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Properties of optical extended fractional Fourier transforms

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Introducing left and right fractional orders separately, we show that the cascading of optical extended fractional Fourier transform systems can be easily calculated. Through such calculations we study the properties of extended fractional Fourier transforms.
Keywords: Optical Fourier transform; fractional Fourier transform; extended fractional Fourier transform.

I. INTRODUCTION

The fractional Fourier transform (FRT) of order $1/N$ means that, if it is applied to a function consecutively N times, the resulting output becomes the ordinary Fourier transform of the function. The FRT was first proposed by Namias [1] and was developed mathematically by McBride and Kerr [2]. Recently, the FRT was rediscovered by Mendlovic and Ozaktas [3,4] and Lohmann [5]. They showed that the FRT can be implemented optically with lenses. Their works stimulated studies on optical FRT and many research results were reported [6-14]. More recently, Hua et al. generalized the idea of FRTs further to extended FRTs (EFRTs) [12,13].

The EFRT of order p of a function $g(x_1)$ denoted by $G(x_2)$ is defined as [13]

$$G(x_2) = \int_{-\infty}^{\infty} g(x_1)\left[\exp\left(j\pi\frac{a^2x_1^2+b^2x_2^2}{\tan\phi_p}\right) \times \exp\left(-2j\pi\frac{abx_1x_2}{\sin\phi_p}\right)\right]dx_1 \quad (1)$$

where $j=\sqrt{-1}$, the angle $\phi_p = p\pi/2$, and a and b are magnification parameters. The operation of Eq. (1) with three parameters a, b, and ϕ_p can be implemented by controlling the distances d_l and d_r, and focal length f (or λ), as shown in Fig. 1. From Fresnel diffraction theory, the relation between the input and the output of the system shown in Fig. 1 can be expressed as Eq. (1). In this case, the three parameters a, b, and ϕ_p become complicated functions of d_l, d_r, and f [13]. Thus it is difficult to understand the cascading properties of EFRT systems in detail.

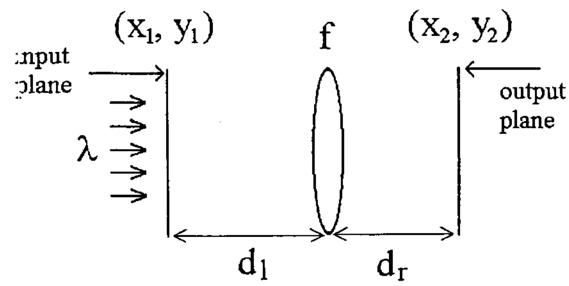

FIG. 1. Optical EFRT system by use of a lens.

In this paper, we define new parameters, i.e., left and right fractional orders instead of a and b. With the two fractional orders, the cascading of EFRT systems can be easily calculated. Through such calculations we study the properties of EFRTs.

II. LEFT AND RIGHT FRACTIONALITY ORDERS IN EFRTS

If the ABCD matrix of a lens system without loss is given, the Fresnel diffraction in the system is described by [11]

$$G(x_2) = \int_{-\infty}^{\infty} g(x_1)\exp\left(j\frac{\pi}{\lambda}\frac{Ax_1^2+Dx_2^2-2x_1x_2}{B}\right)dx_1 \quad (2)$$

where an unessential constant in front of the integral has been omitted, and $AD-BC=1$. The ABCD matrix of the system shown in Fig. 1 is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1-\frac{d_r}{f} & d_l + d_r - \frac{d_l d_r}{f} \\ -\frac{1}{f} & 1-\frac{d_l}{f} \end{bmatrix} \equiv \begin{bmatrix} \cos\phi_r & f(1-\cos\phi_l\cos\phi_r) \\ -\frac{1}{f} & \cos\phi_l \end{bmatrix} \quad (3)$$

where the left and right fractional orders l and r are defined by the following two equations:

$$\cos\phi_l = 1 - \frac{d_l}{f}, \quad \cos\phi_r = 1 - \frac{d_r}{f}. \quad (4)$$

From Eqs. (2)-(4), we can express the EFRT as

$$G(x_2) = \int_{-\infty}^{\infty} g(x_1)\exp\left(j\frac{\pi}{\lambda f}\frac{\cos\phi_r x_1^2 + \cos\phi_l x_2^2 - 2x_1x_2}{1-\cos\phi_l\cos\phi_r}\right)dx_1. \quad (5)$$

Comparing Eq. (1) with Eq. (5), we obtain the following relations between the two parameter sets $\{a, b, \phi_p\}$ and $\{\phi_l, \phi_r, f\}$:

$$\cos\phi_l = \frac{b}{a}\cos\phi_p, \quad (6)$$

$$\cos\phi_r = \frac{a}{b}\cos\phi_p, \quad (7)$$

$$\frac{1}{\lambda f} = ab\sin\phi_p. \quad (8)$$

From Eqs. (6) and (7), we can see the conventional meaning of the fractionality of order p in the EFRT, i.e.,

$$\cos^2\phi_p = \cos\phi_l\cos\phi_r. \quad (9)$$

Like conventional FRTs, the scale factor f_s of the EFRT is defined as

$$f_s = f\sin\phi_p = f\sqrt{1-\cos\phi_l\cos\phi_r} \quad (10)$$

where $\sin\phi_p$ is assumed to be positive. If $\cos\phi_l = \cos\phi_r = \cos\phi_p$, we can see that Eq. (5) becomes an expression of the conventional optical FRT [3-5].

III. CASCADING OF OPTICAL EFRT SYSTEMS

III. A. Parameters of the cascaded system

One advantage of expressing the EFRT as Eq. (5) with $\{\phi_l, \phi_r, f\}$ is that the parameters of the cascaded system are easily calculated by multiplying the ABCD matrices of the component EFRT systems. Suppose two EFRT systems are cascaded as shown in Fig. 2(a). And let us make the parameters of the first and second systems be distinguishable by attaching the subscript 1 and 2 to them, respectively. Then, the ABCD matrix of the whole system in Fig. 2(a) becomes

$$\begin{bmatrix} \cos\phi_r & f(1-\cos\phi_l\cos\phi_r) \\ -\frac{1}{f} & \cos\phi_l \end{bmatrix} = \begin{bmatrix} \cos\phi_{r_2} & f_2(1-\cos\phi_{l_2}\cos\phi_{r_2}) \\ -\frac{1}{f_2} & \cos\phi_{l_2} \end{bmatrix} \times \begin{bmatrix} \cos\phi_{r_1} & f_1(1-\cos\phi_{l_1}\cos\phi_{r_1}) \\ -\frac{1}{f_1} & \cos\phi_{l_1} \end{bmatrix} \quad (11)$$

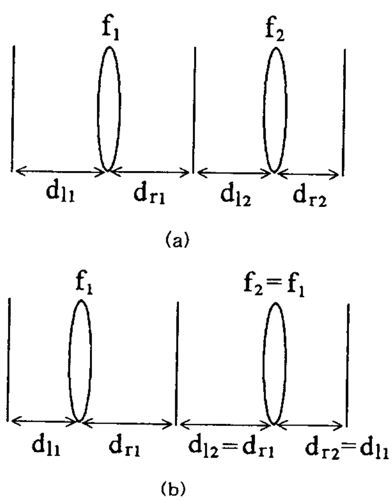

FIG. 2. Cascading of two EFRT systems. (a) General case. (b) Two systems are mutually mirror-symmetric.

From Eq. (11), the resulting three parameters of the cascaded system are

$$\cos\phi_l = \cos\phi_{l_1}\cos\phi_{l_2} - \frac{f_1}{f_2}(1-\cos\phi_{l_1}\cos\phi_{r_1}) = \frac{b_1b_2}{a_1a_2}\cos\phi_{p_1}\cos\phi_{p_2} - \frac{f_{s_1}}{f_{s_2}}\sin\phi_{p_1}\sin\phi_{p_2}, \quad (12)$$

$$\cos\phi_r = \cos\phi_{r_1}\cos\phi_{r_2} - \frac{f_2}{f_1}(1-\cos\phi_{l_2}\cos\phi_{r_2}) = \frac{a_1a_2}{b_1b_2}\cos\phi_{p_1}\cos\phi_{p_2} - \frac{f_{s_2}}{f_{s_1}}\sin\phi_{p_1}\sin\phi_{p_2}, \quad (13)$$

$$\frac{1}{f} = \frac{\cos\phi_{r_1}}{f_2} + \frac{\cos\phi_{l_2}}{f_1}. \quad (14)$$

Because $\cos\phi_l$ and $\cos\phi_r$ in Eq. (12) and (13) are not equal, in general, we can see that, when two EFRT systems with arbitrary parameter values are cascaded, the whole system becomes another EFRT system.

For simplicity, let us express Eq. (5) as

$$G(x_2) = \mathcal{F}_f^{r|l}[g(x_1)] = \mathcal{F}_{(f_s)}^{r|l}[g(x_1)]. \quad (15)$$

Of course, $\mathcal{F}_f^{l|l}[g(x_1)] \equiv \mathcal{F}_f^{l}[g(x_1)] = \mathcal{F}_{(f \sin\phi_l)}^{l}[g(x_1)]$ means a conventional FRT. Then, the transform of the cascaded system shown in Fig. 2(a) is represented by

$$G(x_2) = \mathcal{F}_{f_2}^{r_2|l_2}\mathcal{F}_{f_1}^{r_1|l_1}[g(x_1)] = \mathcal{F}_f^{r|l}[g(x_1)] \quad (16)$$

where l, r, and f are determined according to Eq. (12)-(14), or Eq. (11).

III. B. Additivity of the fractional orders

By equating $\cos\phi_p$ calculated directly from Eq. (9), (12), and (13) to $\cos(\phi_{p_1}+\phi_{p_2}) = \cos\phi_{p_1}\cos\phi_{p_2} - \sin\phi_{p_1}\sin\phi_{p_2}$, we can obtain the condition under which the additivity is satisfied. When $f_{s_1} = f_{s_2}$ [or, $a_1 b_1 = a_2 b_2$ from Eq. (8) and (10)], the condition becomes

$$a_1 a_2 = b_1 b_2 \quad \text{or} \quad \cos\phi_{l_1}\cos\phi_{l_2} = \cos\phi_{r_1}\cos\phi_{r_2}. \quad (17)$$

Unlike conventional FRTs, we can see that the additivity of the fractional orders are not satisfied (i.e., $p \neq p_1 + p_2$) in general, even if $f_{s_1} = f_{s_2}$.

As a special case, if $\cos\phi_{l_1} = \cos\phi_{r_1}$ (or, $a_1 = b_1$), in other words, the first system is a conventional FRT system, Eq. (17) tells us that the second system should be also a conventional FRT system because $\cos\phi_{l_2} = \cos\phi_{r_2}$ (or, $a_2 = b_2$). When the systems of conventional FRT are cascaded, the additivity of fractional orders are satisfied if $f_{s_1} = f_{s_2}$ [9]. In addition, the value of f_s does not change, i.e., $f_s = f_{s_1} = f_{s_2}$.

Let us consider another special case when Eq. (17) is satisfied, for example, when $\cos\phi_{l_1} = \cos\phi_{r_2}$ and $\cos\phi_{r_1} = \cos\phi_{l_2}$. Additional condition $f_{s_1} = f_{s_2}$ implies that $f_1 = f_2$ from Eq. (10). This is the case when the two EFRT systems to be cascaded are mutually mirror-symmetric as shown in Fig. 2(b). From Eq. (9), we see that $\cos\phi_{p_1} = \cos\phi_{p_2}$. The ABCD matrix of the cascaded system in Eq. (11) becomes, from Eq. (12) and (13),

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2\cos\phi_{l_1}\cos\phi_{r_1} - 1 & 2f_1(1-\cos\phi_{l_1}\cos\phi_{r_1})\cos\phi_{l_1} \\ -\frac{2\cos\phi_{r_1}}{f_1} & 2\cos\phi_{l_1}\cos\phi_{r_1} - 1 \end{bmatrix}. \quad (18)$$

Because $A = D$, the cascaded system becomes a conventional FRT system. Thus Eq. (18) can be rewritten as

$$\begin{bmatrix} \cos\phi_p & f_s\sin\phi_p \\ -\frac{\sin\phi_p}{f_s} & \cos\phi_p \end{bmatrix} = \begin{bmatrix} \cos(2\phi_{p_1}) & 2f_{s_1}\sin\phi_{p_1}\cos\phi_{l_1} \\ -\frac{2\sin\phi_{p_1}\cos\phi_{r_1}}{f_{s_1}} & \cos(2\phi_{p_1}) \end{bmatrix}. \quad (19)$$

From Eq. (19), we can see that $\phi_p = 2\phi_{p_1} = 2\phi_{p_2}$. However, the scale factor of the cascaded system is different from that of the component systems ($f_s \neq f_{s_1} = f_{s_2}$), because

$$f_s = f_{s_1}\sqrt{\frac{\cos\phi_{l_1}}{\cos\phi_{r_1}}} = f_{s_2}\sqrt{\frac{\cos\phi_{r_2}}{\cos\phi_{l_2}}}. \quad (20)$$

III. C. Conversion between extended and conventional FRT systems by their cascading

If two conventional FRT systems whose scale factors are different (i.e., $f_{s_1} \neq f_{s_2}$) are cascaded, the resulting system becomes an EFRT system in general. This is because $\cos\phi_l \neq \cos\phi_r$ if $f_{s_1} \neq f_{s_2}$ from Eqs. (12) and (13) even though $a_1 = b_1$ and $a_2 = b_2$. So we can express this fact simply as

$$\mathcal{F}_{(f_{s_2})}^{p_2}\mathcal{F}_{(f_{s_1})}^{p_1}[g(x_1)] = \mathcal{F}_f^{r|l}[g(x_1)] \quad (21)$$

$$\mathcal{F}_{(f_{s_1})}^{p_1}\mathcal{F}_{(f_{s_2})}^{p_2}[g(x_1)] = \mathcal{F}_f^{l|r}[g(x_1)] \quad (22)$$

where the orders l, r and f are determined by Eqs (12), (13), and (14), respectively.

On the contrary, let us consider the general condition under which two EFRT systems become a conventional FRT system when they are cascaded. This condition is obtained easily by equating Eq. (12) to Eq. (13). And from Eq. (4), we get

$$(d_{l_1} - d_{r_2})\left(1 - \frac{f_1 + f_2}{d_{r_1} + d_{r_2}}\right) = f_1 - f_2 \quad (23)$$

under which the following property is satisfied:

$$\mathcal{F}_{f_2}^{r_2|l_2}\mathcal{F}_{f_1}^{r_1|l_1}[g(x_1)] = \mathcal{F}_f^{p}[g(x_1)] = \mathcal{F}_{(f_s)}^{p}[g(x_1)] \quad (24)$$

where the parameter values are also determined by Eqs. (12)-(14).

Especially when $f_1 = f_2$, there are two possible cases that satisfy the condition given in Eq. (23). One is the case when $d_{l_1} = d_{r_2}$ i.e., $\cos\phi_{l_1} = \cos\phi_{r_2} \equiv \cos\phi_\alpha$, regardless of d_{r_1} and d_{l_2}. We can express this as

$$\mathcal{F}_f^{\alpha|l_2}\mathcal{F}_f^{r_1|\alpha}[g(x_1)] = \mathcal{F}_{(f_s)}^{p}[g(x_1)] \quad (25)$$

where $\cos\phi_p = \cos\phi_\alpha(\cos\phi_{r_1} + \cos\phi_{l_2}) - 1$, and $1/f_s = (\cos\phi_{r_1} + \cos\phi_{l_2})/f\sin\phi_p$ from Eqs. (12)-(14). The other is the case when $f_1 + f_2 = d_{r_1} + d_{l_2}$, i.e., $\cos\phi_{r_1} + \cos\phi_{l_2} = 0$, regardless of d_{l_1} and d_{r_2}. This means that $p = 2$ and $f = \infty$ in the cascaded system.

IV. CONCLUSION

In conclusion, we showed that the cascading of EFRT systems can be easily calculated by adopting ϕ_l and ϕ_r parameters. We obtained the condition under which the additivity of the fractional orders is satisfied in EFRTs when the scale factors of the two systems are the same. One interesting case when the additivity is satisfied is that the two systems to be cascaded are mirror-symmetric. The resulting system becomes a conventional FRT system in this case. Unlike the conventional FRT, even if the additivity is satisfied, the scale factor of the cascaded system is different from that of the component systems. When two conventional FRT systems whose scale factors are different are cascaded, we showed that the resulting system becomes an EFRT system, and obtained the left and right fractional orders of the cascaded system. Conversely, we also obtained the general condition under which two EFRT systems, when they are cascaded, become a conventional FRT system.

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