Constrained H_∞ Optimal Control

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Abstract: Recently we have shown based on Lyapunov theorem that the closed loop system with the constrained infinite horizon H_∞ optimal controller is exponentially stable. Moreover the on-line feedback implementation of the constrained infinite horizon H_∞ optimal control based on quadratic programs has been proposed. In this paper, we summarize and discuss these results.

1. Introduction

Since Professor Kalman opened the era of modern control with his powerful state space approach in the early 60's, various different optimal control problems for linear systems has been addressed within state space framework. The first great success of state space approach was the simple Riccati equation based solution for the linear quadratic optimal regulation problem [10]. Although the main driving force for automatic control is the existence of disturbances, it was not explicitly considered in the linear quadratic optimal regulation problem. The existence of disturbances was first addressed in the linear quadaric Gaussian optimal control problem where disturbances were assumed to be white noise. Again solution of this problem was obtained through Riccati equations [11], [12], [17]. Due to the remarkable success including the above results, the state space theory has been predominantly studied and used in control problems until Professor Zames [18] adopted the input/output framework that was prevail in classical control theory, and proposed H_∞ optimal control problem where all the square integrable disturbances are considerd. The H_∞ optimal control problem was quite different from the previously addressed optimal control problems since it was formulated within the input/output framework. Hence the solution of the H_∞ optimal control problem was initially sought within the operator theoretic framework. However, the resulting solution techniques were rather complicated compared to the Riccati equation based solution techniqes for linear quadratic regulation and linear quadratic Gaussian problems. In 1988, combining both state space theoretic and input/output theoretic tools, Doyle, Glover, Khargonekar, and Francis [8] proposed a complete Riccati equation based solution technique for the H_∞ optimal control problem.

Constraints are always present in any practical control problems. For instance, the physical restriction of the actuator limits the value the input can assume. Moreover due to safety, environmental regulation and so on, the states of the plant are desired to lie within a designated area in the state space. Under the presence of these constraints, the closed loop system becomes nonlinear and the current Riccati equation based solution techniques for linear optimal control problmes are no longer valid. However due to the difficulty caused by the nonlinearity, the constrained infinite horizon linear optimal control problems remained unsolved up to recently. As an alternative to constrained infinite horizon linear optimal control, model predictive control was widely used and studied [7], [15], [13], [14], [3]. In model predictive control, constrained finite horizon linear optimal control problem is solved at every time instant and the first control is implemented. In 1987, Sznaier and Damborg [16] pioneered the area of constrained infinite horizon linear quadratic regulation. Indeed, based on the fact that the constrained infinite horizon linear quadratic optimal control problem can be reduced to a finite dimensional quadratic program, they proposed a solution strategy based on a set of quadratic programs and implemented it in receding horizon fashion. In 1996, Chmielewski and Manousiouthakis [6] provided a computationally less demanding technique where only a linear program is required to find a finite dimensional quadratic program equivalent to the constrained infinite horizon linear quadratic optimal control problem. Recently, Choi and Lee [4] established the exponential stability properties of the mixed constrained linear quadratic optimal control based on the Lyapunov theory. Then, based on the exponential envelop associated with the exponential stability, it was shown that the constrained infinite horizon linear quadratic optimal control problem can be reduced to a finite dimensional quadratic program without on-line optimization. More recently, Choi and Lee [5] also established the exponential stability properties and a feedback implementation technique for constrained infinite horizon H_∞ optimal control. In this paper, the core results and the associated underlying principles in Choi and Lee [5] are presented.

2. Preliminaries

2.1 H_∞ Optimal Control

In this section, we briefly summarize some standard results in H_∞ optimal control as exposed in [1], [2].

Consider the system

$$x_{k+1} = Ax_k + Bu_k + Dd_k, \qquad (1)$$
$$y_k = Cx_k + n_k,$$

where $x_k \in R^n$ is the state vector; $u_k \in R^m$ is the manipulated input; $d_k \in R^p$ is the unknown disturbance; $y_k \in R^l$ is the output; $n_k \in R^q$ is the measurement noise. Throughout the paper, the system is assumed to be stabilizable and detectable.

Assumption 2.1: (A, B) and (C, A) are stabilizable and detectable pairs, respectively.

Associated with the linear system (1), consider the H_∞ optimal control problem [1]:

$$J_u(x_0) = \min_u \max_d \left\{ \sum_{k=0}^{\infty} x_k^T Q x_k + \sum_{k=0}^{\infty} u_k^T u_k - \gamma^2 \sum_{k=0}^{\infty} d_k^T d_k \right\}$$

subject to (1) where $Q > 0$. $Q > 0$ is assumed throughout the paper. Using $\frac{1}{\gamma} D$ instead of D, we can assume $\gamma = 1$. We adopt this assumption throughout the paper.

The H_∞ optimal control problem admits the feedback but not necessarily saddle point solution as follows.

Theorem 2.2: Suppose the generalized algebraic Riccati equation:

$$M = Q + A^T M \Lambda^{-1} A$$

where

$$\Lambda = I + (BB^T - \gamma^{-2} DD^T) M$$

admits a positive semi-definite solution M' satisfying

$$\gamma^2 I - D^T M' D > 0.$$

(i) There exists a minimal positive definite solution M.

(ii) the corresponding cost function is

$$J_u(x_0) = x_0^T M x_0.$$

(iii) The controller

$$u_k^* = -B^T M \Lambda^{-1} A x_k =: -F x_k$$

and the corresponding maximizing disturbance

$$d_k^* = D^T M \Lambda^{-1} A x_k =: L x_k$$

results in

$$x_k^* = \Lambda^{-1} A x_k^*.$$

2.2 Maximal Output Admissible Set

We now summarize some results of maximal output admissible sets as exposed in [9].

It is assumed that the control of the system are desired to satisfy the following inequalities:

$$u^{\min} \le u_k \le u^{\max}, \qquad k = 0, 1, \cdots.$$

Also, throughout the paper, the following conditions are assumed to hold for the well posedness of the problem:

Assumption 2.2:

$$0 \in \operatorname{int}\{ v \in R^m, \ u^{\min} \le v \le u^{\max} \}.$$

Note that this condition implies $u^{\min} < 0 < u^{\max}$.

The maximal output admissible set is defined by

$$O_\infty := \{ x \in R^n \mid (A - BF + DL)^k x = \overline{A}^k x \in Y, k \ge 0 \}$$

where

$$Y:=\{x\in R^n \mid u^{\min}\le -Fx\le u^{\max}\}.$$

Theorem 2.2: Suppose the following assumptions hold: i) $A-BF+DL$ is exponentially stable, ii) Y is bounded, iii) $0\in \mathrm{int}\,Y$. Then there exists t^* such that

$$O_\infty=\{x\in R^n \mid u^{\min}\le -F(A-BF+DL)^k x\le u^{\max},\ 0\le k\le t^*\}.$$

3. Main Results

3.1 Exponential Stability

Consider the following constrained H_∞ optimal control problem:

$$J(x_k)=\min_u\max_d\left\{\sum_{j=0}^{\infty}x^T_{k+j|k}Q\,x_{k+j|k}+\sum_{j=0}^{\infty}u^T_{k+j|k}\,u_{k+j|k}-\sum_{j=0}^{\infty}d^T_{k+j|k}\,d_{k+j|k}\right\}$$

subject to

$$x_{k+j+1|k}=Ax_{k+j|k}+Bu_{k+j|k}+Dd_{k+j|k},\quad x_{k|k}=x_k,\qquad (2)$$

$$u_{\min}\le u_{k+j|k}\le u_{\max}.$$

We assume the desired attenuation level is achieved.

Assumption 3.1: $J(0)\le 0$. However, γ is not the infimal performance level such that $J(0)\le 0$.

Since O_∞ contains a neighborhood of the origin, the constrained H_∞ optimal control in this neighborhood is linear and, thus, is exponentially stable. In [5], Choi and Lee established that, for stable plant, the closed-loop system is globally exponentially stable. Indeed they showed that there exist $a, b, c>0$ such that

$$a\,|x_k|^2\le J(x_k)\le b\,|x_k|^2,\quad \forall x_k\in R^n,$$

$$\Delta J(x_k)=J(x_{k+1})-J(x_k)\le -c\,|x_k|^2,\quad \forall x_k\in R^n.$$

Hence, we have the following theorem.

Theorem 3.1: Under Assumption 3.1, the closed-loop system with a stable plant and the constrained H_∞ optimal controller is globally exponentially stable.

3.2 Feedback Implementation

If the solution of the constrained H_∞ optimal control problem exists, the optimal control must converge to zero as $j\to\infty$ because of $\frac{1}{2}\sum_{j=0}^{\infty}u^T_{k+j|k}u_{k+j|k}$ in the cost function. Now consider the following truncated constrained H_∞ optimal control problem:

$$J_N(x_k)=\min_u\max_d\left\{\sum_{j=0}^{\infty}x^T_{k+j|k}Q\,x_{k+j|k}+\sum_{j=0}^{\infty}u^T_{k+j|k}\,u_{k+j|k}-\sum_{j=0}^{\infty}d^T_{k+j|k}\,d_{k+j|k}\right\}\ (P_N)$$

subject to (2),

$$u_{\min}\le u_{k+j|k}\le u_{\max},\qquad 0\le j\le N-1.$$

Since the optimal control of the constrained H_∞ optimal control problem converges to zero, there exists N such that the constraints are not active for all $j\ge N$ and, thus, (P) and (P_N) are equivalent.

In the closed loop information pattern, it is clear that

$$J_N(x_k)=\min_{u_s}\max_{d_s}\left\{x^T_{k+N|k}M\,x_{k+N|k}+\sum_{j=0}^{N-1}x^T_{k+j|k}Q\,x_{k+j|k}+\sum_{j=0}^{N-1}u^T_{k+j|k}\,u_{k+j|k}-\sum_{j=0}^{N-1}d^T_{k+j|k}\,d_{k+j|k}\right\}$$

subject to (2),

$$u_{\min}\le u_{k+j|k}\le u_{\max}\qquad 0\le j\le N-1.$$

To find the solution of this problem in closed loop information pattern, the dynamic programming needs to be employed. However, the solution of this constrained dynamic game problem by dynamic programming is computationally very involved. Hence for feedback implementation, the open loop solution will be sought instead as follows.

As shown in [2], the above dynamic game problem results in the same feedback solution for both open and closed loop information pattern if it admits the unique open loop and the unique closed loop saddle point solutions. The necessary and sufficient condition for the existence for the unique open loop saddle point solution is as follows.

Lemma 4.1: For the linear-quadratic two-person zero-sum dynamic game, introduced above, the objective functional is strictly concave in d if and only if

$$I - D^T S_{j+1} D > 0$$

where S_j is given by

$$S_j = Q + A^T S_{j+1} A + A^T S_{j+1} D[I - D^T S_{j+1} D]^{-1} D^T S_{j+1} A,$$
$$S_N = M.$$

Since the maximization problem in J_N is unconstrained, we have the closed form solution to the problem in open loop information pattern.

Fact 4.1 [2]: Under the assumption for Lemma 4.1, the maximization problem in (P_N) admits the unique solution:

$$d^*_{k+j|k} = P_j S_{j+1} A x_{k+j|k} + P_j[S_{j+1} B u_{k+j|k} - v_{j+1}],$$
$$j \in [0, N-1]$$

where

$$P_j = [I - D^T S_{j+1} D]^{-1} D^T$$
$$S_j = Q + A^T S_{j+1}[I + D P_j S_{j+1}]A, \quad S_N = M$$
$$v_j = A^T[I + D P_j S_{j+1}]^T[v_{j+1} - S_{j+1} B u_{k+j|k}]; \quad v_N = 0$$

Furthermore, the optimal cost of the maximization of (P_N) is

$$x^T_{k|k} S_0 x_{k|k} - 2x^T_{k|k} v_0 - 2 q_0$$

where

$$q_0 = \frac{1}{2}\sum_{j=0}^{N-1}[-u^T_{k+j|k} B^T S_{j+1} B u_{k+j|k} - (S_{j+1} B u_{k+j|k} - v_{j+1})^T P_j^T D^T (S_{j+1} B u_{k+j|k} - v_{j+1}) + 2u^T_{k+j|k} B^T v_{j+1}].$$

Using the above fact, we obtain

$$J_N(x_k) = \min_{u_k}\left\{\sum_{j=0}^{N-1} u^T_{k+j|k} u_{k+j|k} + \left[\frac{1}{2} x^T_{k|k} S_0 x_{k|k} - x^T_{k|k} v_0 - q_0\right]\right\}$$

subject to

$$v_j = A^T[I + D P_j S_{j+1}]^T[v_{j+1} - S_{j+1} B u_{k+j|k}], \quad v_N = 0,$$
$$u_{\min} \le u_{k+j|k} \le u_{\max} \quad 0 \le j \le N-1.$$

To this end (P_N) can be transformed into a quadratic programming problem.

We now summarize the feedback implementation.

Off-line Computation:

1. Compute O_∞ as proposed in [9].

On-line Computation:

1. If $x_k \in O_\infty$, implement $u_k = -Kx_k$.
2. Choose N and solve $J_N(x_k)$.
3. If $x^*_{k+N|k} \in O_\infty$, implement the first control input. Otherwise, increase N and go to Step 2.

Before we close this section, a couple of remarks are in order.

For unconstrained problem, the existence condition for closed loop solution is strictly weaker than that for open loop solution in Lemma 4.1. However, the existence condition for closed loop solution of constrained problem is unknown but stronger than that of unconstrained problem. Hence it is not clear which one is stronger or no one of them is stronger than the other.

Clearly the above implementation is possible whenever the open loop solution exists. However, the stability analysis in the previous subsection holds on the region where both open and closed loop solutions exist. Hence unless the existence condition of the closed loop solution is stronger than that of the open loop solution, the region of stability is reduced. Nevertheless the local exponential stability properties in the previous subsection still hold true because the open solution is not necessary in O_∞. However, the global exponential stability for stable plants fails to hold in any meaningful cases. As N increases, the condition in Lemma 4.1 becomes stronger and stronger and the set of states for which the open loop solution exists shrinks. Indeed, as $N \to \infty$, the condition in Lemma 4.1 requires that the open loop plant satisfy the desired H_∞ performance level without control and thus the H_∞ optimal control be unnecessary. Hence if this requirement is not satisfied, N cannot increase over a certain value. To this end for stable plants, the global exponential stability of the closed loop system will not be attained if the open loop plant doesn't satisfy the desired H_∞ performance level.

4. Conclusions

In this paper, we have summarized and discussed the recently developed constrained H_∞ optimal control theory [5]. The exponential stability properties and the feedback implementation based on quadratic programs have been presented. The application of these results is mainly limited by the on-line computation time for the quadratic programs. Indeed it is only applicable to very slow processes such as chemical processes. Hence it is necessary to develop an implementation technique to reduce on-line computation time. Further theoretical development of stability regions for marginal and/or unstable plants and the closed loop solution existence condition for the constrained problem is desirable as well.

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