CRITERIA FOR A NEW CONCEPT OF STABILITY

V. LAKSHMIKANTHAM

Abstract. A new concept of stability that includes Lyapunov and orbital stabilities and leads to concepts in between them is discussed in terms of a given topology of the function space. The criteria for such new concepts to hold are investigated employing suitably Lyapunov-like functions and the comparison principle.

1. Introduction

Consider the dynamic system

\[(1.1) \quad x' = f(t, x), \quad x(t_0) = x_0, \quad t_0 \geq 0, \quad \frac{d}{dt} = 1' \]

where \(f \in C[R_+ \times R^n, \ R^n]\). Assume, for convenience, that the solutions \(x(t) = x(t, t_0, x_0)\) of (1.1) exist and are unique for \(t \geq t_0\). The original theorems of Lyapunov have been refined, extended and generalized in various directions. See [2; 3, 4, 5, 10] for details.

In the investigation of the initial value problems of differential equations, we have been partial to initial time all along in the sense that we only perturb or change the initial dependent variable or space variable and keep the initial time unchanged for all solutions. It appears, however, important to vary the initial time as well since it is impossible not to make errors in the starting time in any physical phenomena. If we do change the starting time for each solution along with the initial change of the dependent variable, we are faced with the problem of comparing any two solutions which differ in the starting time. There could be several ways of comparing and to each choice of measuring the difference, one may end up with a different set of conditions and a different result. In [6, 7, 8], this approach is initiated.

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Lyapunov stability compares, as we know, the phase-space positions of
the unperturbed and perturbed solutions of (1.1) at exactly simultaneous
times, namely,

\[(LS) \quad |x(t, t_0, y_0) - x_0(t, t_0, x_0)| < \epsilon, \ t \geq t_0,\]

where \(x_0(t, t_0, x_0)\) is the given solution of (1.1) and \(x(t, t_0, y_0)\) is the
perturbed solution of (1.1) with the same initial time \(t_0\). If the solutions
start at different times, then (LS) can be modified as

\[(LS^*) \quad |x(t, \tau_0, y_0) - x_0(t - \eta, t_0, x_0)| < \epsilon, \ t \geq \tau_0,\]

where \(\eta = \tau_0 - t_0\). In both cases, it is a too stringent demand.

Orbital stability, on the other hand, compares at any two unrelated
instants of time, namely,

\[(OS) \quad \inf_{s \in [t_0, \infty)} |x(t, t_0, y_0) - x_0(s, t_0, x_0)| < \epsilon, \ t \geq t_0.\]

As before, if the solutions start at different times as well, then we replace
(OS) by

\[(OS^*) \quad \inf_{s \in [\tau_0, \infty)} |x(t, \tau_0, y_0) - x_0(s - \eta, t_0, x_0)| < \epsilon, \ t \geq \tau_0.\]

In this case, the measurement of time is entirely irregular and hence it
is a too loose a requirement.

The foregoing considerations suggest a unification of the two concepts
so that the notions in between these extreme situations, may have con-
cepts of physical significance. We shall investigate this situation which
was initiated by Messera[9] who discussed the meaning stability and gave
some examples. Here we enlarge his notions and provide criteria.

2. A new concept of stability

Let \(E\) be the space of all functions from \(R_+\) to \(R_+\), each function \(\sigma(t)\)
representing a clock. We call \(\sigma(t) = t\), the perfect clock. Let \(\tau\) be any
topology in \(E\). Given the solution \(x_0(t, t_0, x_0)\), let us define the following
new concept of stability.

**DEFINITION.** The solution \(x_0(t, t_0, x_0)\) of (1.1) is said to be

1. \(\tau\)-stable, if given \(\epsilon > 0, t_0, \tau_0 \in R_+\) and a \(\tau\)-neighborhood \(N\) of the
   perfect clock, there exists a \(\delta = \delta(t_0, \tau_0, \epsilon) > 0\) such that for each
y₀ satisfying |x₀ − y₀| < δ, there is a clock σ ∈ N with σ(τ₀) = t₀ satisfying

|\(x(t, τ₀, y₀) − x₀(σ(t), t₀, x₀)| < ε, \ t ≥ τ₀.

(2) \(τ\)-uniformly stable, if in (1) δ is independent of \(t₀, τ₀\).
(3) \(τ\)-asymptotically stable, if (1) holds and there exists a \(\delta₀ = \delta₀(t₀, τ₀) > 0\) such that |x₀ − y₀| < δ₀ and for each ε > 0 and \(τ\)-neighborhood N of the perfect clock, there exists a \(T = T(t₀, τ₀, ε) > 0\), \(σ ∈ N\) with \(σ(τ₀) = t₀\) satisfying

|\(x(t, τ₀, y₀) − x₀(σ(t), t₀, x₀)| < ε, \ t ≥ τ₀ + T.

(4) \(τ\)-uniformly asymptotically stable, if \(δ₀\) and \(T\) in (3) are independent of \(t₀, τ₀\).

We note that a partial ordering of topologies of \(E\) induces a corresponding partial ordering of stability concepts.

We shall consider the following topologies of \(E\):

(\(τ₁\)) the discrete topology, where every set \(E\) is open;
(\(τ₂\)) the chaotic topology, where the open sets are only the empty set and the entire clock space \(E\);
(\(τ₃\)) the topology defined by the base

\[U_{σ₀, ε} = [σ, σ₀ ∈ C[R₊, R₊] : \sup_{t ∈ [τ₀, ∞)} |σ(t) − σ₀(t)| < ε];\]

(\(τ₄\)) the topology defined by the base

\[U_{σ₀, ε} = [σ, σ₀ = C¹[R₊, R₊] : |σ(τ₀) − σ₀(t₀)| < ε\]

and

\[\sup_{t ∈ [τ₀, ∞)} |σ'(t) − σ₀'(t)| < ε].\]

(\(τ₅\)) the topology consisting of the three open sets, the empty set, the entire clock space, and the set of all continuous increasing functions from \(R₊\) onto \(R₊\).

It is clear that the topologies \(τ₃, τ₄, τ₅\) lie between \(τ₁\) and \(τ₂\).
3. Comparison results

We need the following known result [1] to prove a comparison result in terms of Lyapunov-like function and comparison principle.

**Theorem 3.1.** Let \( g \in C[R^3_+, R] \) and \( g(t, u, v) \) be nondecreasing in \( v \) for each \( (t, u) \). Suppose that \( r(t) = r(t, t_0, u_0) \) is the maximal solution of

\[
\mu' = g(t, u, u), \quad u(t_0) = u_0 \geq 0,
\]

existing on \( [t_0, \infty) \) for each \( (t_0, u_0) \). Let \( m \in C[R_+, R_+] \) and

\[
D_-m(t) \leq g(t, m(t), v), \quad t \geq t_0.
\]

Then for all \( v \leq r(t) \), we have

\[
m(t) \leq r(t), \quad t \geq t_0.
\]

We can now prove the needed comparison result. Let \( w \in C[R^2_+ \times R, R_+] \) and \( r(t) \) be the maximal solution of (3.1). Consider the set \( \Omega \) defined by

\[
\Omega = [\sigma \in C^1[R_+, R] : w(t, \sigma, \sigma') \leq r(t), \quad t \geq \tau_0].
\]

**Theorem 3.2.** Assume that \( V \in C[R^2_+ \times R^n, R_+] \), \( V(t, \sigma, x) \) is locally Lipschitzian in \( x \), and

\[
D_-V(t, \sigma, x - y) = \lim_{h \to 0^-} \inf \frac{1}{h} [V(t + h, \sigma(t + h), x - y + h (f(t, x) - f(\sigma(t), y) \sigma'(t)) - V(t, \sigma(t), x - y)] 
\]

\[
\leq g(t, V(t, \sigma(t), x - y), w(t, \sigma, \sigma'))
\]

where \( g \in C[R^3_+, R] \) and \( g(t, u, v) \) is nondecreasing in \( v \) for each \( (t, u) \). Then \( \sigma(\tau_0) = t_0 \) and \( u_0 = V(\tau_0, \sigma(\tau_0), y_0 - x_0) \) implies

\[
V(t, \sigma(t), x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0) \leq r(t, \tau_0, v(\tau_0, \sigma(\tau_0), y_0 - x_0))
\]

for \( t \geq \tau_0 \).

**Proof.** Let \( x(t, \tau_0, y_0) \) and \( x_0(t, t_0, x_0) \) be the solutions of (1.1) through \( (\tau_0, y_0) \) and \( (t_0, x_0) \) existing on \( [\tau_0, \infty) \) and \( [t_0, \infty) \) respectively. We set

\[
m(t) = V(t, \sigma(t), x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)) \quad \text{for} \quad \sigma \in \Omega.
\]

Then the standard computation [1] yields the differential inequality

\[
D_-m(t) \leq g(t, m(t), w(t, \sigma, \sigma')) \quad t \geq \tau_0.
\]
Since $\sigma \in \Omega$, this implies because of the monotone character of $g(t, u, v)$
in $v$, the inequality

$$D_m(t) \leq g(t, m(t), r(t)), \ t \geq \tau_0,$$

where $r(t) = r(t, \tau_0, u_0)$ is the maximal solution of (3.1). Theorem 3.1
now gives the stated results proving the theorem.

\[\square\]

4. Stability criteria

With the help of comparison theorems, we can now provide criteria
for $\tau_3$, $\tau_4$ stabilities.

In $\tau_1$-topology, one can use the neighborhood consisting solely of the
perfect clock $\sigma(t) = t$ and consequently we get right away Lyapunov
stability from the existing results.

In $\tau_2$-topology, we proceed as follows. We set $B(t_0, x_0) = x_0([t_0, \infty) -
\eta, t_0, x_0)$ $\eta = \tau_0 - t_0$, and obtain using standard results, the stability of
the set $B(t_0, x_0)$ assuming that $B(t_0, x_0)$ is closed, namely

$$d[y_0, B(t_0, x_0)] < \delta \quad \text{implies} \quad d[x(t, \tau_0, y_0), B(t_0, x_0)] < \epsilon, \ t \geq \tau_0.$$

Since

$$d[x(t, \tau_0, y_0), B(t_0, x_0)] = \inf_{s \in [\tau_0, \infty)} |x(t, \tau_0, y_0) - x_0(s - \eta, t_0, x_0)|,$$

denoting the infimum for each $t \geq \tau_0$ by $s_t$ and defining $\sigma(t) = s_t - \eta$ for
each $t > \tau_0$, we see that $\sigma \in E$ in $\tau_2$-topology. Thus, we obtain orbital
stability of $x_0(t, t_0, x_0)$ in terms of $\tau_2$-topology.

Next we shall provide criteria for $\tau_3$-stability.

Theorem 4.1. Assume that condition (1) of Theorem 3.2 is satisfied.
Suppose further that

(a) $b(|x|) \leq V(t, \sigma, x) \leq a(t, \sigma, |x|),$
(b) $d(|t - \sigma|) \leq w(t, \sigma, \sigma'),$ where $b(\cdot)$, $d(\cdot)$ and $a(t, \sigma, \cdot) \in K,$ and
$\ a \in C[R_+, R_+]$ and $K = [a \in C[R_+, R_+] : a(0) = 0 \ and \ a(u) \ is$
increasing in $u$).

Then the stability properties of the trivial solution of (3.1) imply the
corresponding $\tau_3$-stability properties of (1.1) respectively.
Proof. Let \( x_0(t, t_0, x_0) \) be the given solution of (1.1) and let \( \varepsilon > 0, 
\)
\( t_0, \tau_0 \in \mathbb{R}_+ \) and a \( \tau_3 \)-neighborhood of the perfect clock \( \sigma(t) = t \), namely, \( N = \{ \sigma \in C^1[\mathbb{R}_+, \mathbb{R}_+] : |t - \sigma(t)| < \gamma, t \geq \tau_0 \} \) be given for some \( \gamma = \gamma(\varepsilon) > 0 \). Assume that the trivial solution of (3.1) is stable. Then given \( b(\varepsilon) > 0 \) and \( \tau_0 \in \mathbb{R}_+ \) there exists a \( \delta_1 = \delta_1(\tau_0, \varepsilon) > 0 \) such that
\[
0 \leq u_0 < \delta_1 \text{ implies } u(t, \tau_0, u_0) < b(\varepsilon), \quad t \geq \tau_0,
\]
where \( u(t, \tau_0, u_0) \) is any solution of (3.1). Set \( u_0 = V(\tau_0, \sigma(\tau_0), y_0 - x_0) \). Then choosing \( \delta = \delta(t_0, \tau_0, \varepsilon) > 0 \) and \( \eta = \eta(\varepsilon) > 0 \) satisfying
\[
a(\tau_0, t_0, \delta) < \delta_1 \text{ and } \gamma = d^{-1}(b(\varepsilon)),
\]
we have, using (b) and the fact \( \sigma \in \Omega \),
\[
d[|t - \sigma(t)|] \leq w(t, \sigma, \sigma') \leq r(t, \tau_0, V(\tau_0, \sigma(\tau_0), y_0 - x_0)) \leq r(t, \tau_0, \delta_1) < b(\varepsilon).
\]
It then follows that \( |t - \sigma(t)| < \gamma \) and hence \( \sigma \in N \). We now claim that whenever \( |y_0 - x_0| < \delta \) and \( \sigma \in N \), it follows that
\[
|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| < \varepsilon, \quad t \geq \tau_0.
\]
If not, there exist a solution \( x(t, \tau_0, y_0) \) of (1.1) and a \( t_1 > \tau_0 \) such that
\[
|x(t_1, \tau_0, y_0) - x_0(\sigma(t_1), t_0, x_0)| = \varepsilon \quad \text{and} \quad |x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| \leq \varepsilon
\]
for \( \tau_0 \leq t \leq t_1 \). Then by Theorem 3.2, we get
\[
V(t, \sigma(t), x(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)) \leq r(t, \tau_0, V(\tau_0, \sigma(\tau_0), y_0 - x_0))
\]
for \( \tau_0 \leq t \leq t_1 \). It then follows from (a), (4.1) and (4.3),
\[
b(\varepsilon) = b\left(|x(t_1, \tau_0, y_0) - x(\sigma(t_1), t_0, x_0)|\right) \leq V(t_1, \sigma(t_1, \tau_0, y_0) - x_0(\sigma(t_1), t_0, x_0)) \leq r(t_1, \tau_0, V(\tau_0, \sigma(\tau_0), y_0 - x_0)) \leq r(t_1, \tau_0, \delta_1) < b(\varepsilon),
\]
a contradiction, which proves \( \tau_3 \)-stability of (1.1).

Let us next suppose that the trivial solution of (3.1) is asymptotically stable. Then it is stable and given \( \varepsilon > 0 \) and \( \tau_0 \geq 0 \), there exist \( \delta_01 = \delta_01(\tau_0) > 0 \) and a \( T = T(\tau_0, \varepsilon) > 0 \) such that
\[
0 \leq u_0 < \delta_01 \text{ implies } u(t, \tau_0, u_0) < b(\varepsilon), t \geq \tau_0 + T.
\]
The \( \tau_3 \)-stability gives taking \( \varepsilon = \rho > 0 \) and designating \( \delta_0 = \delta(t_0, \tau_0, \rho) \),
\[
|y_0 - x_0| < \delta_0 \quad \text{implies} \quad |x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| < \rho, t \geq \tau_0,
\]
for every $\sigma$ satisfying $|t - \sigma(t)| < \gamma(\rho)$. Thus, Theorem 3.2, we have

$$V(t, \sigma(t), x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)) \leq r(t, \tau_0, \delta_{10}), t \geq \tau_0. \quad (4.5)$$

Since by (4.4), we see that $r(t, \tau_0, \delta_{10}) < b(\epsilon), t \geq \tau_0 + T$, we get

$$d[|t - \sigma(t)|] \leq w(t, \sigma, \sigma') \leq r(t, \tau_0, \delta_{10}) < b(\epsilon), t \geq \tau_0 + T.$$

Thus $|t - \sigma(t)| < d^{-1}(b(\epsilon)) = \gamma(\epsilon), t \geq \tau_0 + T$. Hence there exists a $\sigma \in N$ satisfying

$$b[|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)|] \leq V(t, \sigma(t), x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)) \leq r(t, \tau_0, \delta_{10}) < b(\epsilon), t \geq \tau_0 + T,$$

which yields for $\sigma \in N$,

$$|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| < \epsilon, t \geq \tau_0 + T.$$

This completes the proof of $\tau_3$-asymptotic stability. \hfill \Box

To obtain sufficient conditions for $\tau_4$-stability, we need to make the following changes in Theorem 4.1.

**Theorem 4.2.** Let the conditions of Theorem 4.1 hold except that (b) is changed to

(b*) $d(|1 - \sigma'(t)|) \leq w(t, \sigma, \sigma'), d \in \mathcal{K}.$

Then the stability properties of the trivial solution of (3.1) imply the corresponding $\tau_4$-stability properties (1.1) respectively.

One can construct the proof of Theorem 4.2 following the proof of Theorem 4.1. We omit the details.

For uniform stability concepts, we need to modify condition (a) of Theorem 4.1 as follows:

(a*) $b(|x|) \leq V(t, \sigma, x) \leq a_0(|x|) + a_1(|t - \sigma|), a_0, a, b \in \mathcal{K}.$

**Remark.** The function $g(t, u, v) = -\sigma u + \lambda v$, $\lambda - \alpha = \beta > 0$ is admissible to give $r(t) = u_0 e^{-\beta(t-\tau_0)}$, which yields $\tau_3$-exponential asymptotic stabiliyt by Theorem 4.1. Also, the function $g(t, u, v) = \lambda(t)v$, $\lambda \in L^1[R_+, R_+]$ is admissible to give $r(t) = u_0 \exp \int_{\tau_0}^t \lambda(s)ds \leq u_0 e^{\theta}$, where $\int_0^{\infty} \lambda(s)ds \leq q$. Thus we get $\tau_3$-stability from Theorem 4.1.
References


Florida Institute of Technology
Department of Mathematical Sciences
Melbourne, FL 32901
USA