J. Korean Math. Soc. **37** (2000), No. 4, pp. 565–577

REGULARITY OF NONLINEAR VECTOR VALUED VARIATIONAL INEQUALITIES

DO WAN KIM

ABSTRACT. We consider regularity questions arising in the degenerate elliptic vector valued variational inequalities

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \geq b(x, u, \nabla u)$$

with $p \in (1, \infty)$. It is a generalization of the scalar valued inequalities, i.e., the obstacle problem. We obtain the $C^{1,\alpha}_{\mathrm{loc}}$ regularity for the solution u under a controllable growth condition of $b(x, u, \nabla u)$.

1. Introduction

In this paper, we are concerned with regularity questions arising in the degenerate elliptic vector valued variational inequalities

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \geq b(x, u, \nabla u) \tag{1.1}$$

with $p \in (1, \infty)$. Here we assume u is a vector valued function of dimension N and $b : R^n \times R^N \times M^{nN} \to R^N$ satisfies a controllable growth condition

$$|b(x, u, A)| \leq c(f(x) + |u|^{p-1} + |A|^{p-1}) \tag{1.2}$$

for some c and for all $x \in R^n$, $u \in R^N$ and $A \in M^{nN}$ where f is defined later. Assume $O \subset R^N$ is a closed convex set with smooth boundary and $\Omega \subset R^n$ is a bounded domain. We also assume $u_0 \in [W^{1,p}(\Omega)]^N$ satisfies the following compatibility condition

$$u_0(\Omega) \subset O. \tag{1.3}$$

Received November 27, 1999.

2000 Mathematics Subject Classification: 35J85, 35B99, 58E35.

Key words and phrases: vector valued variational inequalities, regularity, maximum principle.

The author was supported by Sunmoon University.

We define the admissible function class K such that

$$K = \{v \in [W_0^{1,p}(\Omega)]^N + u_0 : v(\Omega) \subset O\} \tag{1.4}$$

and $u \in [W_0^{1,p}(\Omega)]^N + u_0$ is a solution to (1.1). To be more precise, u satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot (\nabla v - \nabla u)\, dx \geq \int_\Omega b(x, u, \nabla u) \cdot (v - u)\, dx \tag{1.5}$$

for all $v \in K$.

The following theorem is our main theorem in this paper.

THEOREM 1. *Suppose $f \in [L^s(\Omega)]^N$ with $s > n$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.*

We recall that when $p = 2$ with a natural growth condition on b, i.e. ,

$$|b(x, u, A)| \leq c(1 + |A|^2) \tag{1.6}$$

for some c, Hildebrandt and Widman[8] proved that $u \in C^{1,\alpha}$ under some smallness condition on $\|u\|_{L^\infty}$. Their smallness condition is optimal for everywhere regularity. Indeed using a hole filling technique, they showed that u satisfies a Morrey type growth condition. Their proof depends on the growth condition of Green's function and the linearity of the principal part of inequality. Hence it seems not applicable to (1.1). Independently Caffarelli[1] proved $C^{1,\alpha}$ regularity for systems with optimal growth condition on b. He showed a decay estimate for $|u - h|$ with appropriate h using a Harnack inequality.

For scalar valued degenerate inequalities, i.e., obstacle problems, a number of authors considered regularity questions (see Choe[2], Choe and Lewis[3], Fuchs[5], Giaquinta[7], Lieberman[10] and Lindquist[11]). Especially, one of the above authors proved $u \in C^{1,\alpha}_{\text{loc}}$ using a perturbation technique.

It is worthwhile to note that the convexity condition for O is necessary for everywhere regularity as the following counterexample shows. Suppose B_1 is a unit ball and u with $|u| \leq 1$ is a minimizer of the functional

$$\int_{B_1} |\nabla v|^2\, dx \tag{1.7}$$

with respect to $\left\{v \in u_0 + \left[W_0^{1,2}(B_1)\right]^N : \frac{1}{2} \le |v| \le 1\right\}$, where $u_0|_{\partial B_1}$ is topologically nontrivial. Then it is relatively easy to see that u has a singular point at interior of B_1 from a topological reason.

For the proof of Theorem 1, we observe a maximum principle which itself is interesting. Once we have a maximum principle, we can construct easily a comparison function v to u. Then a usual perturbations technique such as in Choe[2] can be used to show $u \in C^{\alpha}_{\mathrm{loc}}$. Finally, $C^{1,\alpha}_{\mathrm{loc}}$ regularity follows in the same way as in Choe[2].

We write c as a constant depending only on exterior data. We also use the following notations

$$B_R(x_0) = \{x \in R^n : |x - x_0| < R\}, \tag{1.8}$$

$$(v)_{B_R(x_0)} = \frac{1}{|B_R(x_0)|} \int\limits_{B_R(x_0)} v\,dx, \tag{1.9}$$

$$||g||_{q,B_R(x_0)} = \left[\int\limits_{B_R(x_0)} |g|^q\,dx\right]^{\frac{1}{q}}. \tag{1.10}$$

We drop out the generic point x_0 in various expressions and also the second one of above notations may be briefly written by v_R if there is no confusion.

2. Regularity result

When $p > n$, $C^{0,\alpha}$ regularity is immediate from the Sobolev embedding theorem. Hence we assume $p \in (1, n]$ for Hölder continuity of u. It is relatively well known that a solution v of

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \tag{2.11}$$

is Hölder continuous and satisfies a Campanato type growth condition.

LEMMA 1. *Suppose $v \in [W^{1,p}(B_R)]^N$ is a solution to (2.11), then v satisfies the following integral inequalities*

$$\int\limits_{B_\rho} |\nabla v|^p\,dx \le c\left(\frac{\rho}{R}\right)^n \int\limits_{B_R} |\nabla v|^p\,dx \tag{2.12}$$

and

$$\int_{B_\rho} |v - v_\rho|^p \, dx \le c \left(\frac{\rho}{R}\right)^{n+p} \int_{B_R} |v - v_R|^p \, dx \tag{2.13}$$

for all $\rho \in (0, R/2)$. *Consequently,* u *is* Hölder *continuous.*

In fact, one can prove Lemma 1 using a Cacciopoli type inequality and a weak maximum principle for ∇u. We refer its proof to Choe[2].

The following weak maximum principle is essential in constructing comparison functions.

LEMMA 2. *Suppose* $f : R^N \to R$ *is a* C^2 *convex function. Suppose* $v \in W^{1,p}(B_R)$ *is a solution to* (2.11), *then* $f(v)$ *is a subsolution to*

$$\operatorname{div}(|\nabla v|^{p-2} \nabla w) = 0 \tag{2.14}$$

and $f(v)$ *satisfies the following weak maximum principle*

$$\max_{B_R} f(v) \le \max_{\partial B_R} f(v). \tag{2.15}$$

Proof. Let $\phi \in C_0^\infty(B_R)$ be a nonnegative cutoff function. Then we have

(2.16)

$$\begin{aligned}
\int_{B_R} |\nabla v|^{p-2} \nabla(f(v)) \cdot \nabla \phi \, dx &= \int_{B_R} |\nabla v|^{p-2} f_{v^j}(v) v^j_{x_k} \phi_{x_k} \, dx \\
&= \int_{B_R} |\nabla v|^{p-2} v^j_{x_k} [(f_{v^j} \phi)_{x_k} - f_{v^j v^m} v^m_{x_k} \phi] \, dx \\
&= - \int_{B_R} |\nabla v|^{p-2} f_{v^j v^m} v^j_{x_k} v^m_{x_k} \phi \, dx.
\end{aligned}$$

Since f is a convex function, we conclude

$$\int_{B_R} |\nabla v|^{p-2} \nabla(f(v)) \cdot \nabla \phi \, dx \le 0 \tag{2.17}$$

and $f(v)$ is a subsolution to (2.14). The weak maximum principle follows from a standard argument by taking

$$\phi = \max\{f(v) - \max_{\partial B_R} f(v), 0\} \tag{2.18}$$

as a test function in (2.17). $\square$

From Lemma 2, we have a useful corollary.

Corollary 1. *Suppose that $v \in [w^{1,p}(B_R)]^N$ is a solution to*

$$\begin{cases} \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, \\ v = g \text{ on } \partial B_R \end{cases}$$

then we have

$$v(B_R) \subset \text{the convex hull of } g(\partial B_R).$$

Now using Corollary 1, we prove u is Hölder continuous. Indeed we show ∇u satisfies a Morrey type growth condition using a perturbation argument and Lemma 1. We recall that a similar argument has been used in scalar obstacle problems by one of the authors Choe and Lewis[3].

Theorem 2. *Suppose $B_R \subset \Omega$. Then ∇u satisfies a* Morrey *type growth condition such that*

$$\int_{B_\rho} |\nabla u|^p \, dx \le c\left[R^\varepsilon + \left(\frac{\rho}{R}\right)^n\right] \int_{B_R} |\nabla u|^p \, dx + cR^{n+\frac{p}{p-1}(1-\frac{n}{s})-\varepsilon} \tag{2.19}$$

for some $\varepsilon > 0$ and for all $\rho \in (0, R/2)$. Consequently, if $s > \dfrac{n}{p}$, then we have $u \in C^\alpha_{\mathrm{loc}}$ by Morrey*'s embedding theorem.*

Proof. Since we are assuming ∂O is smooth and O is convex, we can always find a smooth convex function $f : R^N \to R$ such that

$$\{y \in R^N : f(y) \le 1\} = O$$

and

$$\partial O = \{y \in R^N : f(y) = 1\}.$$

Suppose $v \in [W_0^{1,p}(B_R)]^N + u$ is the solution to

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \tag{2.20}$$

in B_R. Then by the maximum principle in Lemma 2, we have

$$\max_{B_R} f(v) \le \max_{\partial B_R} f(v) = \max_{\partial B_R} f(u)$$

and since $u(\partial B_R) \subset O$ and O is convex, we have, from Corollary 1,

$$v(B_R) \subset O.$$

Hence v is an admissible competing function to u in B_R. Moreover, from Lemma 1, we have

$$\int_{B_\rho} |\nabla v|^p \, dx \le c \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla v|^p \, dx \quad \text{for all } \rho \in \left(0, \frac{R}{2}\right) \tag{2.21}$$

for some c. As usual, we also have

$$\begin{aligned} \int_{B_\rho} |\nabla u|^p \, dx &\le c \int_{B_\rho} |\nabla v|^p \, dx + c \int_{B_\rho} |\nabla u - \nabla v|^p \, dx \\ &\le c \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla u|^p \, dx + c \int_{B_R} |\nabla u - \nabla v|^p \, dx \end{aligned} \tag{2.22}$$

for some c, since we know the following fact :

$$\int_{B_R} |\nabla v|^p \, dx \le \int_{B_R} |\nabla u|^p \, dx.$$

Now suppose $p \in [2, \infty)$. Since v is a solution to (2.20) and u is a solution to the vector valued variational inequalities (1.1), we have

(2.23)
$$\begin{aligned} \int_{B_R} |\nabla u - \nabla v|^p \, dx &\le c \int_{B_R} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot (\nabla u - \nabla v) \, dx \\ &= c \int_{B_R} |\nabla u|^{p-2} \nabla u \cdot (\nabla u - \nabla v) \, dx \\ &\le c \int_{B_R} b(x, u, \nabla u) \cdot (u - v) \, dx \\ &\le c \int_{B_R} (f(x) + |\nabla u|^{p-1}) |u - v| \, dx \end{aligned}$$

for some c. We know already u and v are bounded. Using Hölder's inequality and Poincaré's inequality, we can estimate

(2.24)

$$\int_{B_R} f(x)|u-v|\,dx \le \left[\int_{B_R} |f|^{\frac{p}{p-1}}\,dx\right]^{\frac{p-1}{p}} \left[\int_{B_R} |u-v|^p\,dx\right]^{\frac{1}{p}}$$

$$\le R\left[\int_{B_R} |f|^{\frac{p}{p-1}}\,dx\right]^{\frac{p-1}{p}} \left[\int_{B_R} |\nabla u-\nabla v|^p\,dx\right]^{\frac{1}{p}}$$

$$\le \varepsilon \int_{B_R} |\nabla u-\nabla v|^p\,dx + c(\varepsilon)R^{n+\frac{p}{p-1}(1-\frac{n}{s})}\left[\int_{B} |f|^s\,dx\right]^{\frac{p}{s(p-1)}}$$

Again using Hölder's inequality and Poincaré's inequality, we obtain the estimates

(2.25)

$$\int_{B_R} |\nabla u|^{p-1}|u-v|\,dx \le cR\left[\int_{B_R} |\nabla u|^p\,dx\right]^{\frac{p-1}{p}} \left[\int_{B_R} |\nabla u-\nabla v|^p\,dx\right]^{\frac{1}{p}}$$

$$\le \varepsilon \int_{B_R} |\nabla u-\nabla v|^p\,dx + c(\varepsilon)cR^{\frac{p}{p-1}}\int_{B_R} |\nabla u|^p\,dx.$$

Combining (2.23), (2.24) and (2.25), we have

$$(2.26)\qquad \int_{B_R} |\nabla u-\nabla v|^p\,dx \le cR^{\frac{p}{p-1}}\int_{B_R} |\nabla u|^p\,dx + cR^{n+\frac{p}{p-1}(1-\frac{n}{s})}\|f\|_{L^s}^{\frac{p}{p-1}}$$

for some c. Combining (2.22) and (2.26), we prove a Morrey type growth condition (2.19) when $p \in [2,\infty)$. Using a usual iteration lemma, we prove

$$\int_{B_\rho} |\nabla u|^p\,dx \le c\rho^{n-p+\delta}$$

for some $\delta > 0$ and consequently, u is Hölder continuous.

Now suppose $p \in (1,2)$. If we use Hölder inequality, we have

(2.27)

$$\begin{aligned}\int_{B_R} |\nabla u - \nabla v|^p\,dx &= \int_{B_R} \big(|\nabla u| + |\nabla v|\big)^{\frac{p(2-p)}{2}} \left[\big(|\nabla u| + |\nabla v|\big)^{\frac{p(p-2)}{2}} |\nabla u - \nabla v|^p\right] dx \\ &\le \left[\int_{B_R} \big(|\nabla u| + |\nabla v|\big)^p\,dx\right]^{\frac{2-p}{2}} \left[\int_{B_R} \big(|\nabla u| + |\nabla v|\big)^{p-2} |\nabla u - \nabla v|^2\,dx\right]^{\frac{p}{2}}.\end{aligned}$$

Using the property of monotonicity of the operator, we estimate

$$\begin{aligned}\int_{B_R} (|\nabla u| + |\nabla v|)^{p-2} |\nabla u - \nabla v|^2\,dx \\ \le c \int_{B_R} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u - \nabla v)\,dx.\end{aligned} \tag{2.28}$$

Following the same computation as in the previous case $p \in [2,\infty)$, we have, from Hölder's inequality and Poincaré's inequality, the inequalities

$$\begin{aligned}\int_{B_R} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u - \nabla v)\,dx \\ \le c \int_{B_R} (f(x) + |\nabla u|^{p-1})|u - v|\,dx \\ \le cR\,\|f\|_{\frac{p}{p-1}} \left[\int_{B_R} |\nabla u - \nabla v|^p\,dx\right]^{\frac{1}{p}} \\ + cR \left[\int_{B_R} |\nabla u|^p\,dx\right]^{\frac{p-1}{p}} \left[\int_{B_R} |\nabla u - \nabla v|^p\,dx\right]^{\frac{1}{p}}.\end{aligned} \tag{2.29}$$

Using the fact that

$$\int_{B_R} |\nabla v|^p\,dx \le \int_{B_R} |\nabla u|^p\,dx, \tag{2.30}$$

we have

$$\begin{aligned}\int_{B_R} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u - \nabla v)\,dx \\ \le cR\,\|f\|_{\frac{p}{p-1}}\left[\int_{B_R} |\nabla u|^p\,dx\right]^{\frac{1}{p}} + cR\int_{B_R} |\nabla u|^p\,dx.\end{aligned} \tag{2.31}$$

Combining (2.27) and (2.31), we have

(2.32)

$$\int_{B_R} |\nabla u - \nabla v|^p\,dx \le cR^{\frac{p}{2}}\,\|f\|_{\frac{p}{p-1}}^{\frac{p}{2}}\left[\int_{B_R} |\nabla u|^p\,dx\right]^{\frac{3-p}{2}} + cR^{\frac{p}{2}}\int_{B_R} |\nabla u|^p\,dx.$$

Using the assumption $s > \dfrac{n}{p}$, we have

(2.33)

$$R^{\frac{p}{2}}\,\|f\|_{\frac{p}{p-1}}^{\frac{p}{2}}\left[\int_{B_R} |\nabla u|^p\,dx\right]^{\frac{3-p}{2}} \le R^{\varepsilon}\int_{B_R} |\nabla u|^p\,dx + c\,R^{n+\frac{p}{p-1}(1-\frac{n}{s})-\varepsilon}\,\|f\|_{s}^{\frac{p}{p-1}}$$

for some small $\varepsilon > 0$. Therefore, combining (2.22), (2.32) and (2.33), we prove that

(2.34)

$$\int_{B_\rho} |\nabla u|^p\,dx \le c\left(R^{\varepsilon} + \left(\frac{\rho}{R}\right)^{n}\right)\int_{B_R} |\nabla u|^p\,dx + cR^{n+\frac{p}{p-1}(1-\frac{n}{s})-\varepsilon}\,\|f\|_{s}^{\frac{p}{p-1}}$$

for some small $\varepsilon > 0$ and complete the proof. Iterating (2.34), we also prove u is Hölder continuous. □

The following remark is useful for the proof of $C^{1,\alpha}$ regularity.

REMARK 1. From the iteration if $s \geq n$, we have

$$\int_{B_\rho} |\nabla u|^p \, dx \leq c\rho^{n-\varepsilon} \tag{2.35}$$

for all $\varepsilon > 0$.

Now we recall that a solution v for p-Laplacian system satisfies a Campanato type growth condition. Indeed Lieberman[10] proved the lemma for scalar cases and, DiBenedetto and Manfredi[4] proved for systems. The proof depends on a careful iteration on $\sup_{B_R} |\nabla v|$.

LEMMA 3. *Suppose that* $v \in [W^{1,p}(B_R)]^N$ *is a solution to*

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0,$$

then ∇v *satisfies a Campanato type growth condition*

$$\int_{B_\rho} |\nabla v - (\nabla v)_\rho|^p \, dx \leq c\left(\frac{\rho}{R}\right)^{n+\delta} \int_{B_R} |\nabla v - (\nabla v)_R|^p \, dx \tag{2.36}$$

for some $\delta > 0$.

We use a perturbation argument to prove ∇u is Hölder continuous.

THEOREM 3. *Suppose* $B_R \subset \Omega$. *Then* ∇u *satisfies* a Campanato *type growth condition such that*

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^p \, dx \leq c\left(\frac{\rho}{R}\right)^{n+\delta} \int_{B_R} |\nabla u - (\nabla u)_R|^p \, dx + R^{n+\delta'} \tag{2.37}$$

for some $\delta > 0$ *and* $\delta' > 0$, *and for all* $\rho \in (0, R/2)$. *Consequently, if* $s > n$, *then* $\nabla u \in C^\alpha_{\text{loc}}$.

Proof. From the proof of Theorem 2, we note that the solution $v \in [W_0^{1,p}(B_R)]^N + u$ to

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0$$

is an admissible competing function. Hence, from Lemma 3, we have

$$\int_{B_\rho} |\nabla v - (\nabla v)_\rho|^p \, dx \leq c\left(\frac{\rho}{R}\right)^{n+\delta} \int_{B_R} |\nabla v - (\nabla v)_R|^p \, dx \tag{2.38}$$

for some $\delta > 0$. As usual, we also have

(2.39)
$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^p \, dx \le c \int_{B_\rho} |\nabla v - (\nabla v)_\rho|^p \, dx + c \int_{B_R} |\nabla u - \nabla v|^p \, dx$$
$$\le c\left(\frac{\rho}{R}\right)^{n+\delta} \int_{B_R} |\nabla u - (\nabla u)_R|^p \, dx + c \int_{B_R} |\nabla u - \nabla v|^p \, dx$$

for some c.

Suppose $p \in [2, \infty)$. In this case, we note from (2.23)

$$\int_{B_R} |\nabla u - \nabla v|^p \, dx \le c \int_{B_R} f(x)|u - v| \, dx + \int_{B_R} |\nabla u|^{p-1}|u - v| \, dx. \tag{2.40}$$

Again using Hölder's inequality, Poincaré's inequality and (2.30), we have

$$\begin{aligned} \int_{B_R} f(x)|u - v| \, dx &\le cR\|f\|_s \left[\int_{B_R} |\nabla u - \nabla v|^{\frac{s}{s-1}} \, dx\right]^{\frac{s-1}{s}} \\ &\le c\|f\|_s \, R^{1+n(\frac{s-1}{s}-\frac{1}{p})} \left[\int_{B_R} |\nabla u - \nabla v|^p \, dx\right]^{\frac{1}{p}} \\ &\le cR^{1+n(\frac{p-1}{p}-\frac{1}{s})}\|f\|_s \left[\int_{B_R} |\nabla u|^p \, dx\right]^{\frac{1}{p}}. \end{aligned} \tag{2.41}$$

From Remark 1, we have

$$\int_{B_R} f(x)|u - v| \, dx \le cR^{n+1-\frac{n}{s}-\frac{\varepsilon}{p}} \tag{2.42}$$

for some small $\varepsilon > 0$. On the other hand, using Hölder's inequality and Poincare's inequality, we have

$$\int_{B_R} |\nabla u|^{p-1}|u - v| \, dx \le R \int_{B_R} |\nabla u|^p \, dx \le cR^{n+1-\varepsilon}, \tag{2.43}$$

where we used the fact that

$$\int_{B_R} |\nabla v|^p \, dx \le \int_{B_R} |\nabla u|^p \, dx. \tag{2.44}$$

Combining (2.38) through (2.43), we complete the proof when $p \in [2, \infty)$.

Now assume $p \in (1, 2)$. As in the proof of Theorem 2, we have

$$\begin{aligned}
&\int_{B_R} |\nabla u - \nabla v|^p \, dx \\
&\le c \left[\int_{B_R} |\nabla u|^p \, dx \right]^{\frac{2-p}{2}} \left[\int_{B_R} f(x)|u-v| + |\nabla u|^{p-1}|u-v| \, dx \right]^{\frac{p}{2}} \qquad (2.45)\\
&\le c \left[\int_{B_R} |\nabla u|^p \, dx \right]^{\frac{2-p}{2}} \left[R^{1+n(\frac{p-1}{p}-\frac{1}{s})} \|f\|_s \left\{ \int_{B_R} |\nabla u|^p \, dx \right\}^{\frac{1}{p}} \right. \\
&\qquad \left. + R \int_{B_R} |\nabla u|^p \, dx \right]^{\frac{p}{2}}.
\end{aligned}$$

Again, from Remark 1, we have

$$\int_{B_R} |\nabla u - \nabla v|^p \, dx \le c R^{n+\delta'} \tag{2.46}$$

for some $\delta' > 0$ and complete the proof. We prove $C^{1,\alpha}_{\mathrm{loc}}$ regularity from a standard iteration on (2.37). $\square$

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Department of Mathematics
Sunmoon University
Kalsan-ri, Tangjeong-myeon, Asan-si
Chungnam 337-840, Korea
E-mail: dwkim@omega.sunmoon.ac.kr