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On the weak law of large numbers for weighted sums of pairwise negative quadrant dependent random variables†

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of pairwise negative quadrant dependent (NQD) random variables and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $a_n \neq 0$ and $0 < b_n \to \infty$. In this note, for pairwise NQD random variables, a general weak law of large numbers of the form $(\sum |a_j| X_j - \nu_n)/b_n \xrightarrow{P} 0)$ is established, where $\{\nu_n, n \geq 1\}$ is a suitable sequence.

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1. INTRODUCTION

A sequence $\{X_n, n \geq 1\}$ of random variables is called pairwise positive quadrant dependent (PQD) if for each pair $i,\ j\ (i \neq j)$ and for all $r_i,\ r_j \in \mathbb{R}$ $P\{X_i > r_i,\ X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}$ (or $P\{X_i \leq r_i,\ X_j \leq r_j\} \geq P\{X_i \geq r_i\}P\{X_j \geq r_j\}$) and it is called pairwise negative quadrant dependent (NQD) if for each pair $i, j (i \neq j)$ and for all $r_i, r_j \in \mathbb{R}$ $P\{X_i > r_i,\ X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}$ (or $P\{X_i \geq r_i, X_j \geq r_j\} \leq P\{X_i \geq r_i\}P\{X_j \geq r_j\}$. These definitions were introduced by Lehmann(1966).

Let $\{X_n, n \geq 1\}$ be a sequence random variables and $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ sequences of constants with $a_n \neq 0,\ n \geq 1,\ 0 < b_n \to \infty$. Then $\{a_n X_n, n \geq 1\}$ is said to obey the general weak law of large numbers(WLLN) if the normed

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weighted sum $\left(\sum_{j=1}^{n} a_j X_j - \nu_n\right)/b_n$ converges in probability to zero, where $\{\nu_n, n \geq 1\}$ is a suitable sequence.

The WLLN for iid random variables which are stochastically dominated by a random variable X has been derived by Adler and Rosalsky(1991).

In this note we derive the WLLN for the pairwise NQD random variables with the same distribution function $F(x)$.

In section 2 we study some preliminary results and in section 3, we derive the main results for sums of pairwise NQD random variables with the same distribution $F(x)$.

2. PRELIMINARIES

Lemma 2.1. *(Matula, 1992) If $\{X_n, n \geq 1\}$ is a sequence of pairwise NQD random variables, $\{f_n,\ n \geq 1\}$ a sequence of nondecreasing functions $f_n : \mathbb{R} \to \mathbb{R}$, then $\{f_n(X_n),\ n \geq 1\}$ are also pairwise NQD.*

From Lemma 2.1 we obtain the following result : Put

$$\begin{aligned} X_n' = X_n I[|X_n| \leq c_n] + c_n I[X_n > c_n] \\ -c_n I[X_n < -c_n],\ for\ c_n \geq 0 \end{aligned} \tag{2.1}$$

and let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. Then $\{X_n', n \geq 1\}$ is also a sequence of pairwise NQD.

Rosalsky and Taylor (1991) have derived the following results under assumption that $\{X_n, n \geq 1\}$ is a sequence of independent random variables which are stochastically dominated by X. In this section only using the condition that the X_n are identically distributed Lemmas 2.2 and 2.3 will be proved.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with the same distribution function $F(x)$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants with $a_n \neq 0,\ 0 < b_n \to \infty,\ n \geq 1$. Put*

$$c_n = \frac{b_n}{|a_n|}$$

and define

$$X_{nj} = X_j I(|X_j| \leq c_n) + c_n I(X_j > c_n) - c_n I(X_j < -c_n),\ 1 \leq j \leq n,\ n \geq 1.$$

If

$$nP\{|X_1| > c_n\} = o(1) \tag{2.2}$$

then the WLLN

$$\frac{\sum_{j=1}^{n} |a_j|(X_j - X_nj)}{b_n} \xrightarrow{P} 0 \tag{2.3}$$

obtains.

Proof : For arbitrary $\epsilon > 0$,

$$\begin{aligned} P\left\{\frac{\left|\sum_{j=1}^{n} |a_j|(X_j - X_{nj})\right|}{b_n} > \epsilon\right\} &\leq P\left\{\cup_{j=1}^{n}[X_j \neq X_{nj}]\right\} \\ &\leq \sum_{j=1}^{n} P\{|X_j| > c_n\} \\ &\leq P\{|X_1| > c_n\} = o(1) \end{aligned}$$

by (2.2). Hence the desired result follows.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with the same distribution function $F(x)$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants with $a_n \neq 0,\ 0 < b_n \to \infty\ n \geq 1$, and suppose that either*

$$\frac{b_n}{|a_n|} \uparrow,\ \frac{b_n}{n|a_n|} \downarrow,\ \sum_{j=1}^{n} |a_j|^2 = o(b_n^2),\ \textit{and}\ \sum_{j=1}^{n} \frac{b_j^2}{j^2|a_j|^2} = O\left(\frac{b_n^2}{\sum_{j=1}^{n} |a_j|^2}\right) \tag{2.4}$$

or

$$\frac{b_n}{|a_n|} \uparrow,\quad \frac{b_n}{n|a_n|} \to \infty,$$

$$\sum_{j=1}^{n} |a_j|^2 = O(n|a_n|^2),\ \textit{and}\ \sum_{j=1}^{n} \frac{b_j^2}{j^2|a_j|^2} = O\left(\frac{b_n^2}{\sum_{j=1}^{n} |a_j|^2}\right) \tag{2.5}$$

or

$$\frac{b_n}{n|a_n|},\ \textit{and}\ \sum_{j=1}^{n} |a_j|^2 = O(n|a_n|^2) \tag{2.6}$$

hold. Then (2.2) entails that

$$\sum_{j=1}^{n} |a_j|^2 P\{|X_1| > c_n\} = o(|a_n|^2) \tag{2.7}$$

and

$$\sum_{j=1}^{n} |a_j|^2 E|X_1|^2 I(|X_1| \leq c_n) = o(b_n^2) \tag{2.8}$$

hold, where $c_n = \frac{b_n}{|a_n|}$.

Proof : We will use the idea of the proof of Theorem in Rosalsky and Taylor (1991). To prove (2.7), observe that under (2.4)

$$\begin{aligned} &\frac{1}{|a_n|^2} \sum_{j=1}^{n} |a_j|^2 P\{|X_1| > c_n\} \\ &\qquad \leq \frac{C b_n^2 P\{|X_1| > c_n\}}{|a_n|^2 \sum_{j=1} n(c_j^2/j^2)} \\ &\qquad \leq \frac{C c_n^2 P\{|X_1| > c_n\}}{n(c_n^2/n^2)} = CnP\{|X_1| > c_n\} = o(1), \end{aligned}$$

where C is a positive constant, by (2.2). On the other hand, under (2.5) or (2.6)

$$\frac{1}{|a_n|^2} \sum_{j=1}^{n} |a_j|^2 P\{|X_1| > c_n\} \leq CnP\{|X_1| > c_n\} = o(1)$$

again by (2.2) and so (2.7) obtains. To prove (2.8). note that $c_n \uparrow$ under (2.4), (2.5) or (2.6) and that (2.5) and (2.6) individually ensure

$$\sum_{j=1}^{n} |a_j|^2 = o(b_n^2). \tag{2.9}$$

Thus (2.9) holds under (2.4), (2.5) or (2.6). Let $c_0 = 0$ *and* $d_n = c_n/n,\ n \geq 1$. Define an array $\{B_{nk},\ 0 \leq k \leq n,\ n \geq 1\}$ by

$$B_{nk} = \begin{cases} \left(\frac{1}{b_n^2} \sum_{j=1}^{n} |a_j|^2\right) \left(\frac{c_{k+1}^2 - c_k^2}{k}\right) & for\ 1 \leq k \leq n-1,\ n \geq 2 \\ 0, & for\ k = 0,\ n,\ n \geq 1. \end{cases}$$

It will now be shown that $\{B_{nk},\ 0 \leq k \leq n,\ n \geq 1\}$ is a Toeplitz array, that is,

$$\sum_{k=0}^{n} |B_{nk}| = O(1) \tag{2.10}$$

and

$$B_{nk}\ to0\ \ as\ n\to\infty\ for\ all\ fixed\ k\geq 0. \tag{2.11}$$

Clearly (2.9) entails (2.11). To verify (2.10), note that $B_{nk}\geq 0,\ 0\leq k\leq n,\ n\geq 1$, since $c_n\uparrow$ and that $k\geq 1$,

$$\frac{c_{k+1}^2-c_k^2}{k}=\frac{(k+1)^2d_{k+1}^2-k^2d_k^2}{k}\leq (k+3)d_{k+1}^2-kd_k^2 \tag{2.12}$$

Then under (2.4), since $d_n\downarrow$, it follows from (2.12) that

$$\frac{c_{k+1}^2-c_k^2}{k}\leq 3d_k^2=\frac{3c_k^2}{k^2},\ k\geq 1.$$

Hence, for $n\geq 2$,

$$\sum_{k=0}^{n}B_{nk}\leq\left(\frac{3}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}\frac{c_k^2}{k^2}\right)=O(1)$$

and so (2.10) holds. Now under (2.5) or (2.6), for $n\geq 2$,

$$\begin{aligned}
\sum_{k=0}^{n}B_{nk} &\leq \left(\frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}((k+3)d_{k+1}^2-kd_k^2)\right)\quad (by(2.12))\\
&\leq \left(\frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}((k+1)d_{k+1}^2-kd_k^2)\right)\\
&\qquad +\left(\frac{3}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}d_{k+1}^2\right)\\
&\leq \frac{Cn}{c_n^2}nd_n^2+\left(\frac{3}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}d_{k+1}^2\right)\\
&= C+\left(\frac{3}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}d_{k+1}^2\right),
\end{aligned} \tag{2.13}$$

where C is a positive constant.
Under (2.5), for $n\geq 2$,

$$\left(\frac{3}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}d_{k+1}^2\right)\leq\left(\frac{C}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n}\frac{c_k^2}{k^2}\right)=O(1).$$

Under (2.6), for $n \geq 2$,

$$\left(\frac{3}{b_n^2}\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{k=1}^{n-1}d_{k+1}^2\right) \leq \frac{3d_n^2}{b_n^2}\left(\sum_{j=1}^{n}|a_j|^2\right)(n-1) \quad (since\ d_n \uparrow)$$
$$= O(1).$$

Thus, under (2.5) or (2.6), recalling (2.13)

$$\sum_{k=0}^{n} B_{nk} = O(1)$$

and again (2.10) holds, there by proving that $\{B_{nk},\ 0 \leq k \leq n,\ n \geq 1\}$ is a Toeplitz array. By (2.2) and the Toeplitz lemma (see, e.g., Knopp p74 or Loeve p250)

$$\sum_{k=0}^{n} B_{nk} kP\{|X_1| > c_k\} = o(1). \tag{2.14}$$

Next, note that

$$\begin{aligned}
&\frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2 E|X_1|^2 I(|X_1| \leq c_n) \\
&= \frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2 \sum_{k=1}^{n} E|X_1|^2 I(c_{k-1} < |X_1| \leq c_k) \\
&\leq \frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2 \sum_{k=1}^{n} c_k^2 P\{c_{k-1} \leq |X_1| \leq c_k\} \\
&= \frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2 \sum_{k=1}^{n} c_k^2 (P\{|X_1| > c_{k-1}\} - P\{|X_1| > c_k\}) \\
&= \frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2 (c_1^2(P\{|X_1| > 0\} - c_n^2 P\{|X_1| > c_n\} \\
&\qquad\qquad + \sum_{k=1}^{n-1}(c_{k+1}^2 - c_k^2)P\{|X_1| > c_k\}) \\
&\leq \frac{1}{b_n^2}\sum_{j=1}^{n}|a_j|^2 \sum_{k=1}^{n-1}\frac{c_{k+1}^2 - c_k^2}{k} kP\{|X_1| > c_k\} + o(1) \\
&= \sum_{k=0}^{n} B_{nk} kP\{|X_1| > c_k\} + o(1) \\
&= o(1) \quad (by\ (2.14)),
\end{aligned}$$

thereby establishing (2.8) and the proof is complete.

Remark Note that assumption of independence(or pairwise NQD) is not required in Lemmas 2.2-2.3

3. MAIN RESULTS

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution function $F(x)$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $a_n \neq 0$, $0 < b_n \to \infty$, $n \geq 1$ and suppose that either (2.4) or (2.5) or (2.6) hold. If (2.2) holds then the WLLN*

$$\frac{\sum_{j=1}^{n} |a_j|(X_{nj} - EX_{nj})}{b_n} \xrightarrow{P} 0 \tag{3.1}$$

obtains, where X_{nj} is defined as in Lemma 2.2.

Proof : First note that $\{|a_j|(X_{nj} - EX_{nj})\}$'s are pairwise NQD by Lemma 2.1. It follows from Lemma 2.3 and pairwise negative quadrant dependence condition that for arbitrary $\epsilon > 0$,

$$\begin{aligned}
&P\left\{\frac{\left|\sum_{j=1}^{n} |a_j|(X_{nj} - EX_{nj})\right|}{b_n} > \epsilon\right\} \\
&\leq \frac{1}{\epsilon^2 b_n^2} E\left|\sum_{j=1}^{n} |a_j|(X_{nj} - EX_{nj})\right|^2 \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} |a_j|^2 E(X_{nj} - EX_{nj})^2 \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} |a_j|^2 E(X_{nj}^2) \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} |a_j|^2 EX_j^2 I(|X_j| \leq c_n) + \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} |a_j|^2 c_n^2 P\{|X_j| > c_n\} \\
&\leq \frac{1}{\epsilon^2 |a_n|^2} \sum_{j=1}^{n} |a_j|^2 P\{|X_1| > c_n\} + \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} |a_j|^2 |X_1|^2 I(|X_1| \leq c_n) \\
&= o(1),
\end{aligned}$$

by (2.7) and (2.8). Thus the desired result (3.1) follows.

Finally from Lemma 2.2 and Theorem 3.1 we obtain the following result :

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution function $F(x)$. Let $\{a_n, n \geq 1\}$ and $\{b_n; n \geq 1\}$ be constants with $a_n \neq 0,\ 0 < b_n \to \infty,\ n \geq 1$, and suppose that either (2.4) or (2.5) or (2.6) holds. If (2) holds then the WLLN*

$$\frac{\sum_{j=1}^{n} |a_j|(X_j - EX_{nj})}{b_n} \xrightarrow{P} 0 \tag{3.2}$$

obtains, where $X_{nj} = X_j I(|X_j| \leq c_n) + c_n I(X_j > c_n) - c_n I(X_j < -c_n),\ 1 \leq j \leq n,\ n \geq 1$ and $c_n = \frac{b_n}{|a_n|}$.

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