Phi(t,\tau) \right] = 0, \tag{47}$$

and, by property ii) and the boundedness of $\Phi(t,\tau)$ and $t\phi'(t,\tau)$ as $t \to 0+$, the same term also tends to zero as $t \to 0+$. Therefore,

$$(-\tau^2)^{n+1}G(\tau) = \int_0^1 (D^{n+1}g)(t)\Phi(t,\tau)\,\frac{dt}{t},$$

and the induction is complete. □

LEMMA 2. *Let $g(t)$ be the Φ-transform of a function $G(\tau)$ as given by (39)0. Let $\tau^n G(\tau) \in L_2(R^+, d\xi)$ for all $n = 0, 1, 2, \dots$. Then*

$$\lim_{n\to\infty} \|D^n g\|_{L_2([0,1],t^{-1}dt)}^{1/(2n)} = \sup_{\tau \in \operatorname{supp} G} \tau. \tag{48}$$

Proof. From the relation

$$(D^n g)(t) = (-1)^n \int_0^\infty \tau^{2n} G(\tau)\Phi(t,\tau)\,d\xi(\tau),$$

and the Parseval equation (43) we have

$$\|D^n g\|_{L_2([0,1],t^{-1}dt)}^2 = \int_0^\infty \tau^{4n}|G(\tau)|^2\,d\xi(\tau).$$

First, let G have a compact support: $\sup_{\tau\in\operatorname{supp} G} \tau = \delta < \infty$. Then

$$\int_0^\infty \tau^{4n}|G(\tau)|^2\,d\xi(\tau) = \int_0^\delta \tau^{4n}|G(\tau)|^2\,d\xi(\tau) \le \delta^{4n}\int_0^\delta |G(\tau)|^2\,d\xi(\tau).$$

Hence,

$$\limsup_{n\to\infty} \|D^n g\|_{L_2([0,1],t^{-1}dt)}^{1/(2n)} \le \delta\ \limsup_{n\to\infty}\left\{\int_0^\delta |G(\tau)|^2\,d\xi(\tau)\right\}^{1/(4n)} = \delta.$$

On the other hand, since δ is the supremum of the support of G, we have, for any ϵ, $0 < \epsilon < \delta$,

$$\int_{\delta-\epsilon}^\delta |G(\tau)|^2\,d\xi(\tau) > 0.$$

Therefore,

$$\liminf_{n\to\infty} \|D^n g\|_{L_2([0,1],t^{-1}dt)}^{1/(2n)} \ge \liminf_{n\to\infty}\left\{\int_{\delta-\epsilon}^\delta \tau^{4n}|G(\tau)|^2\,d\xi(\tau)\right\}^{1/(4n)}$$

$$\ge\ (\delta-\epsilon)\ \liminf_{n\to\infty}\left\{\int_{\delta-\epsilon}^\delta |G(\tau)|^2\,d\xi(\tau)\right\}^{1/(4n)} = \delta - \epsilon.$$

Because $\epsilon > 0$ is arbitrary, we obtain

$$\lim_{n\to\infty} \|D^n f\|^{1/(2n)}_{L_2([0,1],t^{-1}dt)} = \delta.$$

Now let G have an unbounded support. Then for any N large enough

$$\int_N^\infty |G(\tau)|^2\, d\xi(\tau) > 0.$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \|D^n g\|^{1/(2n)}_{L_2([0,1],t^{-1}dt)} &\geq \lim_{n\to\infty} \left\{ \int_N^\infty \tau^{4n} |G(\tau)|^2\, d\xi(\tau) \right\}^{1/(4n)} \\ &\geq N \lim_{n\to\infty} \left\{ \int_N^\infty |G(\tau)|^2\, d\xi(\tau) \right\}^{1/(4n)} = N.\end{aligned}$$

Because N is arbitrary, we obtain

$$\lim_{n\to\infty} \|D^n g\|^{1/(2n)}_{L_2([0,1],t^{-1}dt)} = \infty.$$

□

THEOREM 2. (Paley-Wiener-type Theorem) *A function $g(t)$ is the Φ-transform* (39) *of a function $G(\tau) \in L_2(R^+, d\xi)$ with a compact support if and only if $g(t)$ satisfies conditions* i)-iii) *of* Lemma 1 *and*

$$\lim_{n\to\infty} \|D^n g\|^{1/(2n)}_{L_2([0,1],t^{-1}dt)} < \infty. \tag{49}$$

Proof. Let $G(\tau) \in L_2(R^+, d\xi)$ have a compact support. Then $\tau^n G(\tau) \in L_2(R^+, d\xi)$ for all $n = 0, 1, 2, \ldots$ Consequently, by Lemma 1, $g(t)$ satisfies conditions i)-iii), and by Lemma 2

$$\lim_{n\to\infty} \|D^n g\|^{1/(2n)}_{L_2([0,1],t^{-1}dt)} = \sup_{\tau \in \text{supp } G} \tau < \infty.$$

Conversely, let $g(t)$ satisfy conditions i)-iii) of Lemma 1 and (49). By Lemma 1, $g(t)$ is the Φ-transform (39) of a function $G(\tau)$ such that $\tau^n G(\tau) \in L_2(R^+, d\xi)$ for all $n = 0, 1, 2, \ldots$ By Lemma 2 and equation (52) we have

$$\sup_{\tau \in \text{supp } G} \tau = \lim_{n\to\infty} \|D^n g\|^{1/(2n)}_{L_2([0,1],t^{-1}dt)} < \infty.$$

Hence, $G(\tau)$ has a compact support. □

4. Acknowledgement

This work was initiated during the first author's visit to the Department of Applied Mathematics, Faculty of Science, Fukuoka University, and he would like to take this opportunity to thank the Department and Faculty for hospitality and Fukuoka University and Science Research Promotion Fund, Japanese Private School Promotion Foundation, for financial support. The support of Kuwait University Research Administration under the Grant SM 187 is acknowledged.

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