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EQUIVARIANT K-GROUPS OF SPHERES WITH INVOLUTIONS

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ABSTRACT. We calculate the $R(G)$-algebra structure on the reduced equivariant K-groups of two-dimensional spheres on which a compact Lie group G acts as a reflection. In particular, the reduced equivariant K-groups are trivial if G is abelian, which shows that the previous Y. Yang's calculation in [8] is incorrect.

1. Introduction

Let G be a compact Lie group. As one of (stable) equivariant cohomology theories, $RO(G)$-graded cohomology theory has been considered a full equivariant generalization of ordinary singular cohomology, which is built in the interplay relating the Burnside ring, the real character ring $RO(G)$, and G-homotopy theory (see [5] for more details). The main difference from the other equivariant cohomology theories is that the coefficient ring is indexed by all virtual real representations instead of integers, and equivariant K-theory is one of the adequate known examples.

In 1995, Y. Yang [8] proved that, for any compact Lie group G, the coefficient groups $\widetilde{K}_G^\star(S^0) \equiv \widetilde{K}_G(S^\star)$ of $RO(G)$-graded equivariant K-theory can only have 2-torsion. He calculated the reduced equivariant

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K-groups of S^1 and S^2 with a finite cyclic group G acting as involutions, and deduced his main result using the Bott periodicity theorem and J. E. McClure's results [7].

More precisely, if a compact G-space X has a base point $*$ fixed by the G-action, then the reduced equivariant K-group $\widetilde{K}_G(X)$ is defined to be the kernel of the restriction homomorphism $K_G(X) \to K_G(*)$ induced from the inclusion map $* \hookrightarrow X$. In fact, $K_G(X)$ and $\widetilde{K}_G(X)$ are algebras over the complex character ring $R(G)$ (although there is no identity element in $\widetilde{K}_G(X)$). Let $\lambda\colon G \to O(1) = \{\pm 1\}$ be a surjective homomorphism regarded as a one-dimensional real representation of G. Denote by 1 the trivial one-dimensional real representation of G. Actually Y. Yang calculated $\widetilde{K}_G(S^\lambda)$ and $\widetilde{K}_G(S^{1\oplus\lambda})$ for finite cyclic groups G, where S^λ and $S^{1\oplus\lambda}$ denote the one-point compactifications of the real representations λ and $1 \oplus \lambda$, respectively.

The authors proved recently in [2, Theorem 10.1] that $\widetilde{K}_G(S^\lambda)$ is isomorphic to the ideal in $R(G)$ generated by $(1-\lambda)\otimes\mathbb{C}$, which extends Y. Yang's result [8, Theorem A] for G finite cyclic to any compact Lie group. In this paper we apply the same technique to calculate the $R(G)$-algebra structure of $\widetilde{K}_G(S^{1\oplus\lambda})$ when G is a compact Lie group. In particular, we will prove that $\widetilde{K}_G(S^{1\oplus\lambda})$ is trivial for any compact abelian Lie group G, which shows that Y. Yang's result [8, Theorem B] is not true.

In the following our main results are stated. Denote by H the kernel of the real representation λ. Given a character χ of H and $g \in G$, a new character ${}^g\chi$ of H is defined by ${}^g\chi(h) = \chi(g^{-1}hg)$ for $h \in H$. Choose and fix an element $b \in G \setminus H$. Since a character is a class function and G/H is of order two, ${}^b\chi = {}^g\chi$ for all $g \in G \setminus H$ so that ${}^b\chi$ is independent of the choice of the element $b \in G \setminus H$. Note that $R(H)$ has a canonical $R(G)$-module structure given by $\varphi \cdot \chi = res_H\varphi \otimes \chi$ for $\varphi \in R(G)$ and $\chi \in R(H)$.

THEOREM 1.1. *Let G be a compact Lie group and let $\lambda\colon G \to O(1)$ be a surjective homomorphism. Denote by H the kernel of λ and choose an element $b \in G \setminus H$. Then $\widetilde{K}_G(S^{1\oplus\lambda})$ is isomorphic to the $R(G)$-module consisting of the elements $\chi - {}^b\chi$ in $R(H)$ for all characters χ of H. Moreover, the ring structure on $\widetilde{K}_G(S^{1\oplus\lambda})$ is given by $\alpha\beta = 0$ for all elements $\alpha, \beta \in \widetilde{K}_G(S^{1\oplus\lambda})$.*

COROLLARY 1.2. *In particular,* $\widetilde{K}_G(S^{1\oplus\lambda})$ *is trivial if there exists an element in* $G \setminus H$ *commuting with all elements in* H *(for example,* G *is abelian).*

2. Complex $\mathbb{Z}_2$-vector bundles over $S^{1\oplus\lambda}$

We will begin by considering the structure of complex $\mathbb{Z}_2$-vector bundles over $S^{1\oplus\lambda}$. In this case $\lambda\colon \mathbb{Z}_2 \to O(1) = \{\pm 1\}$ becomes an isomorphism.

LEMMA 2.1. *Every complex* $\mathbb{Z}_2$*-vector bundle over* $S^{1\oplus\lambda}$ *decomposes into the* Whitney *sum of* $\mathbb{Z}_2$*-invariant sub-line bundles.*

Proof. The set of $\mathbb{Z}_2$-fixed points in $S^{1\oplus\lambda}$ constitutes a circle, denoted by S^1, containing 0 and $*$. Then S^1 divides the sphere $S^{1\oplus\lambda}$ into two hemispheres, say H_1 and H_2. Let E be a complex $\mathbb{Z}_2$-vector bundle over $S^{1\oplus\lambda}$. Note that the restriction of E to the subspace $S^1 \subset S^{1\oplus\lambda}$ decomposes into the Whitney sum of $\mathbb{Z}_2$-invariant sub-line bundles. Choose a $\mathbb{Z}_2$-invariant sub-line bundle, say F, over S^1. Then it is always possible to extend F to a non-equivariant sub-line bundle over H_1, since the restriction of E to H_1 is non-equivariantly trivial and the set of one-dimensional subspaces of the fiber is homeomorphic to the Grassmann manifold $G_{\mathbb{C}}(k,1) \cong \mathbb{C}P^{k-1}$, the fundamental group of which is trivial. We now extend it over the other hemisphere H_2 using the $\mathbb{Z}_2$-action on E to get a resulting $\mathbb{Z}_2$-invariant sub-line bundle of E. □

LEMMA 2.2. *Every complex* $\mathbb{Z}_2$*-vector bundle over* $S^{1\oplus\lambda}$ *is trivial.*

Proof. It suffices to show that every complex $\mathbb{Z}_2$-line bundle E over $S^{1\oplus\lambda}$ is trivial by Lemma 2.1. Choose two $\mathbb{Z}_2$-invariant hemispheres, denoted by H_0 and H_* of $S^{1\oplus\lambda}$ containing 0 and $*$, respectively, such that $H_0 \cup H_* = S^{1\oplus\lambda}$ and $H_0 \cap H_*$ is a $\mathbb{Z}_2$-invariant circle. Since H_0 (resp. H_*) is equivariantly contractible to 0 (resp. $*$), E restricted to H_0 (resp. H_*) is equivariantly isomorphic to the product bundle $H_0 \times E_0$ (resp. $H_* \times E_*$) where E_0 (resp. E_*) denotes the fiber of E at 0 (resp. $*$). Then E induces a $\mathbb{Z}_2$-equivariant clutching map of $H_0 \cap H_*$ into the set $\mathrm{Hom}(E_0, E_*)$ of linear maps between the two fibers E_0 and E_*. Note that E_0 and E_* are isomorphic as complex representations of $\mathbb{Z}_2$ since 0 and $*$ are connected by $\mathbb{Z}_2$-fixed points, and thus the induced $\mathbb{Z}_2$-action on $\mathrm{Hom}(E_0, E_*)$ is trivial. Since $\mathbb{Z}_2$ acts on the circle $H_0 \cap H_*$ as a reflection, the clutching map is equivariantly null-homotopic, that is, E is equivariantly trivial. $\square$

3. Induced vector bundles

In this section we review the notion of induced vector bundles defined in a way similar to the definition of induced representations.

Let G be a compact Lie group and let X be a G-space. The set of isomorphism classes of (real or complex) G-vector bundles over X, usually denoted by $\mathrm{Vect}_G(X)$, is a semi-group under the Whitney sum operation. For a closed subgroup H of G, the inclusion map $H \hookrightarrow G$ induces a semi-group homomorphism $\mathrm{res}_H\colon \mathrm{Vect}_G(X) \to \mathrm{Vect}_H(X)$ called the *restriction* homomorphism. On the other hand, there is another homomorphism $\mathrm{ind}_H^G\colon \mathrm{Vect}_H(X) \to \mathrm{Vect}_G(X)$ called the *induction* homomorphism (see for instance [4] or [6]). In addition, if H has finite index in G, then the induction homomorphism can be defined explicitly, which we will explain below.

Let E be an H-vector bundle over X. Choose a set of representatives $\{t_0, t_1, \ldots, t_l\}$ of G/H. The identity element of G will always be selected as t_0. Define a group isomorphism $\imath_g\colon gHg^{-1} \to H$ by $\imath_g(ghg^{-1}) = h$ for $h \in H$ and $g \in G$.

We first construct a $t_iHt_i^{-1}$-vector bundle t_iE over X for each $0 \le i \le l$. Consider the following pull-back diagram

$$\begin{array}{ccc} (t_i^{-1})^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{t_i^{-1}} & X, \end{array}$$

where t_i^{-1} denotes the action of t_i^{-1} on X. Since the action of t_i^{-1} on X is $\imath_{t_i}$-equivariant, i.e., $t_i^{-1}(hx) = \imath_{t_i}(h)t_i^{-1}(x)$ for $h \in t_iHt_i^{-1}$ and $x \in X$, the pull-back bundle $(t_i^{-1})^*E$ has a canonical $t_iHt_i^{-1}$-action and it becomes a $t_iHt_i^{-1}$-vector bundle over X, simply denoted by t_iE. Note that every element in the fiber of t_iE at $x \in X$ is represented by t_iv for some v in the fiber of E at $t_i^{-1}x$. We now define $\mathrm{ind}_H^G E$ by the Whitney sum (without action) of all the bundles t_iE and then give a G-action on it as follows.

Given $g \in G$, each element gt_i has a unique form $t_{i'}h_i$ for some representative $t_{i'} \in \{t_0, \dots, t_l\}$ and $h_i \in H$. So we define an action of g on $\mathrm{ind}_H^G E$ by

$$g \cdot \left(\sum t_i v_i\right) = \sum t_{i'}(h_i v_i).$$

If $\sum t_i v_i$ is an element in the fiber of $\mathrm{ind}_K^G E$ at $x \in X$, then each v_i is contained in the fiber of E at $t_i^{-1}x$ and $h_i v_i$ is in the fiber of E at $t_{i'}^{-1}gx$ since $h_i t_i^{-1} = t_{i'}^{-1}g$. Thus the action of g gives a linear map between the two fibers at x and gx for all $x \in X$. It is easy to see that the definition gives actually an action of G on $\mathrm{ind}_K^G E$. Moreover the action is compatible with the $t_iHt_i^{-1}$-action already defined on t_iE.

We next show that the induced bundle is independent of the choice of the representatives $\{t_0, \dots, t_l\}$ of G/H. Choose another set of representatives $\{s_0 = e, s_1, \dots, s_l\}$ of G/H. We may assume that each $s_i^{-1}t_i$ is contained in H by arranging the order of representatives if necessary, and that every element in the induced bundle constructed with $\{s_0, \dots, s_l\}$ is written by $\sum s_i v_i$. It is easy to check that the bundle map defined by $\sum t_i v_i \mapsto \sum s_i(s_i^{-1}t_i)v_i$ gives an equivariant isomorphism between the two induced bundles.

LEMMA 3.1. *If two H-vector bundles E and E' over X are isomorphic, then so are the induced bundles $\mathrm{ind}_H^G E$ and $\mathrm{ind}_H^G E'$. The converse is not true in general.*

Proof. Given an H-vector bundle isomorphism $\Phi_0 \colon E \to E'$, the map

$$\Phi \colon \mathrm{ind}_H^G E \to \mathrm{ind}_H^G E'$$

sending $\sum t_i v_i$ to $\sum t_i \Phi_0(v_i)$ gives a G-vector bundle isomorphism. $\square$

LEMMA 3.2. *Let W be a representation of H. Then the induced bundle of the product bundle $X \times W \to X$ is isomorphic to the product bundle $X \times \mathrm{ind}_H^G W \to X$.*

Proof. The construction of induced bundles is the same as that of induced representations. $\square$

LEMMA 3.3. *Let L be a G-vector bundle and let E be an H-vector bundle over the same base space X. Then $L \otimes \operatorname{ind}_H^G E$ is isomorphic to $\operatorname{ind}_H^G(\operatorname{res}_H L \otimes E)$.*

Proof. For $l \in L$ and $\sum t_i v_i \in \operatorname{ind}_H^G E$, the map

$$\Phi\colon L \otimes \operatorname{ind}_H^G E \to \operatorname{ind}_K^G(\operatorname{res}_H L \otimes E)$$

sending $l \otimes \sum t_i v_i$ to $\sum t_i(t_i^{-1} l \otimes v_i)$ gives a (non-equivariant) vector bundle map. Given $g \in G$ we have $gt_i = t_{i'} h_i$ for some representative $t_{i'}$ and $h_i \in H$. Then the equalities

$$\begin{aligned}
\Phi\big(g(l \otimes \sum t_i v_i)\big) &= \Phi\big(gl \otimes \sum t_{i'}(h_i v_i)\big) = \sum t_{i'}(t_{i'}^{-1} gl \otimes h_i v_i) \\
&= \sum t_{i'} h_i (t_i^{-1} l \otimes v_i) = \sum g t_i (t_i^{-1} l \otimes v_i) \\
&= g\Phi(l \otimes \sum t_i v_i)
\end{aligned}$$

imply that Φ is G-equivariant. On the other hand, the inverse bundle map is given by the map sending $\sum t_i(l_i \otimes v_i)$ to $\sum(t_i l_i \otimes t_i v_i)$ for $l_i \in L$ and $v_i \in E$. □

4. Decomposition of equivariant vector bundles

In this section we rephrase the relevant material from [2, Section 2] to decompose equivariant vector bundles for the readers' convenience.

Let G be a compact Lie group and H a closed normal subgroup of G. Given a character χ of H and $g \in G$, a new character ${}^g\chi$ of H is defined by ${}^g\chi(h) = \chi(g^{-1}hg)$ for $h \in H$. This defines an action of G on the set $\operatorname{Irr}(H)$ of irreducible characters of H. Since a character is a class function, H acts trivially on $\operatorname{Irr}(H)$. Therefore, the isotropy subgroup of G at $\chi \in \operatorname{Irr}(H)$, denoted by G_χ, contains H. We choose a representative from each G-orbit in $\operatorname{Irr}(H)$ and denote by $\operatorname{Irr}(H)/G$ the set of those representatives.

Let X be a connected G-space on which H acts trivially. Then all the fibers of a G-vector bundle E over X are isomorphic as representations of H. We call the unique (up to isomorphism) representation of H the *fiber H-representation* of E.

As is well-known, E decomposes according to irreducible representations of H. For $\chi \in \mathrm{Irr}(H)$, we denote by $E(\chi)$ the χ-isotypical component of E, that is, the largest H-subbundle of E with a multiple of χ as the character of the fiber H-representation. Note that $gE(\chi)$, that is $E(\chi)$ mapped by $g \in G$, is ${}^g\chi$-isotypical component of E. This means that $E(\chi)$ is actually a G_χ-vector bundle and that $\bigoplus_{\chi' \in G(\chi)} E(\chi')$, where $G(\chi)$ denotes the G-orbit or χ, is a G-subbundle of E. Since $\bigoplus_{\chi' \in G(\chi)} E(\chi')$ is nothing but the induced bundle $\mathrm{ind}_{G_\chi}^G E(\chi)$, we have the following decomposition

$$(*) \qquad E = \bigoplus_{\chi \in \mathrm{Irr}(H)/G} \mathrm{ind}_{G_\chi}^G E(\chi)$$

as G-vector bundles.

LEMMA 4.1. *Two G-vector bundles E and E' are isomorphic if and only if $E(\chi)$ and $E'(\chi)$ are isomorphic as G_χ-vector bundles for each $\chi \in \mathrm{Irr}(H)/G$. In particular, E is trivial if and only if $E(\chi)$ is trivial for each $\chi \in \mathrm{Irr}(H)/G$.*

Proof. The necessity is obvious since a G-vector bundle isomorphism $E \to E'$ restricts to a G_χ-vector bundle isomorphism $E(\chi) \to E'(\chi)$, and the sufficiency follows from the fact that $\mathrm{ind}_{G_\chi}^G$ is functorial. □

COROLLARY 4.2. *Two G-vector bundles E and E' are stably isomorphic if and only if $E(\chi)$ and $E'(\chi)$ are stably isomorphic for each $\chi \in \mathrm{Irr}(H)/G$.* □

The observation above can be restated in K-theory as follows. Denote by $K_{G_\chi}(X, \chi)$ the subgroup of $K_{G_\chi}(X)$ generated by G_χ-vector bundles over X with a multiple of χ as the character of fiber H-representations. The reduced version $\widetilde{K}_{G_\chi}(X, \chi)$ can be defined accordingly. Then the map sending E to $\prod_{\chi \in \mathrm{Irr}(H)/G} E(\chi)$ gives group isomorphisms

$$K_G(X) \to \prod_{\chi \in \mathrm{Irr}(H)/G} K_{G_\chi}(X, \chi) \quad \text{and} \quad \widetilde{K}_G(X) \to \prod_{\chi \in \mathrm{Irr}(H)/G} \widetilde{K}_{G_\chi}(X, \chi)$$

by Corollary 4.2.

LEMMA 4.3. *If there is a complex G_χ-vector bundle L over X with χ as the character of the fiber H-representation, then the map sending E*

to $\mathrm{Hom}_H(L, E)$ *gives group isomorphisms*

$$K_{G_\chi}(X,\chi) \to K_{G_\chi/H}(X) \quad \text{and} \quad \widetilde{K}_{G_\chi}(X,\chi) \to \widetilde{K}_{G_\chi/H}(X).$$

Proof. It is easy to check that the map

$$K_{G_\chi/H}(X) \to K_{G_\chi}(X,\chi)$$

sending F to $L \otimes F$ gives the inverse of the map in the lemma. □

REMARK. The lemma above does not hold in the real category in general, but it does if χ is the character of a real irreducible representation of H with the endomorphism algebra isomorphic to the set of real numbers.

5. Proof of Theorem 1.1

We now return to our original setting to prove the main result. Hereafter we omit the adjective "complex" for complex vector bundles and complex representations since we work in the complex category. At first consider the additive structure on $\widetilde{K}_G(S^{1\oplus\lambda})$.

Let G be a compact Lie group. Denote by H the kernel of the surjective homomorphism $\lambda\colon G \to O(1) = \{\pm 1\}$. Choose and fix an element $b \in G \setminus H$. Since G_χ contains H and G/H is of order two, G_χ is either H or G for each irreducible character $\chi \in \mathrm{Irr}(H)/G$. Note that, in each case, the condition of Lemma 4.3 holds, since there exists a G_χ-extension of χ [Proposition 4.2][3] and it gives a trivial G_χ-vector bundle with χ as the character of the fiber H-representation. Therefore, according to the arguments in the previous section, we have a group isomorphism

$$(**) \qquad \widetilde{K}_G(S^{1\oplus\lambda}) \cong \prod_{\substack{\chi\in\mathrm{Irr}(H)/G \\ \text{s.t. } G_\chi = H}} \widetilde{K}(S^2) \times \prod_{\substack{\chi\in\mathrm{Irr}(H)/G \\ \text{s.t. } G_\chi = G}} \widetilde{K}_{G/H}(S^{1\oplus\lambda}).$$

It is well-known that $\widetilde{K}(S^2)$ is infinite cyclic with generator $\zeta - 1$, where ζ and 1, respectively, denote the dual bundle of the canonical line bundle and the trivial line bundle over $S^2 \cong \mathbb{C}P^1$ (see for instance [Theorem 2.3.14][1]). Moreover, $\widetilde{K}_{G/H}(S^{1\oplus\lambda})$ is trivial since $G/H \cong \mathbb{Z}_2$ and every $\mathbb{Z}_2$-vector bundle over $S^{1\oplus\lambda}$ is trivial by Lemma 2.2. Therefore the decomposition $(*)$ and Lemma 4.3 in the previous section imply the following lemma.

LEMMA 5.1. *$\widetilde{K}_G(S^{1\oplus\lambda})$ is a free abelian group generated by* $\mathrm{ind}_H^G\bigl(\chi\otimes(\zeta-1)\bigr)$ *for each* $\chi\in\mathrm{Irr}(H)/G$ *such that* $G_\chi=H$. *Here, the same notation χ is used for the product H-vector bundle over $S^{1\oplus\lambda}$ with χ as the character of the fiber representation.* □

Proof of THEOREM 1.1. We now consider the map

$$\Psi\colon \widetilde{K}_G(S^{1\oplus\lambda})\to R(H)$$

sending each generator $\mathrm{ind}_H^G\bigl(\chi\otimes(\zeta-1)\bigr)$ of $\widetilde{K}_G(S^{1\oplus\lambda})$ to $\chi-{}^b\chi\in R(H)$. Since both $\widetilde{K}_G(S^{1\oplus\lambda})$ and $R(H)$ are free abelian groups, Ψ is a well-defined group homomorphism. Moreover it is injective, since the set of the elements $\chi-{}^b\chi$ for all $\chi\in\mathrm{Irr}(H)/G$ such that $G_\chi=H$ can be extended to an additive basis of $R(H)$.

For each $\chi\in\mathrm{Irr}(H)$, either $G_\chi=H$ or G. In case that $G_\chi=G$, we have $\chi-{}^b\chi=0$. Thus the image of Ψ is generated by the elements $\chi-{}^b\chi$ for all $\chi\in\mathrm{Irr}(H)$, that is, the $R(G)$-submodule of $R(H)$ consisting of the elements $\chi-{}^b\chi$ for all characters χ of H.

Given a character φ of G, as in Lemma 5.1, we use the same notation φ for the product G-vector bundle over $S^{1\oplus\lambda}$ with φ as the character of the fiber representation. Then Lemma 5.3 implies that

$$\varphi\otimes\mathrm{ind}_H^G\bigl(\chi\otimes(\zeta-1)\bigr)=\mathrm{ind}_H^G\bigl(\mathrm{res}_H\varphi\otimes\chi\otimes(\zeta-1)\bigr).$$

Since ${}^b(\mathrm{res}_H\varphi)=\mathrm{res}_H\varphi$, we have the equalities

$$\begin{aligned}\Psi\bigl(\varphi\otimes\mathrm{ind}_H^G(\chi\otimes(\zeta-1))\bigr)&=\mathrm{res}_H\varphi\otimes\chi-{}^b(\mathrm{res}_H\varphi)\otimes{}^b\chi\\&=\mathrm{res}_H\varphi\otimes(\chi-{}^b\chi)\\&=\varphi\cdot\Psi\bigl(\mathrm{ind}_H^G(\chi\otimes(\zeta-1))\bigr)\end{aligned}$$

showing that Ψ is an $R(G)$-module homomorphism.

It remains to show the ring structure on $\widetilde{K}_G(S^{1\oplus\lambda})$. It suffices to show that the tensor product of any two generators in $\widetilde{K}_G(S^{1\oplus\lambda})$ is zero. Note that, given an induced bundle $\mathrm{ind}_H^G(\chi\otimes\zeta)$, the image of $\chi\otimes\zeta$ mapped by the b-action, that is the ${}^b\chi$-isotypical component of $\mathrm{ind}_H^G(\chi\otimes\zeta)$, is isomorphic to ${}^b\chi\otimes\zeta^*$ where ζ^* denotes the dual bundle of ζ. Indeed, the pull-back bundle $(b^{-1})^*(\zeta)$ is isomorphic to ζ^*, since the action $b^{-1}\colon S^{1\oplus\lambda}\to S^{1\oplus\lambda}$ is a reflection so that it induces the multiplication by

-1 on the second cohomology level. It follows that

$$\begin{aligned}\mathrm{res}_H\mathrm{ind}_H^G\big(\chi\otimes(\zeta-1)\big) &= \chi\otimes(\zeta-1)+{}^b\chi\otimes(\zeta^*-1)\\ &= \chi\otimes(\zeta-1)-{}^b\chi\otimes(\zeta-1)\\ &= (\chi-{}^b\chi)\otimes(\zeta-1),\end{aligned}$$

since $\zeta^*-1=-(\zeta-1)$ in $\widetilde{K}(S^2)$. Therefore, by Lemma 3.3, we have the equalities

$$\begin{aligned}&\mathrm{ind}_H^G\big(\chi\otimes(\zeta-1)\big)\otimes\mathrm{ind}_H^G\big(\eta\otimes(\zeta-1)\big)\\ &\qquad= \mathrm{ind}_H^G\big(\mathrm{res}_H\mathrm{ind}_H^G(\chi\otimes(\zeta-1))\otimes\eta\otimes(\zeta-1)\big)\\ &\qquad= \mathrm{ind}_H^G\big(((\chi-{}^b\chi)\otimes\eta\otimes(\zeta-1)^2\big)\\ &\qquad= 0,\end{aligned}$$

since $(\zeta-1)^2=0$ in $\widetilde{K}(S^2)$ (see [1, Theorem 2.3.14]). □

Proof of COROLLARY 1.2. By assumption there exists an element $b\in G\setminus H$ such that $bh=hb$ for all $h\in H$. It follows that ${}^b\chi(h)=\chi(b^{-1}hb)=\chi(h)$ for any character χ of H, which completes the proof. □

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