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WEIGHTED ORLICZ SPACE INTEGRAL INEQUALITIES FOR POTENTIAL MAXIMAL OPERATORS

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Abstract. We characterize a condition for $\mathcal{M}_\varphi$ to be of weak type (Φ_1, Φ_2) in terms of Orlicz norms.

1. Introduction

Given a function f in $\mathbb{R}^n$, we define a function $\mathcal{M}f$ in $\overline{\mathbb{R}^{n+1}_+} = \{(x,s) : x \in \mathbb{R}^n, s \geq 0\}$ by setting

$$\mathcal{M}f(x,s) = \sup\left\{\frac{1}{|Q|}\int_Q |f(y)|\,dy : x \in Q \text{ and sidelength}(Q) \geq s\right\}.$$

It is well known that this maximal operator $\mathcal{M}$ controls the Poisson integral defined by, for $x \in \mathbb{R}^n$ and $s \geq 0$,

$$P(f)(x,s) = \int_{\mathbb{R}^n} f(y)P(x-y,s)\,dy,$$

where

$$P(x,s) = \frac{c_n s}{(|x|^2 + s^2)^{n+1/2}}$$

is the Poisson kernel. For a given positive measure ν on $\mathbb{R}^n \times [0,\infty)$, under what condition on ν can we assert that $\mathcal{M}$ is bounded from $\mathcal{L}^p(\mathbb{R}^n)$ into weak-$\mathcal{L}^p(\mathbb{R}^n \times [0,\infty), \nu)$? Carleson([3]) showed that this was equivalent to the Carleson condition and later Feffermann-Stein([5]) found a sufficient condition, and Ruiz([13]), Ruiz-Torrea([14]) unified all these

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results. Recently, Gallardo([6]) and Chen([4]) obtained characterizations in terms of the Orlicz norm and in [8], we obtained a characterization for the fractional maximal operator. In this paper, we characterize a condition for $\mathcal{M}_\varphi$ to be of weak type (Φ_1, Φ_2) having four weights in the Orlicz norm. In the next theorem, we shall assume that φ is essentially nondecreasing, i.e., there exists a positive constant ρ for which

$$\varphi(t) \le \rho\varphi(s),\ t \le s$$

and

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0 .$$

Our result is as follows.

THEOREM 3.1. *Let $\mathcal{M}_\varphi f$ be the potential maximal operator on $\overline{\mathbb{R}^{n+1}_+}$. $\mathcal{M}_\varphi f$ is defined by*

$$\mathcal{M}_\varphi f(x,s) = \sup_{x\in Q} \frac{\varphi(|Q|)}{|Q|} \int_Q |f(y)|\, dy, \quad l(Q) \ge s\,, \tag{1}$$

where $l(Q)$ denotes the sidelength of Q. Let u, v be weights on $\mathbb{R}^n$, w be a weight on $\overline{\mathbb{R}^{n+1}_+}$ and μ be a nonnegative measure on $\overline{\mathbb{R}^{n+1}_+}$. Φ_1 and Φ_2 are N-functions with complementary N-functions Ψ_1 and Ψ_2, respectively. Assume further that $\Phi_2 \circ \Phi_1^{-1}$ is convex. Then weak type boundedness, i.e.,

(2)
$$\Phi_2^{-1}\Big[\int_{\{(x,s)\in\overline{\mathbb{R}^{n+1}_+}:\mathcal{M}_\varphi f(x,s)>\lambda\}} \Phi_2(\lambda w(x,s))\, d\mu(x,s)\Big] \le \Phi_1^{-1}\Big[\int_{\mathbb{R}^n} \Phi_1\big(C|f(x)|u(x)\big)v(x)\, dx\Big]$$

holds if and only if

$$\int_Q \Psi_1\left[\frac{\gamma(\lambda,\widetilde{Q})}{C\lambda u(y)v(y)} \frac{\varphi(|Q|)}{|Q|}\right] v(y)\, dy \ \le\ \gamma(\lambda,\widetilde{Q})\ <\ \infty \tag{3}$$

holds for each cube Q, where

(4)
$$\gamma(\lambda,\tilde{Q}) = \Phi_1 \circ \Phi_2^{-1}\left[\int_{\tilde{Q}} \Phi_2\Big(\lambda w(x,s)\Big)\, d\mu(x,s)\right],\ \tilde{Q} = Q \times \Big(0, l(Q)\Big].$$

When $\varphi(|Q|) = 1$, the Hardy-Littlewood maximal operator is obtained. The fractional maximal operator $\mathcal{M}_\alpha (0 < \alpha < n)$ is given by $\varphi(|Q|) = |Q|^{\frac{\alpha}{n}}$. Maximal operators connected to the Bessel potential operator are defined by $\varphi(|Q|) = \int_0^{|Q|^{\frac{1}{n}}} \psi(s)\,ds$, where ψ is the derivative of φ.

2. Preliminaries

DEFINITION 2.1. Let $\Phi : [0,\infty) \to \mathbb{R}$ be a function satisfying

(i) $\Phi(s) > 0$ for all $s \geq 0$;
(ii) $\lim_{s\to 0} \Phi(s)/s = 0$;
(iii) $\lim_{s\to\infty} \Phi(s)/s = \infty$.

Then Φ is called an N-function. Each N-function has the integral representation: $\Phi(s) = \int_0^s \phi(t)\,dt$, where $\phi(s) > 0$ for $s > 0, \phi(0) = 0$ and $\phi(s) \to \infty$ as $s \to \infty$. Further, ϕ is right-continuous and nondecreasing. ϕ is called the density function of Φ. Define $\rho : [0,\infty) \to \mathbb{R}$ by $\rho(t) = \sup\{s : \phi(s) \leq t\}$. Then ρ is called the generalized inverse of ϕ. Finally, define

$$\Psi(t) = \int_0^t \rho(s)\,ds$$

and Ψ is called the complementary N-function of Φ. For further details, see [10].

DEFINITION 2.2. An N-function Φ is said to satisfy the Δ_2-condition in $[0,\infty)$ if $\sup_{s>0} \frac{\Phi(2s)}{\Phi(s)} < \infty$.

REMARK 1. If ϕ is the density function of Φ, then Φ satisfies the Δ_2-condition if and only if there exists a constant $\alpha > 1$ such that $s\,\phi(s) < \alpha\Phi(s)$, for any $s > 0$.

REMARK 2. If Ψ is the complementary N-function of Φ, then $st \leq \Phi(s) + \Psi(t)$ for all $s, t \geq 0$. Futher, $=$ holds if and only if $\phi(s-) \leq t \leq \phi(s)$ or else $\rho(t-) \leq s \leq \rho(t)$.

DEFINITION 2.3. Let $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and Φ be an N-function. Then the Orlicz space $\mathcal{L}_\Phi(d\mu)$ and $\mathcal{L}^*_\Phi(d\mu)$ are defined by

$$\mathcal{L}_\Phi(d\mu) = \left\{ f : \int_X \Phi(|f|)\, d\mu < \infty \right\}$$

and

$$\mathcal{L}^*_\Phi(d\mu) = \{ f : fg \in \mathcal{L}_1(d\mu) \quad \text{for all} \quad g \in \mathcal{L}_\Psi \},$$

where Ψ is the complementary N-function of Φ.

Keeping these definitions and notions, the following properties about the Orlicz space will be used in the proof of the Theorem 3.1.

PROPOSITION 2.4.

(i) *The* Orlicz space $\mathcal{L}^*_\Phi(d\mu)$ *is a* Banach space *with the* Orlicz norm

$$\|f\|_\Phi = \sup\left\{ \int |fg|\, d\mu : g \in \mathcal{S}_\Psi \right\},$$

where $\mathcal{S}_\Psi = \left\{ g \in \mathcal{L}_\Psi : \int \Psi(|g|)\, d\mu \le 1 \right\}$, *or with the* Luxemburg norm

$$\|f\|_{(\Phi)} = \inf\left\{ \lambda > 0 : \int \Phi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \le 1 \right\}.$$

(ii) (Hölder's inequality) *If* $f \in \mathcal{L}^*_\Phi(d\mu)$ *and* $g \in \mathcal{L}^*_\Psi(d\mu)$, *then*

$$\|fg\|_\Phi \le 2\|f\|_{(\Phi)}\|g\|_{(\Psi)}. \tag{5}$$

(iii) (Young's inequality)

$$ab \le \Phi(a) + \Psi(b) \quad \text{for all} \quad a, b > 0. \tag{6}$$

LEMMA 2.5. *Let* Φ *be an* N*-function with complementary function* Ψ. *Let* x *and* $y > 0$. *Then*

$$\Phi(x) \le x\phi(x) \le \Phi(2x), \tag{7}$$

$$\Phi(x) + \Phi(y) \le \Phi(x+y) \tag{8}$$

and

$$\Phi\left[\frac{\Psi(x)}{x}\right] \le \Psi(x). \tag{9}$$

LEMMA 2.6. *Suppose that f is an integrable function in $\mathbb{R}^n$. For each $\lambda > 0$, let $E_\lambda = \{(x,s) \in \mathbb{R}^{n+1}_+ : M_\varphi f(x,s) > \lambda\}$. Then, if E_λ is not empty, we have*

$$E_\lambda \subset \bigcup_j \widetilde{Q_j^3}, \tag{10}$$

where Q_j is the family of nonoverlapping maximal dyadic cubes satisfying

$$\frac{\lambda}{4^n \rho} < \frac{\varphi(|Q_j|)}{|Q_j|} \int_{Q_j} f(y)\,dy \le \frac{\lambda}{2^n} \tag{11}$$

for each integer j. Furthermore, we have that

$$\left\{x \in \mathbb{R}^n \ : \ M_\varphi^d f(x) > \frac{\lambda}{4^n \rho}\right\} = \bigcup_j Q_j.$$

Proof. Following [7](p.160), we let $C_\lambda = \{P_j\}$ be the family of the dyadic maximal nonoverlapping cubes satisfying the condition

$$\lambda < \frac{\varphi(|P_j|)}{|P_j|} \int_{P_j} f(y)\,dy.$$

To show that there is such a family C_λ, observe that

$$\frac{\varphi(|Q|)}{|Q|} \int_Q f(y)\,dy \to 0$$

as $Q \uparrow \mathbb{R}^n$, since f is integrable and since $\lim_{t\to\infty} \varphi(t)/t = 0$. If, for some dyadic cube Q,

$$\lambda < \frac{\varphi(|Q|)}{|Q|} \int_Q f(y)\,dy,$$

then Q is contained in dyadic cubes satisfying this condition, which are maximal with respect to the inclusion. Thus, there is a family of maximal nonoverlapping dyadic cubes $\{P_j\}$ yield

$$\lambda < \frac{\varphi(|P_j|)}{|P_j|} \int_{P_j} f(y)\,dy \le 2^n \rho \frac{\varphi(|P_j'|)}{|P_j'|} \int_{P_j'} f(y)\,dy \le 2^n \rho \lambda, \tag{12}$$

where P_j' denotes the only dyadic cube containing P_j. From this discussion, it is clear that

$$\{x \in \mathbb{R}^n : M_\varphi^d f(x) > \lambda\} = \bigcup_j P_j.$$

Let $(x, s) \in E_\lambda$; by definition, there is a cube R containing x with $l(R) \geq s$ such that

$$\lambda < \frac{\varphi(|R|)}{|R|} \int_R f(y)\, dy.$$

Let k be the unique integer such that $2^{(k+1)n} < |R| \leq 2^{-kn}$. There are some dyadic cubes with side length 2^{-k}, and at most 2^n of them, $\{J_i : i = 1, \ldots, 2^n\}$ meeting the interior of R. It is easy to see that, for one of these cubes, say J_1,

$$\frac{\lambda}{2^n} < \frac{\varphi(|R|)}{|R|} \int_{R \cap J_1} f(y)\, dy.$$

Now, since $|R| \leq |J_1| < 2^n |R|$, $\varphi(|R|) \leq \rho\varphi(|J_1|)$ and

$$\frac{\lambda}{4^n}|J_1| < \frac{\lambda}{2^n}|R| < \varphi(|R|) \int_{R \cap J_1} f(y)\, dy \leq \rho\varphi(|J_1|) \int_{J_1} f(y)\, dy.$$

Hence,

$$\frac{\lambda}{4^n \rho} < \frac{\varphi(|J_1|)}{|J_1|} \int_{J_1} f(y)\, dy.$$

By letting $C_{t/4^n\rho} = \{Q_j\}$, we see that $J_1 \subset Q_k$, for some k, and $x \in R \subset J_1^3 \subset Q_k^3$.

On the other hand, $s \leq l(R) \leq l(Q_j^3)$. From this, we conclude that

$$E_\lambda \subset \bigcup_j \widetilde{Q_j^3}.$$

Finally, it follows from (12) that

$$\frac{\lambda}{4^n \rho} < \frac{\varphi(|Q_j|)}{|Q_j|} \int_{Q_j} f(y) dy \leq \frac{\lambda}{2^n},$$

for each j, concluding the proof of the lemma. □

3. Main Results

THEOREM 3.1. *Let $\mathcal{M}_\varphi f$ be the potential maximal operator on $\overline{\mathbb{R}^{n+1}_+}$. $\mathcal{M}_\varphi f$ is defined by*

$$\mathcal{M}_\varphi f(x,s) = \sup_{x\in Q} \frac{\varphi(|Q|)}{|Q|} \int_Q |f(y)|\,dy, \quad \text{where } l(Q) \geq s.$$

Let u and v be weights on $\mathbb{R}^n$, w be a weight on $\overline{\mathbb{R}^{n+1}_+}$ and μ be a nonnegative measure on $\overline{\mathbb{R}^{n+1}_+}$. Φ_1 and Φ_2 are N-functions with complements Ψ_1 and Ψ_2, respectively. Assume further that $\Phi_2 \circ \Phi_1^{-1}$ is convex. Then weak type boundedness, i.e.

$$\Phi_2^{-1}\left[\int_{\{(x,s)\in\overline{\mathbb{R}^{n+1}_+}:\mathcal{M}_\varphi f(x,s)>\lambda\}} \Phi_2\big(\lambda w(x,s)\big)\,d\mu(x,s)\right] \leq \Phi_1^{-1}\left[\int_{\mathbb{R}^n} \Phi_1\Big(C|f(y)|u(y)\Big)v(y)\,dy\right]$$

holds if and only if

$$\int_Q \Psi_1\left[\frac{\gamma(\lambda,\widetilde{Q})}{C\lambda u(x)v(x)}\frac{\varphi(|Q|)}{|Q|}\right] v(x)\,dx \leq \gamma(\lambda,\widetilde{Q}) < \infty$$

holds for each cube Q, where

$$\gamma(\lambda,\tilde{Q}) = \Phi_1\circ\Phi_2^{-1}\left[\int_{\tilde{Q}} \Phi_2\Big(\lambda w(x,s)\Big)\,d\mu(x,s)\right], \quad \tilde{Q} = Q\times\Big(0, l(Q)\Big].$$

Proof. For the necessity, we follow the idea of [2]. Since $\frac{\Psi_1(\varepsilon)}{\varepsilon}$ is increasing in ε and has full range $\mathbb{R}^+$, for given $\lambda > 0$, we can choose ε such that

$$\int_E \Psi_1\left(\frac{\varepsilon}{u(y)v(y)}\right)\frac{v(y)}{\varepsilon}\,dy = 2C\lambda\frac{|Q|}{\varphi(|Q|)}, \quad E\subset Q$$

$$f(x) = \frac{1}{C}\Psi_1\left(\frac{\varepsilon}{u(x)v(x)}\right)\frac{v(x)}{\varepsilon}\chi_E(x).$$

If $(y,s) \in \tilde{Q}$, then $y \in Q$ and $s \le l(Q)$. So

$$\begin{aligned}\mathcal{M}_\varphi f(y,s) &= \sup_{y\in Q} \frac{\varphi(|Q|)}{|Q|}\int_Q |f(y)|\,dy \\ &\ge \frac{\varphi(|Q|)}{|Q|}\int_Q |f(y)|\,dy \\ &= \frac{\varphi(|Q|)}{|Q|}\int_Q \frac{1}{C}\Psi_1\Big(\frac{\varepsilon}{u(y)v(y)}\Big)\frac{v(y)}{\varepsilon}\chi_E(y)\,dy \\ &= 2\lambda > \lambda.\end{aligned}$$

Thus if $(y,s) \in \tilde{Q}$, then $(y,s) \in E_\lambda = \big\{(x,r) \in \mathbb{R}^{n+1}_+ : \mathcal{M}_\varphi f(x,r) > \lambda\big\}$. Hence

$$\begin{aligned}\gamma(\lambda,\tilde{Q}) &= \Phi_1\circ\Phi_2^{-1}\Big[\int_{\tilde{Q}} \Phi_2\Big(\lambda w(x,s)\Big)\,d\mu(x,s)\Big] \\ &\le \Phi_1\circ\Phi_2^{-1}\Big[\int_{\{\mathcal{M}_\varphi f>\lambda\}} \Phi_2\Big(\lambda w(x,s)\Big)\,d\mu(x,s)\Big] \\ &\le \int_E \Phi_1\left[\frac{u(y)v(y)}{\varepsilon}\Psi_1\left(\frac{\varepsilon}{u(y)v(y)}\right)\right]v(y)dy \\ &\le \int_E \Psi_1\left(\frac{\varepsilon}{u(y)v(y)}\right)v(y)\,dy \\ &= 2C\lambda\frac{|Q|}{\varphi(|Q|)}\varepsilon.\end{aligned}$$

The third inequality follows from (2). Put

$$\Delta = \int_E \psi_1\left(\frac{\gamma(\lambda,\tilde{Q})}{4C\lambda u(y)v(y)}\frac{\varphi(|Q|)}{|Q|}\right)\frac{1}{u(y)}\,dy.$$

Then

$$\begin{aligned}\Delta &\le \int_E \psi_1\left(\frac{\varepsilon}{2u(y)v(y)}\right)\frac{1}{u(y)}\,dy \\ &\le 2\int_E \Psi_1\left(\frac{\varepsilon}{u(y)v(y)}\right)\frac{v(y)}{\varepsilon}\,dy \\ &= 4C\lambda\frac{|Q|}{\varphi(|Q|)}.\end{aligned}$$

Also

$$\Delta \geq \frac{4C\lambda|Q|}{\varphi(|Q|)\gamma(\lambda,\tilde{Q})}\int_E \Psi_1\left[\Big(\frac{\gamma(\lambda,\tilde{Q})}{4C\lambda u(y)v(y)}\Big)\frac{\varphi(|Q|)}{|Q|}\right]v(y)\,dy.$$

Thus

$$\int_E \Psi_1\left[\Big(\frac{\gamma(\lambda,\tilde{Q})}{4C\lambda u(y)v(y)}\Big)\frac{\varphi(|Q|)}{|Q|}\right]v(y)\,dy \leq \gamma(\lambda,\tilde{Q}) < \infty.$$

As $E \to Q$, (3) holds.

For the sufficiency, we follow the idea of [4] and [11]. From Lemma 2.6, for each $\lambda > 0$, let $E_\lambda = \{(x,s) \in \mathbb{R}^{n+1}_+ \ : \ M_\varphi f(x,s) > \lambda\}$. Then, if E_λ is not empty, we have

$$E_\lambda \subset \bigcup_j \widetilde{Q_j^3},$$

where Q_j is the family of nonoverlapping maximal dyadic cubes satisfying

$$\frac{\lambda}{4^n\rho} < \frac{\varphi(|Q_j|)}{|Q_j|}\int_{Q_j} f(y)\,dy \leq \frac{\lambda}{2^n}$$

for each integer j. Then $\frac{\lambda}{4^n\rho} < \frac{\varphi(|Q_j|)}{|Q_j|}\int_{Q_j} f(y)\,dy$ and $\{\widetilde{Q_j^3}\}$ is a covering of E_λ. Hence it follows from (6) that

$$\begin{aligned}
2\gamma(\lambda,\widetilde{Q_j^3}) &\leq \int_{Q_j} \frac{4^n\rho|f(x)|}{\lambda}\frac{\varphi(|Q_j|)}{|Q_j|}2\gamma(\lambda,\widetilde{Q_j^3})\,dx \\
&= \int_{Q_j} 23^n4^nC\rho|f(x)|u(x)\frac{\gamma(\lambda,\widetilde{Q_j^3})}{C\lambda u(x)v(x)}\frac{\varphi(|Q_j^3|)}{|Q_j^3|}v(x)\,dx \\
&\leq \int_{Q_j} \Phi_1\big(23^n4^nC\rho|f(x)|u(x)\big)v(x)\,dx \\
&\quad + \int_{Q_j} \Psi_1\left(\frac{\gamma(\lambda,\widetilde{Q_j^3})}{C\lambda u(x)v(x)}\frac{\varphi(|Q_j^3|)}{|Q_j^3|}\right)v(x)\,dx \\
&\leq \int_{Q_j} \Phi_1\big(23^n4^n\rho C|f(x)|u(x)\big)v(x)\,dx + \gamma(\lambda,\widetilde{Q_j^3}).
\end{aligned}$$

So

$$\gamma(\lambda, \widetilde{Q_j^3}) \le \int_{Q_j} \Phi_1(23^n 4^n C\rho |f(x)| u(x)) v(x)\, dx$$

and thus

$$\int_{\widetilde{Q_j^3}} \Big(\Phi_2\big(\lambda w(x,s)\big)\Big)\, d\mu(x,s) \le \Phi_2 \circ \Phi_1^{-1}\Big[\int_{Q_j} \Phi_1\Big(23^n 4^n C\rho |f(x)| u(x)\Big) v(x)\, dx\Big].$$

Summing over j gives

$$\int_{\{(x,s)\in\mathbb{R}_+^{n+1}: \mathcal{M}_\varphi f(x,s) > \lambda\}} \Phi_2\big(\lambda w(x,s)\big)\, d\mu(x,s) \le \sum_j \Phi_2 \circ \Phi_1^{-1}\Big[\int_{Q_j} \Phi_1\Big(23^n 4^n C\rho |f(x)| u(x)\Big) v(x)\, dx\Big].$$

By using that $\Phi_2 \circ \Phi_1^{-1}$ is convex, this last sum is bounded by

$$\begin{aligned} &\Phi_2 \circ \Phi_1^{-1}\Big[\sum_j \int_{Q_j} \Phi_1\Big(23^n 4^n C\rho |f(x)| u(x)\Big) v(x)\, dx\Big] \\ &\le \Phi_2 \circ \Phi_1^{-1}\Big[C_n' \int_{\mathbb{R}^n} \Phi_1\Big(23^n 4^n C\rho |f(x)| u(x)\Big) v(x)\, dx\Big] \\ &\le \Phi_2 \circ \Phi_1^{-1}\Big[\int_{\mathbb{R}^n} \Phi_1\Big(C_n |f(x)| u(x)\Big) v(x)\, dx\Big]. \end{aligned}$$

□

Corollary 3.2. *From (3), put $w = u = 1, \Phi_1 = \Phi_2$. Then (3) gives*

(13)
$$\Big[\mu(\tilde{Q}) \frac{\varphi(|Q|)}{|Q|}\Big] \phi\Big[\frac{\varphi(|Q|)}{C|Q|} \int_Q \psi\Big(\frac{\varepsilon}{v(y)}\Big)\, dy\Big] \le C\varepsilon, \quad \text{for each} \quad \varepsilon > 0.$$

Proof. We follow the proof of Corollary 3.2 of [8]. □

Corollary 3.3. *If $\Phi_1 \circ \Phi_2^{-1}$ has the Δ' condition and (3) holds for $w = u = 1$, then there exist constants $C', C'' > 0$ such that, for any $\varepsilon > 0$,*

$$\Phi_1 \circ \Phi_2^{-1}\Big(\mu(\tilde{Q})\Big) \frac{\varphi(|Q|)}{|Q|} \phi_1\Big[\frac{C'}{|Q|} \int_Q \psi_1\Big(\frac{\varepsilon}{v(y)}\Big)\, dy\Big] \le C''\varepsilon. \tag{14}$$

Proof. We follow the proof of corollary 3.3 of [8]. □

EXAMPLE 3.4. Let $\varphi(|Q|) = 1$ and $\varepsilon = \frac{1}{t}$. From Corollary 3.2, (13) gives A_Φ^+ of [4], i.e.

$$\sup_{Q,t>0} \phi\left[\frac{1}{|Q|}\int_Q \psi\Big(\frac{1}{tv(x)}\Big)dx\right]\frac{t\mu(\tilde{Q})}{|Q|} < \infty.$$

But (13) is weaker than A_Φ^+, because an N-function Φ of A_Φ^+ in [4] satisfies the Δ_2-condition.

EXAMPLE 3.5. From Corollary 3.3, put $d\mu(x,t) = u(x)dx \otimes d\delta(t)$, where δ is the Dirac mass on $[0,\infty)$, concentrated at 0. Also set $\Phi_1(x) = \frac{x^p}{p}$ and $\Phi_2(x) = \frac{x^q}{q}$, where $1 < p \le q < \infty$. Then (14) gives $(u,v) \in A(\varphi,p,q)$ of [12], i.e.

$$\varphi\big(|Q|\big)|Q|^{\frac{1}{q}-\frac{1}{p}}\left[\frac{1}{|Q|}\int_Q u(x)\,dx\right]^{\frac{1}{q}}\left[\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\,dx\right]^{1-\frac{1}{p}} \le A \quad \text{for all} \quad Q.$$

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