MULTIPLICITY RESULT FOR PERIODIC SOLUTIONS OF SEMILINEAR DISSIPATIVE HYPERBOLIC EQUATIONS WITH COERCIVE GROWTH NONLINEARITY

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ABSTRACT. The multiplicity of periodic solutions of semilinear dissipative hyperbolic equations is treated.

1. Introduction

Let \( R \) be the set of all reals and \( \Omega \subseteq R^n, \ n \geq 1, \) be a bounded domain with smooth boundary \( \partial \Omega \) which is assumed to be of class \( C^2. \)

Let \( Q = (0, 2\pi) \times \Omega \) and \( L^2(Q) \) be the space of measurable and Lebesgue square integrable real-valued functions on \( Q \) with usual inner product \( \langle \cdot, \cdot \rangle \) and corresponding norm \( \| \cdot \|_2. \)

By \( H^1_0(\Omega) \) we mean the completion of \( C^1_0(\Omega) \) with respect to the norm \( \| \cdot \|_1 \) defined by

\[
\| \phi \|_1^2 = \int_{\Omega} \sum_{|\alpha| \leq 1} |D^\alpha \phi(x)|^2 dx.
\]

\( H^2(\Omega) \) stands for the usual Sobolev space; i.e., the completion of \( C^2(\bar{\Omega}) \) with respect to the norm \( \| \cdot \|_2 \) defined by

\[
\| \phi \|_2^2 = \int_{\Omega} \sum_{|\alpha| \leq 2} |D^\alpha \phi(x)|^2 dx.
\]

Let \( g : R \to R \) be a continuous function. Moreover, we assume that there exist constants \( a_0 \) and \( b_0 \) such that

\[
(H_1) \quad |g(u)| \leq a_0 |u| + b_0 \quad \text{for all} \ u \in R.
\]

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The purpose of this work is to investigate the multiplicity for periodic solutions of the semilinear dissipative hyperbolic equations

\[
\beta \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \Delta_x u - \lambda_1 u + g(u) = h(t, x) \quad \text{in } Q,
\]

\[(B_1) \quad u(t, x) = 0 \quad \text{on } (0, 2\pi) \times \partial \Omega,\]

\[(B_2) \quad u(0, x) = u(2\pi, x) \quad \text{on } \Omega\]

where \(\lambda_1\) and \(\lambda_2\) denotes the first and second eigenvalues of \(-\Delta\) with zero Dirichlet boundary data and \(\phi_1\) is the positive normalized eigenfunction corresponding to \(\lambda_1\) and \(h \in L^2(Q)\).

The purpose of this paper is to give a multiplicity result for semilinear dissipative hyperbolic equations. Originally, the linear dissipative hyperbolic equations are derived from physical principle (see [4]). The existence and asymptotic theory of dissipative hyperbolic equations have been developed by several authors for initial value problems, boundary value problems, or mixed problems. For information on dissipative hyperbolic equations, we refer to [24]. On the existence of doubly-periodic solutions of semilinear dissipative hyperbolic equations have been done by Mawhin [22], Fucik and Mawhin [7]. Mawhin treat the existence of double-periodic solutions for semilinear dissipative hyperbolic equations of several types of \(g(u)\) with at most linear growth in connection with the set \(\Sigma = \{k^2 - j^2 | k, j \text{ integers}\}\). Fucik and Mawhin consider also the existence double-periodic solutions of semilinear dissipative hyperbolic equations with nonlinear term of the form \(g(u) = \mu u^+ - \nu u^- - \phi(u)\), where \(\phi\) is a continuous and bounded function, and \(\mu, \nu\) are real numbers related to the set \(\Sigma\). In [9, 15], the existence of solutions for Dirichlet-periodic problem for semilinear dissipative hyperbolic equations at resonances, in [13, 14], the existence of Dirichlet-periodic solutions for semilinear dissipative hyperbolic problems with superlinear growth, in [16], the existence of double-periodic solutions for semilinear dissipative hyperbolic equations with non-decreasing type of non-linear term, in [19, 20], the multiple existence of double-periodic and Dirichlet-periodic problem, respectively,
for semilinear dissipative hyperbolic equations and, in [17], the asymptotic behavior of Dirichlet-initial problem of semi-linear dissipative hyperbolic equations are discussed. Our result is related to the results in [19, 20] which are so called the Ambrosetti-prodi type multiplicity result which has been initiated by Ambrosetti-Prodi [1] in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. For more information on this problem for semilinear elliptic, parabolic and ordinary equations, we refer to [3, 5, 6, 10, 11, 12, 18, and 21] and their references.

In our result, we will treat a multiplicity result for Dirichlet-periodic solutions of semilinear dissipative hyperbolic equations in n-dimensional space. we assume the coercive growth on g with restriction on the left-hand and our proof based on Mawhin’s continuation theorem in [8].

2. Preliminary results

Let us define the linear operator

\[ L : \text{Dom} L \subseteq L^2(Q) \to L^2(Q) \]

by

\[ \text{Dom} L = \{ u \in L^2((0, 2\pi), H^2(\Omega) \cap H^1_0(\Omega)) | \frac{\partial u}{\partial t} \in L^2(Q), \frac{\partial^2 u}{\partial t^2} \in L^2(Q), \]

\[ u(0, x) = u(2\pi, x), x \in \Omega \} \]

and

\[ Lu = \beta \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \Delta u - \lambda_1 u. \]

Using Fourier series and Parseval inequality, we get easily

\[ < Lu, \frac{\partial u}{\partial t} > = \beta \| \frac{\partial u}{\partial t} \|^2_{L^2} \text{ for all } u \in \text{Dom} L. \]

Hence ker\( L = \text{ker}(\Delta + \lambda_1 I) = \text{ker} L^* \) since \( \Delta + \lambda_1 I \) is self-adjoint and ker\((\Delta + \lambda_1 I)\) is one space dimension generated by the eigenfunction \( \phi_1 \). Therefore \( L \) is a closed, densely defined linear operator and
\text{Im}(L) = [\ker L]^\perp; \text{i.e., } L^2(Q) = \ker L \oplus \text{Im} L. \text{ Let's consider a continuous projection } P_1 : L^2(Q) \to L^2(Q) \text{ such that } \text{Im} P_1 = \ker L. \text{ Then } L^2(Q) = \ker L \oplus \ker P_1. \text{ We consider another continuous projection } P_2 : L^2(Q) \to L^2(Q) \text{ defined by}

\[(P_2h)(t, x) = \frac{1}{2\pi} \int \int_Q h(t, x)\phi(x)dtdx\phi(x).\]

Then we have \(L^2(Q) = \text{Im} P_1 \oplus \text{Im} L, \ker P_2 = \text{Im} L, \text{ and } L^2(Q)/\text{Im} L \text{ is isomorphism to } \text{Im} P_2.\)

Since \(\dim[L^2(Q)/\text{Im} L] = \dim[\text{Im} P_2] = \dim[\ker L] = 1, \text{ we have an isomorphism } J : \text{Im} P_2 \to \ker L.\)

By the closed graph theorem, the generalized right inverse of \(L\) defined by

\[K = [L|_{\text{Dom} L \cap \text{Im} L}]^{-1} : \text{Im} L \to \text{Im} L\]

is continuous. If we equip the space \(\text{Dom} L\) with the norm

\[\|u\|_{\text{Dom} L} = \int \int_Q [u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial^2 u}{\partial t^2}\right)^2 + \sum_{|\beta| \leq 2} (D^\beta_x u)^2]dt dx.\]

Then there exist a constant \(c > 0\) independently of \(h \in \text{Im} L, u = Kh\) such that

\[\|Kh\|_{\text{Dom} L} \leq c\|h\|_{L^2}.\]

Therefore \(K : \text{Im} L \to \text{Im} L\) is continuous and by the compact imbedding of \(\text{Dom} L\) in \(L^2(Q)\), we have that \(K : \text{Im} L \to \text{Im} L\) is compact.

\textbf{Lemma 2.1.} \(L\) is closed, densely defined linear operator such that \(\ker L = [\text{Im} L]^\perp\) and such that the right inverse \(K : \text{Im} L \to \text{Im} L\) is completely continuous.

\textit{Proof.} See [2, 23].

\[\square\]

\textbf{3. Multiplicity result}

Let us consider the following

\((E^h)h\quad \beta \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \Delta_x u - \lambda_1 u + \mu g(u) = \mu h(t, x) \text{ in } Q,\)
(B₁) \[ u(t, x) = 0 \text{ on } (0, 2\pi) \times \partial \Omega, \]

(B₂) \[ u(0, x) = u(2\pi, x) \text{ on } \Omega \]

where \( \mu \in [0, 1] \).

Let \( L : \text{Dom}L \subseteq L^2(Q) \to L^2(Q) \) be defined as before. If we define a substitution operator \( N^\mu_h : L^2(Q) \to L^2(Q) \) by

\[
(N^\mu_h)(t, x) = \mu g(u) - \mu h(t, x)
\]

for \( u \in L^2(Q) \) and \((t, x) \in Q\), then \( N^\mu_h \) maps continuously into itself and take bounded sets into bounded set. Let \( G \) be any open bounded subset of \( L^2(Q) \), then \( P_2 N^\mu_h : \bar{G} \to L^2(Q) \) is bounded and \( K(I - P_2) : \bar{G} \to L^2(Q) \) is compact and continuous. Thus \( N^\mu_h \) is \( L \)-compact on \( \bar{G} \).

The coincidence degree \( D_L(L + N^\mu_h, G) \) is well defined and constant in \( \mu \) if \( Lu + N^\mu_h u \neq 0 \) for \( \mu \in [0, 1] \) and \( u \in \text{Dom}L \cap \partial G \). It is easy to check that \((u, \mu)\) is a weak solution of \((E^\mu_h)\) if and only if \( u \in \text{Dom}L \) and

\[(3.1^\mu_h)

Lu + N^\mu_h u = 0.\]

Here, we assume the following

(H₂) \[
\lim_{|u| \to \infty} \inf g(u) = +\infty,
\]

(H₃) \[
\lim_{u \to -\infty} \sup |\frac{g(u)}{u}| < \lambda_2 - \lambda_1.
\]

From (H₂) and (H₃), we may assume that

\[
m = \inf_{u \in R} g(u) > 0
\]

and there exist \( a \in (0, \lambda_2 - \lambda_1) \) and \( b \geq 0 \) such that

\[
|g(u)| \leq a|u| + b \text{ for all } u \leq 0.
\]

For \( h \in L^2(Q) \), we write \( \wedge h = \int_Q h(t, x) \phi(x) dt dx \).
Lemma 3.1. If \((H_1), (H_2)\) and \((H_3)\) are satisfied, then, for each \(h^* \in L^2(Q)\), there exists \(M(h^*) > 0\) independently of \(\mu\) such that
\[
\|\bar{u}\|_{L^2} \leq M
\]
holds for each possible weak solution \(u = \alpha \phi + \bar{u}\), with \(\alpha \in R\) and \(\bar{u} \in \text{Im}L\), of \((E^\mu_h)\) with \(\mu \in [0.1]\), and with \(\wedge h \leq \wedge h^*\) and \(\|h\|_{L^2} \leq \|h^*\|_{L^2}\).

Proof. Suppose there exists \(h \in L^2(Q)\) with \(\wedge h \leq \wedge h^*\) and \(\|h\|_{L^2} \leq \|h^*\|_{L^2}\) and the corresponding sequence of solutions \(\{(u_n, \mu_n)\}\), with \(\mu \in [0,1]\), of \((3.1^\mu_h)\) such that
\[
\lim_{n \to \infty} \|\bar{u}_n\|_{L^2} = \infty,
\]
then clearly
\[
\lim_{n \to \infty} \|u_n\|_{L^2} = \infty.
\]
For each \(n \geq 1\), we put \(u_n(t, x) = \alpha\phi(x) + \bar{u}_n(t, x)\).

First, we are going to prove that
\[
\lim_{n \to \infty} \frac{|\alpha_n|}{\|\bar{u}_n\|_{L^2}} = c < \infty.
\]
If it is not the case, we may assume that, by extracting subsequence if it is necessary,
\[
\lim_{n \to \infty} \frac{\|\bar{u}_n\|_{L^2}}{|\alpha_n|} = 0.
\]
Therefore, we may have a subsequence, say again, \(\{\bar{u}_n\}\) such that we have easily
\[
\lim_{n \to \infty} |u_n(t, x)| = \infty \text{ a.e. on } Q.
\]
By taking the inner product with \(\phi\) on both sides of \((3.1^\mu_h)\), we have
\[
\iint_Q g(u_n(t, x))\phi(x)dtdx = \iint_Q h\phi(x)dtdx \leq \wedge h^*.
\]
On the other hand, by \((H_2)\) and Fatou's lemma, we have
\[
\lim_{n \to \infty} \iint_Q g(u_n(t, x))\phi(x)dtdx = \infty
\]
which leads to a contradiction. First, we assume that $0 < c < \infty$, then there exist $n_0 \in \mathbb{N}$ such that

$$(c/2)\|\tilde{u}_n\|_{L^2} \leq |\alpha_n| \leq (3c/2)\|\tilde{u}_n\|_{L^2} \text{ for all } n \geq n_0.$$ 

For given $\epsilon > 0$, we may choose $\delta > 0$ such that

$$\iint_A |\phi|^2 dtdx < \epsilon\|\phi\|_{L^2}^2$$

for any measurable set $A \subset \bar{Q}$ with $|A| \leq \delta$.

Let $0 < \gamma < \|\phi\|_{\infty}$ and $\Omega_0 = \{x \in \Omega : \phi(x) \geq \gamma\}$. Choose $M_0 > 0$ such that

$$\delta M_0 - |m| \iint_Q \phi dtdx > \iint_Q h^* \phi(x) dtdx.$$ 

Then, since $\lim_{u \to \infty} g(u) = \infty$, we have that

$$m_0 = \sup\{|u| : \gamma g(u) < M_0\} < \infty.$$ 

We put

$$Q_n = \{(t, x) \in [0, 2\pi] \times \Omega_0 : |u_n(t, x)| \geq m_0\}.$$ 

Then we have $|Q_n| \leq \delta$. In fact, if $|Q_n| > \delta$, then from the definition of $m_0$ we have

$$\iint_Q g(u_n(t, x))\phi(x)dtdx = \iint_{Q_n} g(u_n)\phi(x)dtdx + \iint_{Q \setminus Q_n} g(u_n)\phi(x)dtdx > \delta M_0 - m \iint_Q \phi(x) dtdx$$

and this leads to a contradiction. Therefore, we have

$$\iint_{Q \setminus Q_n} |\alpha_n\phi|^2 \geq (1 - \epsilon) \iint_Q |\alpha_n\phi|^2.$$
On the other hand,
\begin{align*}
0 &= \iint_Q \alpha_n \phi \tilde{u}_n \\
&= \iint_{Q\setminus Q_n} \alpha_n \phi \tilde{u}_n + \iint_{Q_n} \alpha_n \phi \tilde{u}_n \\
&\leq (1/2) \iint_{Q\setminus Q_n} \left( |\alpha_n \phi + \tilde{u}_n|^2 - |\alpha_n \phi|^2 - |	ilde{u}_n|^2 \right) + \iint_{Q_n} |\alpha_n \phi| |	ilde{u}_n|.
\end{align*}
From the definition of $m_0$ and the above facts, we have, for all $n \geq n_0$,
\begin{align*}
0 &\leq (1/2)m_0^2 - (1/2)(1 - \epsilon)(c/2)\|\tilde{u}_n\|_{L^2}^2 + \epsilon(3c/2)\|\tilde{u}_n\|_{L^2}^2 \\
&= (1/2)m_0^2 - (c/4)(1 + 5\epsilon c)\|\tilde{u}_n\|_{L^2}^2.
\end{align*}
Therefore, $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded which leads to a contradiction.

Next, we assume $c = 0$, then $\lim_{n \to \infty} \frac{\|\tilde{u}_n\|}{\|u_n\|_{L^2}} = 1$.

Multiplying (3.1$^\mu_h$) by $\frac{\partial u}{\partial t}$ and integrate over $Q$, we find from the periodicity of $u$ that
\begin{equation*}
\|\frac{\partial u}{\partial t}\|_{L^2} \leq \frac{1}{|\beta|} \|h\|_{L^2}.
\end{equation*}
Again, taking the inner product with $u_n$ on both sides of (3.1$^\mu_h$), we have
\begin{equation*}
(\lambda_2 - \lambda_1)\|\tilde{u}_n\|_{L^2}^2 - \|\frac{\partial u_n}{\partial t}\|_{L^2}^2 < g(u_n), u_n > \leq \|h\|_{L^2}\|\tilde{u}_n\|_{L^2}
\end{equation*}
and hence
\begin{equation*}
\lim_{n \to \infty} \sup(\lambda_2 - \lambda_1 - a)\|\tilde{u}_n\|_{L^2}^2 \leq \left[ \max\{m, b\} |Q|^{1/2} + \frac{1}{|\beta|} \|h^*\|_{L^2}^2 + \|h^*\|_{L^2} \right].
\end{equation*}
Thus $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded which leads to another contradiction. \qed

**Lemma 3.2.** If $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied, then, for each $h^* \in L^2(Q)$, there exists $r = r(h^*) > 0$ independently of $\mu$ such that
\begin{equation*}
|\tilde{u}| \leq r
\end{equation*}
holds for each possible weak solution $u = \tilde{u} + \bar{u}$, with $\bar{u} = \alpha \phi(x)$, $\alpha \in R$ and $\bar{u} \in \text{Im} L$, of (3.1$^\mu_h$) where $\mu \in [0, 1]$, and with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$.
Proof. Suppose there exists \( h \in L^2(Q) \) with \( \wedge h \leq \wedge h^* \) and \( \|h^*\|_{L^2} \leq \|h\|_{L^2} \), and the corresponding sequence of weak solutions \( \{(u_n, \mu_n)\} \) of (3.1) with \( \{\mu_n\} \) is unbounded. Then \((u_n, \mu_n)\) is a solution of (3.1) where \( u_n = \tilde{u}_n + \alpha_n \phi(x) \) and \( \tilde{u}_n \in \text{Im} L \). We may choose a subsequence, say again \( \{\tilde{u}_n\} \) with \( \tilde{u}_n = \alpha_n \phi(x) \) such that \( |\alpha_n| \to +\infty \) as \( n \to +\infty \). Now, let \( M > M \) which is given in Lemma 3.1. Let
\[
Q_0 = \{(t, x) \in Q | |\tilde{u}(t, x)| \geq \frac{1 + \tilde{M}}{|Q|}\}.
\]
Then
\[
\tilde{M}^2 \geq \iint_Q |\tilde{u}(t, x)|^2 dt \, dx \\
\geq \iint_{Q_0} |\tilde{u}(t, x)|^2 dt \, dx \\
\geq |Q_0| \left( \frac{1 + \tilde{M}}{|Q|} \right)^2.
\]
Therefore \( |Q_0| \leq |Q_0| \leq \left( \frac{1 + \tilde{M}}{|Q|} \right)^2 |Q| \) and hence \( |Q \setminus Q_0| = |\{(t, x) \in Q | |\tilde{u}(t, x)| \geq \frac{1 + \tilde{M}}{|Q|}\}| \leq \left( 1 - \frac{\tilde{M}}{1 + \tilde{M}} \right)^2 |Q| > 0. \)
Let \( W = (0.2\pi) \times \Omega_0 \). Then we have \( |\alpha_n \phi(x)| \to \infty \) for each \( x \in \Omega_0 \) as \( n \to \infty \). Hence, by Fatou’s lemma and (H2), we have
\[
\liminf_{n \to \infty} \iint_Q g(\alpha_n \phi(x) + \tilde{u}(t, x)) \phi(x) \, dt \, dx \\
= \liminf_{n \to \infty} \iint_Q g(\alpha_n \phi(x) + \tilde{u}(t, x)) \phi(x) \, dt \, dx \\
\geq \iint_{W \cap (Q \setminus Q_0)} \liminf_{n \to \infty} g(\alpha_n \phi(x) + \tilde{u}(t, x)) \phi(x) \, dt \, dx \\
= \infty.
\]
Hence, there exists \( r_0(h^*) > 0 \) such that, for \( |\alpha_n| > r_0 \), we have
\[
(3.1) \quad \iint_Q g(\alpha_n \phi(x) + \tilde{u}_n(t, x)) \phi(x) \, dt \, dx > \iint_Q h^* \phi(x) \, dt \, dx.
\]
On the other hand, by taking the inner product with $\phi(x)$ on the both sides of (3.1)\textsuperscript{*$\text{mu}_h$}, we have

\[
\iint_Q g(\alpha_n \phi(x) + \bar{u}_n(t,x))\phi(x)dt\,dx = \iint_Q h\phi(x)dt\,dx \leq \land h^*
\]

which is impossible. The proof is complete. \hfill \Box

**Lemma 3.3.** If (H\textsubscript{1}), (H\textsubscript{2}) and (H\textsubscript{3}) are satisfied, then, for each $h^* \in L^2(Q)$, we can find an open bounded set $G(h^*)$ in $L^2(Q)$ such that, for each open bounded set $G$ in $L^2(Q)$ such that $G \supseteq G(h^*)$, we have

\[
D_L(L + N^h_k, G) = 0 \quad \text{for all} \quad h \in L^2(Q)
\]

with $\land h \leq \land h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$.

**Proof.** By similar fashion as we did in the proof of Lemma 3.2 to get (3.1), there exists $\bar{r}(h^*) > 0$ such that, for $|\alpha| > \bar{r}$, we have

\[
\iint_Q g(\alpha \phi(x))\phi(x)dt\,dx > \iint_Q h^*\phi(x)dt\,dx.
\]

Let

\[
G(h^*) = \{u \in L^2(Q) | -\bar{r}\phi(x) < \alpha \phi(x) < \bar{r}\phi(x) \text{ for } x \in \Omega, \|\bar{u}\|_{L^2} < \tilde{M}\}
\]

where $u = \alpha \phi(x) + \bar{u}$ with $\bar{r}(h^*) > \max\{r(h^*), r_0(h^*), \bar{r}(h^*)\}$ and $\tilde{M} > \tilde{M}$ which are given in Lemma 3.1 and Lemma 3.2. If (3.1)\textsuperscript{*$\text{mu}_h$} has a solution $u$ for some $\bar{h} \in L^2(Q)$ such that $\land \bar{h} < 2\pi m \int_{\Omega} \phi(x)dx$ and $\mu \in [0,1]$, then by taking the inner product with $\phi$ on the both sides of the equation (3.1)\textsuperscript{*$\text{mu}_h$}, we have

\[
2\pi m \int_{\Omega} \phi(x)dx \leq \iint_Q g(u(t,x))\phi(x)dt\,dx = \iint_Q \bar{h}\phi(x)dt\,dx.
\]

Thus (3.1)\textsuperscript{*$\text{mu}_h$} has no solution for $\bar{h} \in L^2(Q)$ such that $\land \bar{h} < 2\pi m \int_{\Omega} \phi(x)dx$. 
Hence, for each open bounded set $G \supseteq G(h^*)$, we have

$$D_L(L + N_h^1, G) = 0 \quad \text{for} \quad \bar{h} \in L^2(Q)$$

such that $\wedge \bar{h} < 2\pi m \int_\Omega \phi(x)dx$. Choose $\bar{h} \in L^2(Q)$ with $\wedge \bar{h} \leq 2\pi m \int_\Omega \phi(x)dx$ and $\|\bar{h}\|_{L^2} \leq \|h^*\|_{L^2}$, and define

$$F : (D(L) \cap G) \times [0, 1] \to L^2(Q) \quad \text{by}$$

$$F(u, \lambda) = Lu + N_{(1-\lambda)}\bar{h} + \lambda h(u) \quad \text{for} \quad h \in L^2(Q)$$

with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$. Then by Lemma 3.1 and Lemma 3.2, we have

$$0 \notin F(D(L) \cap \partial G) \times [0, 1] \quad \text{for} \quad h \in L^2(Q)$$

with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$. By the homotopy invariance of degree, we have, for all $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$,

$$D_L(L + N_h^1, G) = D_L(F(\cdot, 1), G) = D_L(F(\cdot, 0), G) = D_L(L + N_h^1, G) = 0$$

and the proof is completed. \(\square\)

**Theorem.** Assume $(H_1), (H_2)$ and $(H_3)$. Then there exist a constant $\alpha_0$ such that the boundary value problem $(E), (B_1)$, and $(B_2)$ has at least two solutions for $h$ such that

$$\int_Q g(\alpha_0 \phi(x) + \bar{u}(t, x))\phi(x)dt\,dx < \int_Q h\phi(x)dt\,dx$$

for every $\bar{u} \in L^2(\Omega)$ having mean value zero on $\Omega$, satisfying the conditions $(B_1)$, and $(B_2)$ such that

$$\|\bar{u}\|_{L^2} < \bar{M},$$

where $\bar{M}$ is given Lemma 3.3.
Proof. Let

\[ g(\alpha \phi(x_0) + \bar{u}_0) = \min_{\|x\| \leq \bar{r}, \|\bar{u}\| \leq \bar{M}} g(\alpha \phi(x) + \bar{u}). \]

Define

\[ \Delta(G(h)) = \{ u \in L^2(Q) | \alpha_0 \phi(x) < \alpha \phi(x) < \bar{r}_0 \phi(x) \text{ for } x \in \Omega, \|\bar{u}\|_{L^2} < \bar{M} \} \]

where \( \bar{r}_0(h) > \bar{r} \) which is given in Lemma 3.3.

If \( u \in \partial \Delta G(h) \), then necessary \( u = \alpha_0 \phi(x) + \bar{u} \) or \( u = \bar{r}_0 \phi(x) + \bar{u} \). If \( u = \alpha_0 \phi(x) + \bar{u} \) with \( \|\bar{u}\|_{L^2} < \bar{M} \), then, by taking inner product with \( \phi \) on the both sides of (3.1\( ^* \)), we have

\[ \iint_Q g(\alpha \phi(x) + \bar{u}(t,x)) \phi(x) dt dx = \iint_{\Omega} h \phi(x) dt dx \]

which, from (3.2) and (3.3), is impossible. If \( u = \bar{r}_0 \phi(x) + \bar{u} \) with \( \|\bar{u}\|_{L^2} < \bar{M} \), then, by the choice of \( \bar{r}_0 > 0 \), we have

\[ \iint_Q g(\bar{r}_0 \phi(x) + \bar{u}) \phi(x) dt dx > \iint_{\Omega} h \phi(x) dt dx \]

which is also impossible. Thus for \( \mu \in [0, 1] \), \( D_L(L + N^\mu_h, \Delta G(h)) \) is well defined and

\[ D_L(L + N^\mu_h, \Delta G(h)) = D_B(JP_2 N^\mu_h, \Delta G(h) \cap \ker L, 0) \]

where \( D_B \) ia Brouwer degree and \( P_2 N^\mu_h : L^2(Q) \to \ker L \) is an operator defined by

\[ (P_2 N^\mu_h u)(t,x) = \mu \iint_Q g(u(t,x)) \phi(x) dt dx - \iint_{\Omega} h dt dx \phi(x). \]

Now let \( T : \ker L \to R \) be defined by

\[ T(\alpha \phi(x)) = \alpha. \]
Then, for $\mu = 1$,

$$D_L(L + N^1_h, \Delta G(h)) = D_B(JP_2 N^1_h, \Delta G(h) \cap \ker L, 0)$$

$$= D_B(T(JP_2 N^1_h)T^{-1}, T(\Delta G(h)) \cap \ker L), 0).$$

If we let $J : \text{Im} P_2 \to \ker L$ be the identity map, then the operator $\Phi = T(JP_2 N^1_h)T^{-1}$ will be defined by

$$\Phi(\alpha) = \iint_Q g(\alpha \phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx.$$ 

Thus, we have

$$\Phi(\alpha_0) = \iint_Q g(\alpha_0 \phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx < 0$$

and by the choice of $\tilde{r}_0$, we have

$$\Phi(\tilde{r}_0) = \iint_Q g(\tilde{r}_0 \phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx > 0.$$ 

Hence, the coincidence degree exists and the corresponding value

$$|D_L(L - N, \Delta G(h))| = |D_B[J P_2 N^1_h, \Delta \cap \ker L, 0]| = 1.$$ 

Therefore, the equation $(3.1_1)$ has at least one solution in $\Delta G(h)$

Choose $G \supseteq \Delta G(h)$, where $G$ is defined in Lemma 3.3. By the additivity of degree, we have

$$0 = D_L(L + N^1_h, G) = D_L(L + N^1_h, \Delta G(h)) + D_L(L + N^1_h, G - \overline{\Delta G(h)}$$

and hence

$$|D_L(L + N^1_h, G - \overline{\Delta G(h)})| = 1.$$ 

Therefore $(3.1^1_h)$ has another solution in $G - \overline{\Delta G(h)}$. This proves our assertion. $\square$
References


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