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STABILITY AND CONSTRAINED CONTROLLABILITY OF LINEAR CONTROL SYSTEMS IN BANACH SPACES

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ABSTRACT. For linear time-varying control systems with constrained control described by both differential and discrete-time equations in Banach spaces we give necessary and sufficient conditions for exact global null-controllability. We then show that for such systems, complete stabilizability implies exact null-controllability.

1. Introduction

Consider a linear time-varying control system described by differential equations of the form

$$(1) \qquad \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0, \\ x(t) \in X, \; u(t) \in U, \end{cases}$$

where X and U are infinite-dimensional Banach spaces; $A(.)$ and $B(.)$ are linear operators. From the classical control theory, it is defined that system (1) is exactly null-controllable if every point $x \in X$ can be controllable to the origin by some control $u \in U$ in finite time; exponentially stabilizable (in Lyapunov sense) if there exists an operator function $K(t)(.) : X \to U, t \geq 0$, such that all solutions $x(t, x_0)$ of the closed-loop control system $\dot{x} = [A(t) + B(t)K(t)]x$, with the initial condition $x(0) = x_0$, satisfy

$$\|x(t, x_0)\| \leq Me^{-\alpha t}\|x_0\|, \quad \forall t \geq 0,$$

for some $M > 0$ and $\alpha > 0$.

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The problems of controllability and stabilizability of control system (1) or of its discrete analog were studied widely by many researchers in control systems theory(see, e.g. [2, 5, 9-13, 16, 19, 22]) and references therein. In particular, the relationship between controllability and stabilizability was presented in [1] for time-invariant control systems, where X, U are finite-dimensional and it was shown that the exact null-controllability implies exponential stabilizability. It is obvious that all exactly null-controllable systems are exponentially stabilizable(see [4, 23]), however the exponentially stabilizable system is, in general, not exactly null-controllable. If the time-invariant system is completely stabilizable in a sense of Wonham [20], i.e., for an arbitrary $\alpha > 0$ there is a matrix K such that the matrix $A + BK$ is exponentially stable with the stability exponent α, then the system is exactly null-controllable. A natural question is: To what extent does complete stabilizability imply exact null-controllability for infinite-dimensional control systems? In the infinite-dimensional control theory [3] characterizations of controllability and stabilizability are complicated and therefore their relationships are much more complicated and require more sophisticated methods. The difficulties increase to the same extent as passing from time-invariant systems to time-varying systems as well as from unconstrained control systems to systems with constrained controls. For time-invariant control systems without constrained controls in Hilbert spaces, in a stronger type of complete stabilizability, it was shown in Megan [7] and then Zabczyk [21, 23] that complete stabilizability implies exact null-controllability.

In this paper, we first give necessary and sufficient conditions for exact global null-controllability of continuous and discrete-time time-varying systems, where the control $u(k)$ is restricted to lie in some subset Ω in a Banach space U. Secondly, and more importantly, as an extension of [7, 21], we show that for such control systems complete stabilizability implies exact global null-controllability. Unlike usual constrained controllability conditions expressed by the spectrum of the adjoint operator A^*(see, e.g. [10, 18]), our controllability conditions are described in terms of Schmitendorf and Barmish-type conditions [17], which can be applicable to the stability analysis. To our knowledge, the main results of this paper are also new for the case of finite-dimensional control systems. Some constrained controllability results, Corollaries

3.1 and 3.2, are well-known for the finite-dimensional control systems, but the proofs are different and our approach can be extended to the study of the controllability and stability of some other classes of both continuous-time and discrete-time systems with constrained controls in Banach spaces.

The paper is organized as follows. In Section 2, we review main notations, definitions and give some auxiliary lemmas needed later. Global null-controllability conditions for both continuous-time and discrete-time control systems with constrained controls in Banach spaces are given in Section 3. In Section 4, we show that complete stabilizability implies exact global null-controllability. Discrete analog of the result is also given in this section.

2. Notations, definitions and preliminaries

Let X and U be infinite-dimensional Banach spaces. X^* denotes the topological dual space of X and $< y^*, x >$ denotes the value of $y^* \in X^*$ at $x \in X$. The adjoint and the inverse operator of an operator A are denoted by A^* and A^{-1}, respectively. I denotes the identity operator.

In this paper, we use the following standard notations from [3, 6].

- $R-$ the set of all real numbers, Z^+- the set of all non-negative integers,

- $B_1^* = \{x^* \in X^* : \|x^*\|_{X^*} = 1\}$, $X_0^* = X^* \setminus \{0\}$,

- $\mathcal{L}(X,Y)-$ the Banach space of all linear bounded operators mapping X into Y,

- $L_\infty([0,t],U)-$ the Banach space of all $U-$valued strongly measurable functions $u(.) \in U$ such that $\|u\|_U$ is essentially bounded on $[0,t]$,

- $H_\Omega(y^*)-$ the support function of a set Ω defined by

$$H_\Omega(y^*) = \sup_{u\in\Omega} < y^*, u >,$$

- M^0- the polar set of a set M at $0 \in M$ defined by

$$M^0 = \{y^* \in X^* :< y^*, x > \leq 1, \quad \forall x \in M\}.$$

We shall consider control system (1), where $x(t) \in X,\ u(t) \in \Omega \subset U$; Ω is a given nonempty subset in U. Throughout this paper, we assume that $A(t)$ and $B(t)$ are strongly measurable and locally integrable in $t \geq 0$ and for every $t \geq 0,\ A(t) \in \mathcal{L}(X,X),\ B(t) \in \mathcal{L}(U,X)$.

For any $T > 0$, the set of admissible controls on $[0, T]$ for system (1) is defined as

$$\mathcal{U}_T = \{u(.) \in L_\infty([0,T], U) : u(t) \in \Omega \quad \text{a.e. on } [0,T]\}.$$

Thus, as in [6], for each $u(t) \in \mathcal{U}_T$ and $x_0 \in X$, the unique solution of (1) initiated at x_0 is given by

$$x(t, x_0, u) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)B(s)u(s)\,ds,$$

where $\Phi(t,s)$ is a nonsingular evolution operator of the linear system $\dot{x} = A(t)x$ satisfying

$$\Phi(s,s) = I, \quad \Phi^{-1}(t,s) = \Phi(s,t),$$

$$\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s), \quad t \geq \tau \geq s \geq 0.$$

DEFINITION 2.1. The time-varying system (1) is completely stabilizable if for every $\alpha > 0$, there exist a linear bounded operator function $K(t)(.) : X \to \Omega, t \geq 0$ and a number $M > 0$ such that the solution $x(t, x_0)$ of the system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t), \quad x(0) = x_0, \tag{1.1}$$

satisfies the condition

$$\|x(t, x_0)\| \leq Me^{-\alpha t}\|x_0\|, \quad \forall\, t \geq 0.$$

In other words, if $\Phi_K(t,s)$ is the evolution operator of system (1.1), then the complete stabilizability is equivalent to

$$\forall\, \alpha > 0, \exists\, K(t)(.) : X \to \Omega,\ \exists M > 0 : \|\Phi_K(t,0)\| \leq Me^{-\alpha t},\ \forall\, t \geq 0.$$

Note that if the operator K and number M do not depend on α, then the complete stabilizability implies exponential stabilizability in usual Lyapunov sense (see [1, 23]).

We now give analogous definition of the stabilizability for discrete-time control systems of the form

$$(2)\qquad \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), & k \in Z^+, \\ x(k) \in X, u(k) \in \Omega \subset U, \end{cases}$$

where $A(k) \in \mathcal{L}(X,X)$ and $B(k) \in \mathcal{L}(U,X)$. It is obvious that for every $u \in \mathcal{U}_k = \{u^k = (u(0), u(1), \ldots\ldots, u(k-1) \in \Omega^k\}$ and $x_0 \in X$, the discrete-time control system (2) always has a solution $x(k, x_0, u)$ with $x(0) = x_0$ given by

$$x(k, x_0, u) = \Psi(k, 0)x_0 + \sum_{i=0}^{k-1} \Psi(k, i+1)B(i)u(i),$$

where $\Psi(k,i)$ is the transition operator defined by

$$\Psi(k,i) = A(k-1)A(k-2)\ldots A(i), \ k > i, \quad \Psi(k,k) = I.$$

DEFINITION 2.2. The discrete-time system (2) is completely stabilizable if for every $q \in (0,1)$, there exist a linear bounded operator function $Q(k)(.) : X \to \Omega, k \in Z^+$ and a number $M > 0$ such that the solution $x(k, x_0)$ of the system

$$(2.1)\qquad x(k+1) = [A(k) + B(k)Q(k)]x(k), \quad x(0) = x_0, \quad k \in Z^+,$$

satisfies the condition

$$\|x(k, x_0)\| \leq Mq^k \|x_0\|, \quad \forall\ k \in Z^+,$$

or equivalently,

$$\|\Psi_Q(k, 0)\| \leq Mq^k, \quad \forall\ k \in Z^+,$$

where $\Psi_Q(k,i)$ is the transition operator of the system (2.1).

To give controllability definition, let us define the reachable set of the control system (1) from x_0 in time $T > 0$ by

$$R_T(x_0) = \{x \in X : \exists\ u(t) \in \mathcal{U}_T, \ x(T, x_0, u) = x\},$$

and by

$$R(x_0) = \bigcup_{T>0} R_T(x_0),$$

the reachable set of the system in finite time.

DEFINITION 2.3. A point $x_0 \in X$ is said to be null-controllable by the system (1) in time $T > 0$, if $0 \in R_T(x_0)$. The system (1) is globally null-controllable if every point $x_0 \in X$ is null-controllable in some finite time $T > 0$, i.e., $0 \in R(x_0)$ for all $x_0 \in X$. Similar definition is applied for global null-controllability of discrete-time system (2).

We need the following lemma for later use.

LEMMA 2.1. *Assume that Ω is a convex, compact subset in U. Then*

$$\forall\, y^* \in X^*: \quad \sup_{u \in \mathcal{U}_T} < y^*, L_T u >= \int_0^T H_\Omega\Big(B^*(s)\Phi^*(T,s)y^*\Big)\, ds,$$

where $L_T u = \int_0^T \Phi(T,s)B(s)u(s)\, ds$.

Proof. We shall prove the lemma based on the same arguments used in the proof of Theorem 5.3.8 in [1]. By the assumption that Ω is a convex compact set, it is easy to see that the admissible control set $\mathcal{U}_T$ is convex, weakly compact in $L_\infty([0,T],U)$ (e.g., [1, 3]). For every fixed y^*, the linear function $u \longmapsto F(y^*,u) =< y^*, L_T u >$ is weakly continuous and it attains its supremum on the convex weakly compact set $\mathcal{U}_T$ at some $u^*(t) \in \mathcal{U}_T$:

$$\sup_{u \in \mathcal{U}_T} F(y^*,u) = F(y^*,u^*).$$

Let $t \in (0,T) = I_T$ and for every $v \in \Omega$, we define

$$I_T^n = \{s \in R : t - \frac{1}{n} \le s \le t + \frac{1}{n}\} \bigcap I_T,$$

and

$$\bar{u}(t) = \begin{cases} u^*(t) & \text{if } t \in I_T \setminus I_T^n, \\ v & \text{if } t \in I_T^n. \end{cases}$$

For the admissible control $\bar{u}(t) \in \mathcal{U}_T$, we have

$$F(y^*,u^*) \ge F(y^*,\bar{u}),$$

and consequently,

$$\int_{I_T^n} < B^*(s)\Phi^*(T,s)y^*, u^*(s) > ds \ge \int_{I_T^n} < B^*(s)\Phi^*(T,s)y^*, v > ds.$$

Denoting $\mu(I_T^n)$ the Lebesgue length measure of I_T^n, we have

$$\frac{1}{\mu(I_T^n)}\int_{I_T^n} < B^*(s)\Phi^*(T,s)y^*, u^*(s) > ds \geq \frac{1}{\mu(I_T^n)}\int_{I_T^n} < B^*(s)\Phi^*(T,s)y^*, v > ds.$$

Letting $n \to \infty$, we get, for all $v \in \Omega$, the estimate

$$< B^*(t)\Phi^*(T,t)y^*, u^*(t) > \geq < B^*(t)\Phi^*(T,t)y^*, v >,$$

and hence

$$< B^*(t)\Phi^*(T,t)y^*, u^*(t) > \geq H_\Omega(B^*(t)\Phi^*(T,t)y^*).$$

Since $u^*(t) \in \Omega$, for every $t \in (0,T)$, we also get

$$< B^*(t)\Phi^*(T,t)y^*, u^*(t) > \leq H_\Omega(B^*(t)\Phi^*(T,t)y^*).$$

Therefore

$$< B^*(t)\Phi^*(T,t)y^*,\ u^*(t) > = H_\Omega(B^*(t)\Phi^*(T,t)y^*). \tag{3}$$

Integrating both sides of the equation (3) over $[0,T]$, we obtain

$$F(y^*, u^*) = \int_0^T H_\Omega(B^*(t)\Phi^*(T,t)y^*)\, dt$$

which proves the lemma. □

We first give a necessary and sufficient condition for the controllability of a point $x_0 \in X$ to an arbitrary convex set in a fixed time by the system (1). For this, we consider the following operator equation

$$x = \Phi x_0 + \mathcal{T}u, \quad u \in \mathcal{U} \subset U, \tag{4}$$

where $\Phi \in \mathcal{L}(X,X), \mathcal{T} \in \mathcal{L}(U,X), x_0 \in X$ is a fixed element. As in [8], the system (4) is controllable with respect to $(x_0, \mathcal{U}, M)$ if there is a $u \in \mathcal{U}$ such that $\Phi x_0 + \mathcal{T}u \in M$.

LEMMA 2.2. ([8]) *Assume that $\mathcal{U}$ is a convex, weakly compact subset in U. The system (4) is controllable with respect $(x_0, \mathcal{U}, M)$ if and only if*

$$\forall y^* \in M^0 : \sup_{u \in \mathcal{U}} < y^*, \mathcal{T}u > \geq < y^*, \Phi x_0 > -1.$$

REMARK 2.1. Note that if $M = \{0\}$, then $M^0 = X^*$, and if we take

$$\Phi = \Phi(T, 0), \quad \mathcal{T} = L_T, \quad \mathcal{U} = \mathcal{U}_T,$$

then from Lemmas 2.1 and 2.2 it follows that a point x_0 is null-controllable at time $T > 0$ by system (1), where Ω is a convex, compact subset in U, if and only if

$$y^* \in X^* : \quad J(T, x_0, y^*) \geq 0,$$

where

$$J(T, x_0, y^*) = \int_0^T H_\Omega(B^*(s)\Phi^*(T, s)y^*)\, ds - < x_0, \Phi^*(T, 0)y^* > +1.$$

The similar result is obtained for discrete-time control systems (2), where we define

$$G(k, x_0, y^*) = \sum_{i=0}^{k-1} H_\Omega(B^*(i)\Psi^*(k, i+1)y^*) + < x_0, \Psi^*(k, 0)y^* >, k \in Z^+.$$

LEMMA 2.3. *Assume that Ω is a convex, compact subset in U. A point $x_0 \in X$ is null-controllable at step $K > 0$ by the discrete-time control system (2) if and only if*

$$\forall y^* \in X^* : \quad G(K, x_0, y^*) \geq 0.$$

Proof. Assume that x_0 is null-controllable at step $K > 0$, i.e., $0 \in R_K(x_0)$, where

$$R_K(x_0) = \Big\{ \Psi(K, 0)x_0 + \sum_{i=0}^{K-1} \Psi(K, i+1)B(i)u(i) : u(i) \in \Omega, i = 0, 1, ..., K-1 \Big\}.$$

Therefore, for all $y^* \in X^*$, we get

$$\sup_{x \in R_K(x_0)} < y^*, x > \geq < y^*, 0 > = 0.$$

By the definition of $R_K(x_0)$, we then have $G(K, x_0, y^*) \geq 0$ for all $y^* \in X^*$. Conversely, we assume that $G(K, x_0, y^*) \geq 0$ for all $y^* \in X^*$. By the assumption that Ω is a convex compact set, it is easy to see that the reachable set $R_K(x_0)$ is also convex and compact. If a point x_0 is not null-controllable at step $K > 0$, then $0 \notin R_K(x_0)$. By the separation theorem of convex sets in Banach space [14], there exist $y_0^* \in X^*$ and some $\epsilon > 0$ such that

$$< y_0^*, x > < -\epsilon, \quad \forall x \in R_K(x_0).$$

Therefore, we have

$$\sup_{x \in R_K(x_0)} < y_0^*, x > < 0,$$

which contradicts the assumption and then completes the proof. □

3. Global null-controllability

Consider the time-varying control system (1), where, throughout this section we assume that Ω is a convex compact subset in U. We are now in position to prove the following theorem of the global null-controllability of system (1).

THEOREM 3.1. *If*

(5)
$$\forall\, c > 0, \exists\, T > 0 : \int_0^T H_\Omega(B^*(s)\Phi^*(T,s)y^*)\, ds \geq c\|\Phi^*(T,0)y^*\|, \ \forall y^* \in X^*,$$

then the system (1) *is globally null-controllable. Conversely, if the system* (1) *is globally null-controllable, then*

$$\forall y^* \in X_0^* : \int_0^\infty H_\Omega(B^*(s)\Phi^*(0,s)y^*)\, ds = +\infty. \tag{6}$$

Proof. Assume that the condition (5) holds but the system (1) is not globally null-controllable. Then we can find a point x_0 which can not be null-controllable in any time $t > 0$. By Lemma 2.2 and Remark 2.1, for every $t > 0$, there is $y_t^* \in X^*$ such that

$$J(t, x_0, y_t^*) = 1 - < x_0, \Phi^*(t,0)y_t^* > + \int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y_t^*)\,ds < 0.$$

Therefore

$$\int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y_t^*)\,ds << x_0, \Phi^*(t,0)y_t^* >\leq |< x_0, \Phi^*(t,0)y_t^* >|$$

$$\leq \|x_0\|\|\Phi^*(t,0)y_t^*\|,$$

which contradicts the condition (5).

Conversely, we assume that the system (1) is globally null-controllable, but the condition (6) is not satisfied, i.e.,

$$\text{(7)} \qquad \exists a > 0, \exists y_0^* \in X_0^* : \int_0^\infty H_\Omega(B^*(s)\Phi^*(0,s)y_0^*)\,ds \leq a < +\infty.$$

For every $t \geq 0$, we define

$$y_t^* = \Phi^{*^{-1}}(t,0)y_0^*.$$

Since $\Phi^*(t,0)$ is nonsingular, we get $y_t^* \in X_0^*$. Let us consider for any $x_0 \in X$, the following relation

$$J(t, x_0, y_t^*) = 1 - < x_0, \Phi^*(t,0)y_t^* > + \int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y_t^*)\,ds.$$

Since

$$\Phi^*(t,0)y_t^* = y_0^*, \quad \Phi^*(t,s) = \Phi^*(0,s)\Phi^*(t,0),$$

we have

$$J(t, x_0, y_t^*) = 1 - < x_0, y_0^* > + \int_0^t H_\Omega(B^*(s)\Phi^*(0,s)y_0^*)\,ds.$$

From (7), it follows that

$$J(t, x_0, y_t^*) \leq 1 - < x_0, y_0^* > + \quad a.$$

Therefore, for any fixed $\epsilon > 0$, we take

$$x_0 = -(\epsilon + a + 1)x_1,$$

where $x_1 \in X$ is chosen by the Hahn-Banach theorem [14], due to $y_0^* \neq 0$, such that $< y_0^*, x_1 >= 1$. We then obtain

$$J(t, x_0, y_t^*) \leq -\epsilon < 0,$$

which implies, by Lemma 2.2, the point x_0 can not be null-controllable in any time. This contradicts the global null-controllability of the system. □

REMARK 3.1. Note that if X, U are finite-dimensional, we can check that the conditions (5) and (6) are equivalent and then for finite-dimensional systems we derive the following global null-controllability criterion.

COROLLARY 3.1. [1] *Let dim* $X < +\infty$, *dim* $U < +\infty$. *The system* (1) *is globally null-controllable if and only if*

$$(8) \qquad \forall y^* \in B_1^* : \quad \int_0^\infty H_\Omega\Big(B^*(s)\Phi^*(0,s)y^*\Big)\, ds = +\infty.$$

It is worth to note that the condition (8) can be formulated in terms of the solution $\psi^*(t)$ of the adjoint equation

$$(9) \qquad \dot{\psi}^*(t) = -A^*(t)\psi^*(t), \quad t \geq 0.$$

Indeed, since ${\Phi^*}^{-1}(t, 0)$ is the evolution operator of the adjoint system (9) and

$$\Phi^*(0, s)\psi^*(0) = \psi^*(s),$$

we can take the solution of (9) in the form $\psi^*(t) = {\Phi^*}^{-1}(t, 0)y_0^*$ with the initial condition $\psi^*(0) = y_0^* \neq 0$. We then obtain the null-controllability criterion in terms of nontrivial solutions of the adjoint system (9) presented in [1, 17] as follows.

COROLLARY 3.2. [17] *Assume that dim $X < +\infty$ and dim $U < +\infty$. The system* (1) *is globally null-controllable if and only if*

$$\int_0^\infty H_\Omega\Big(B^*(s)\psi^*(s)\Big)\,ds = +\infty,$$

for all nontrivial solutions $\psi^*(t)$ of system (9).

Based on Lemma 2.3 and using the same arguments that used in the proof of Theorem 3.1, the following discrete analog of Theorem 3.1, which gives a sufficient condition for the global null-controllability of discrete-time control system (2), is consequently derived.

THEOREM 3.2. *The discrete-time control system* (2) *is globally null controllable if*

$$\forall\, c > 0, \exists\, K > 0, \forall y^* \in X^* : \quad \sum_{i=0}^{K-1} H_\Omega\Big(B^*(i)\Psi^*(K, i+1)y^*\Big) \geq c\|\Psi^*(K,0)y^*\|.$$

4. Complete stabilizability implies null-controllability

Consider system (1), where Ω is a nonempty subset in U. In [4] it was shown that if the system (1), where A, B are constants and $\Omega = U; X, U$ are Hilbert spaces, is null-controllable then the system is exponentially stabilizable and the converse is, in general, not true. In a stronger type of complete stabilizability, i.e., if for any $\alpha > 0$ there exist a linear bounded operator $K : X \to U$ and a number $M > 0$ such that the generator $S_K(t)$ of the system $\dot{x} = (A + BK)x$ satisfies the condition

$$\|S_K(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0,$$

it was proved in [7, 23] that the complete stabilizability implies exact null controllability. In this section we extend these results to the time-varying systems (1), (2) with constrained controls in Banach space U.

Let us first remark that if A is a constant operator, the evolution operator $\Phi(t, 0) = S(t)$ is, as in [14, 22], the generator of the infinitesimal operator A and has the following important property:

$$\exists N > 0, \exists \delta \in R : \quad \|S(t)\| \leq Ne^{\delta|t|}, \quad \forall\, t \in R.$$

In time-varying case, it is, in general, not true and therefore, we shall assume, throughout this section, that for every $t, s \geq 0$, the evolution operator $\Phi(t,s)$ generated by $A(t)$ is a linear operator acting in X and satisfies the following condition.

$$\textbf{(A)} \quad \exists N > 0,\ \exists \delta \in R: \quad \|\Phi(t,s)\| \leq N e^{\delta|t-s|} \quad \text{for all } t, s \geq 0.$$

It is obvious that if $A(t)$ is uniformly bounded for all $t \geq 0$, then $\Phi(t,s)$ satisfies condition (A).

THEOREM 4.1. *Assume the condition* (A). *If the system* (1), *where* X, U *are* Banach *spaces,* Ω *is a convex compact subset of* U, *is completely stabilizable, then the system is globally null-controllable.*

Proof. Let $N > 0, \delta \in R$ be given numbers defined by the condition (A) such that

$$\|\Phi^*(0,t)\| = \|\Phi(0,t)\| \leq N e^{\delta t}, \quad t \geq 0.$$

For all $y^* : \|y^*\| = 1$, we get

$$1 = \|\Phi^*(0,t)\Phi^*(t,0)y^*\| \leq \|\Phi^*(0,t)\|\|\Phi^*(t,0)y^*\|,$$

and hence

$$\frac{1}{\|\Phi^*(t,0)y^*\|} \leq \|\Phi^*(0,t)\| \leq N e^{\delta t}, \quad \forall t \geq 0. \tag{10}$$

Taking any $\alpha > \max\{0, \delta\}$, by the complete stabilizability of system (1), there exist a linear bounded operator function $K(t) : X \to \Omega, t \geq 0$, and $M > 0$ such that

$$\|\Phi_K(t,0)\| \leq M e^{-\alpha t}, \quad t \geq 0,$$

where $\Phi_K(t,s)$ is the evolution operator of system (1.1). For all y^* : $\|y^*\| = 1$ we get the estimate

$$\|\Phi_K^*(t,0)y^*\| \leq \|\Phi_K^*(t,0)\| = \|\Phi_K(t,0)\| \leq M e^{-\alpha t} \quad \forall\, t \geq 0. \tag{11}$$

For the operator $K(t)$ we consider the following linear differential system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)K(t)x(t), & t \geq 0, \\ x(0) = x_0. \end{cases}$$

The solution of this system is defined either by

$$x(t,x_0) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)B(s)K(s)x(s)\,ds,$$

or by

$$x(t,x_0) = \Phi_K(t,0)x_0.$$

Therefore, we get

$$\Phi_K(t,0)x_0 = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)B(s)K(s)x(s)\,ds.$$

For every $y^* \in X^*$, we also get

$$\begin{aligned} < \Phi_K^*(t,0)y^*, x_0 > &=< \Phi^*(t,0)y^*, x_0 > \\ &\quad + \int_0^t < B^*(s)\Phi^*(t,s)y^*, K(s)x(s) > \, ds. \end{aligned}$$

Since $K(s)x(s) \in \Omega$, we have

$$\int_0^t < B^*(s)\Phi^*(t,s)y^*, K(s)x(s) > \, ds \leq \int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y^*)\,ds$$

and hence for every $y^* \in X^*$ we also have

$$< \Phi_K^*(t,0)y^*, x_0 > \leq < \Phi^*(t,0)y^*, x_0 > + \int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y^*)\,ds$$

or equivalently,

$$< \Phi^*(t,0)y^*, -x_0 > \leq < \Phi_K^*(t,0)y^*, -x_0 > + \int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y^*)\,ds.$$

Therefore, since x_0 is an arbitrary, we obtain the relation

(12)

$$< \Phi^*(t,0)y^*, x > \leq < \Phi_K^*(t,0)y^*, x > + \int_0^t H_\Omega(B^*(s)\Phi^*(t,s)y^*)\,ds$$

for all $x \in X, y^* \in X^*$.

We now assume to the contrary that the system (1) is not globally null-controllable. By Theorem 3.1, there is a number $c > 0$ and for any sequence $t_k \to +\infty$ and $y_k^* \in X^*$ such that

$$\int_0^{t_k} H_\Omega(B^*(s)\Phi^*(t_k, s)y_k^*)\, ds < c\|\Phi^*(t_k, 0)y_k^*\|. \tag{13}$$

From the above strict inequality it follows that $y_k^* \neq 0$. Combining (12) and (13) gives

$$< \Phi^*(t_k, 0)y_k^*, x > < < \Phi_K^*(t_k, 0)y_k^*, x > +c\|\Phi^*(t_k, 0)y_k^*\|,$$

for all $x \in X$. Since $y_k^* \neq 0$, the above estimate holds for all $y_k^* \in B_1^*$. On the other hand, since $\|\Phi^*(t_k, 0)y_k^*\| \neq 0$, we then get

$$< \bar{y}_k^*, x > < < F_K(t_k), x > +c,$$

where

$$F_K(t_k) = \frac{\Phi_K^*(t_k, 0)y_k^*}{\|\Phi^*(t_k, 0)y_k^*\|}, \quad \bar{y}_k^* = \frac{\Phi^*(t_k, 0)y_k^*}{\|\Phi^*(t_k, 0)y_k^*\|} \in B_1^*.$$

Consequently,

$$< \bar{y}_k^* - F_K(t_k), x > < c$$

for all $x \in X$. Let us take a number $a > 0$ such that $ac = \epsilon < 1$. We get

$$< \bar{y}_k^* - F_K(t_k), y > < \epsilon,$$

for all $y = ax \in X$. The above estimate holds for all $y \in X$, we then get

$$\|\bar{y}_k^* - F_K(t_k)\| \leq \epsilon. \tag{14}$$

Let us set

$$W_K(k) = \|\bar{y}_k^* - F_K(t_k)\|.$$

As remarked above, $\|\bar{y}_k^*\| = 1$, then from (14) it follows that

$$1 - \|F_K(t_k)\| = \|\bar{y}_k^*\| - \|F_K(t_k)\| \leq W_K(k) \leq \epsilon. \tag{15}$$

On the other hand, since $y_k^* \in B_1^*$, taking (10), (11) and (15) into account we finally obtain

$$1 - \epsilon \leq \|F_K(t_k)\| \leq Ce^{(-\alpha+\delta)t_k},$$

where $C = NM$. Since $\alpha > \delta$ as chosen before, letting $k \to \infty$, the right-hand side of the above inequality tends to 0, while the left-hand side is $1 - \epsilon > 0$, which leads to a contradiction. The theorem is proved. □

REMARK 4.1. Note that Theorem 4.1 is an extension of a result of [20] for finite-dimensional systems without constrained controls and of [7, 21] for unconstrained control systems in a Hilbert space, where one requires $\alpha \in R$.

We now consider the discrete-time control system (2). Let us make the following assumption :

(B) $\quad \exists N > 0, \ \exists p > 0 : \quad \|\Psi(k,i)\| \leq Np^{|k-i|}$ for all $k, i \geq 0$.

Note that the condition (B) holds if the operator $A(k)$ is uniformly bounded for all $k \in Z^+$.

THEOREM 4.2. *Assume the condition* (B). *If the discrete-time system* (2), *where X and U are* Banach *spaces and Ω is a convex compact subset of U, is completely stabilizable, then the system is globally null-controllable.*

Proof. As in the proof of Theorem 4.1, for numbers $N > 0$, $p > 0$ given by the assumption (B), for every $k \in Z^+$ and for all $y^* \in X^*$: $\|y^*\| = 1$, we get

$$\frac{1}{\|\Psi^*(k,0)y^*\|} \leq \|\Psi^*(0,k)\| = \|\Psi(0,k)\| \leq Np^k. \tag{16}$$

Let us take a number $q > 0$ such that $q < \min\{1, 1/p\}$. By the definition of the complete stabilizability of system (2), there exist a linear operator $Q(k) : X \to \Omega, k \in Z^+$ and a number $M > 0$ such that for every $k \in Z^+$ and for all $y^* \in X^* : \|y^*\| = 1$, we have

$$\|\Psi_Q^*(k,0)y^*\| \leq \|\Psi_Q^*(k,0)\| = \|\Psi_Q(k,0)\| \leq Mq^k, \tag{17}$$

where $\Psi_Q(k,i)$ is the transition operator of system (2.1). We now assume to the contrary that the system (2) is not globally null-controllable. By Theorem 3.2, there is a number $c > 0$ such that for any sequence $t_k \to \infty$ and $y_k^* \in X^*$ such that

$$\sum_{i=0}^{t_k-1} H_\Omega(B^*(i)\Psi^*(t_k, i+1)y_k^*) < c\|\Psi^*(t_k,0)y_k^*\|.$$

By the same arguments that used in the proof of Theorem 4.1, we will arrive at the following estimate

$$1 - \|F_Q(t_k)\| \leq \epsilon, \tag{18}$$

where $\epsilon < 1$ and

$$F_Q(t_k) = \frac{\Psi_Q^*(t_k,0)y_k^*}{\|\Psi^*(t_k,0)y_k^*\|}, \quad y_k^* \in B_1^*.$$

Taking into (16), (17) and (18) into account, we obtain

$$1 - \epsilon \leq MN(qp)^k.$$

Since $\epsilon < 1$, and $0 < q < 1/p$ as choosen before, letting $k \to \infty$ we again arrived at a contradiction, which completes the proof. □

5. Conclusions

Global null-controllability of linear time-varying control systems with constrained controls in Banach spaces was studied. New necessary and sufficient conditions for global null-controllability of the systems were established. We showed that complete stabilizability implies exact global null-controllability for both continuous and discrete-time systems with restrained controls. The obtained results can be considered as extensions of the results obtained earlier for linear time-invariant control systems without constrained controls in finite dimensional spaces.

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References

[1] N. U. Ahmed, *Elements of Finite-dimensional Systems and Control Theory Pitman SPAM*, Longman Sci. Tech. Publ. **37** (1990).

[2] O. Carja, *On constrained controllability of linear systems in Banach spaces*, J. Optim. Theory Appl. **56** (1988), 215–225.

[3] R. F. Curtain and A. J. Pritchard, *Infinite-dimensional Linear Systems Theory, Lecture Notes in Contr. Inform. Sci.*, Springer-Verlag **8** (1981).

[4] R. Datko, *Uniform asymptotic stability of evolutionary process in a Banach space*, SIAM J. Math. Anal. **3** (1972), 428–445.

[5] V. Komornik, *Exact Controllability and Stabilization*, John Wiley & Sons, New York, 1994.

[6] S. G. Krein, *Linear Differential Equation in Banach Spaces*, Nauka, Moscow (in Russian), 1967.

[7] G. Megan, *On the stabilizability and controllability of linear dissipative systems in Hilbert space*, S.E.F., Universitate din Timisoara, **3** (1975).

[8] S. K. Mitter, *Theory of inequalities and the controllability of linear systems In: "Math. Control Theory"*, Academic Press, 1967, pp. 203–212.

[9] S. Nikitin, *Global Controllability and Stabilization of Nonlinear Systems*, World Scientific, Singapore, 1994.

[10] V. N. Phat, *Constrained Control Problems of Discrete Processes*, World Scientific, Singapore, 1996.

[11] ———, *Weak asymptotic stabilizability of discrete-time systems given by set-valued operators*, J. Math. Anal. Appl. **202** (1996), 363–378.

[12] V. N. Phat and J. Y. Park, *Further generations of Farkas' theorem and applications in optimal control*, J. Math. Anal. Appl. **216** (1997), 23–39.

[13] A. Pritchard and J. Zabczyk, *Stability and stabilizability of infinite-dimensional systems*, SIAM Review **23** (1981), 25–52.

[14] S. Rolewics, *Functional Analysis and Control Theory*, Polish Sci. Publ., Warszawa, 1987.

[15] Robert val Til P., *Constrained controllability of discrete-time systems, Ph.D thesis*, Northwestern University, Evanston IL, 1985.

[16] T. Schanbacher, *Aspect of positivity in control theory*, SIAM J. Contr. Optim. **27** (1989), 457–475.

[17] W. Schmitendorf and B. Barmish, *Null-controllability of linear systems with contrained controls*, SIAM J. Contr. Optim., **18** (1980), 327–345.
[18] N. K. Son, *A unified approach to constrained controllability for the heat equations and the retarded equations*, J. Math. Anal. Appl. **150** (1990), 1–19.
[19] Y. J. Sun and J. G. Hsieh, *Robust stabilization for a class of uncertain systems with time-varying delays*, J. Math. Anal. Appl. **98** (1998), 161–173.
[20] W. M. Wonham, *On pole assignment in multi-input controllable linear systems*, IEEE Trans. Automat. Control **12** (1967), 660–665.
[21] J. Zabczyk, *Complete stabilizability implies exact controllability*, Seminarul Ecuati Functionale **38** (1977), 1–10.
[22] ———, *Some comments on stabilizability*, Appl. Math. Optim. **19** (1988), 1–9.
[23] ———, *Mathematical Control Theory: An Introduction*, Birkhauser, 1992.

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