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On the Autocovariance Function of INAR(1) Process with a Negative Binomial or a Poisson marginal

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Abstract

We show asymptotic normality of the sample mean and sample autocovariances function generated from first-order integer valued autoregressive process(INAR(1)) with a negative binomial or a Poisson marginal. It is shown that a Poisson INAR(1) process is a special case of a negative binomial INAR(1) process.

Key Words: Negative binomial and Poisson INAR(1) processes; asymptotic normality; sample autocovariance.

1. Introduction

It frequently occurs that a sequence of count observations exhibits dependency which must be appropriately modeled. Several authors have studied integer valued analogues of ARMA models. Mckenzie (1986) proposed autoregressive moving-average processes with negative binomial and geometric marginal distributions as the counter part of models for continuous time stationary processes with gamma and exponential marginals. Alzaid and AL-Osh (1990), and Mckenzie (1988) studied ARMA processes with Poisson marginals. Park and Kim (1997), and McCormick and Park (1997) investigated asymptotic properties of sample autocovariance function in MA processes with Poisson marginals. In this paper we study the asymptotic properties of sample autocovariance function in a negative binomial/Poisson autoregressive process with order 1 which is introduced by Mckenzie (1986,1988). The model is defined as

$$X_n = \alpha * X_{n-1} + W_n \tag{1.1}$$

where $X_n, n = 0, \pm 1, \pm 2, \cdots$ is a stationary sequence with a negative binomial distribution($NB(r,\theta)$) or a Poisson(λ). The star operation is referred to as binomial thinning $\alpha * X = \sum_{i=1}^{X} B_i(\alpha)$, where $B_i(\alpha)$ are i.i.d Bernoulli r.v's independent of X with $P(B_i(\alpha) = 1) = \alpha$. As in a usual continuous $AR(1)$ model, X_{n-1} is independent of W_n and $\{W_n, n = 0, \pm 1, \pm 2, \cdots\}$ are i.i.d r.v's.

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Model (1.1) can be interpreted as a queueing process when X_n is defined as the queue size in a system at time n. In particular (1.1) is equivalent to the $M/M/\infty$ queueing process when X_n is a Poisson r.v (Mckenzie, 1988). Furthermore, (1.1) can be also interpreted a branching process with offspring mean α and immigration r.v, W_n.

It can be easily verified by the properties of binomial thinning and the stationarity of X_n that

$$X_n = \alpha * X_{n-1} + W_n \stackrel{\text{d}}{=} \sum_{i=0}^{\infty} \alpha^i * W_{n-i} \tag{1.2}$$

Let X_n be the number of objects in a system at time n. And also let $\mathcal{W}_n$ be a set of objects at time n and W_n be the number of objects in $\mathcal{W}_n$. Define $Y_{ji}^{(n)} = 1$ if the jth object of $\mathcal{W}_n$ is in the system at time $n+i$ and 0 otherwise. Then $\alpha^i * W_n$ can be defined as $\sum_{j=1}^{W_n} Y_{j,i}^{(n)}$ where, for each fixed n and i, $Y_{j,i}^{(n)}$ are i.i.d Bernoulli sequence with $P[Y_{j,i}^{(n)} = 1] = \alpha_i$. Then by (1.1) and (1.2) the following construction is an alternative representation of the binomial thinning; for each fixed n and j, $P[Y_{j,i_1}^{(n)} = 1, Y_{j,i_2}^{(n)} = 1, \cdots, Y_{j,i_k}^{(n)} = 1] = \alpha^{i_k}, i_1 \le i_2 \le \cdots \le i_k$. Observe that from (1.1), the event $(Y_{j,i}^{(n)} = 1)$ can be occurred only when $(Y_{j,i-1}^{(n)} = 1)$. This means that the event $(Y_{j,i_1}^{(n)} = 1, Y_{j,i_2}^{(n)} = 1, \cdots, Y_{j,i_k}^{(n)} = 1)$ is equivalent to the event $(Y_{j,i_k}^{(n)} = 1)$ for $0 \le i_1 \le i_2 \le \cdots \le i_k < \infty$. Therefore, we have $P[Y_{j,i_k}^{(n)} = 1] = \alpha^{i_k}$.

Summarizing these, we have

$$\begin{aligned} &P[Y_{j,i_1}^{(n)} = 1, Y_{j,i_2}^{(n)} = 1, \cdots, Y_{j,i_k}^{(n)} = 1] = P[Y_{j,i_k}^{(n)} = 1] = \alpha^{i_k} \\ &\text{and } Y_{j,i}^{(n)} \text{and} Y_{j',i'}^{(n')} \text{are independent whenever} (n,j) \neq (n',j'). \end{aligned} \tag{1.3}$$

Note that the independence between $Y_{j,i}^{(n)}$ and $Y_{j',i'}^{(n')}$ for $(n,j) \neq (n',j')$ is obtained by the independent assumption for $\{W_n\}$.

Let X_n^{NB} and W_n^{NB} be X_n and W_n in (1.2), respectively when $X_n \sim NB(r,\theta)$. Similarly, X_n^{PN} and W_n^{PN} are the Poisson section of X_n and W_n, respectively in which $X_n \sim Poisson(\lambda)$. Then the following moment calculation is easily found:

$$\begin{aligned} E(X_n^{NB}) &= \frac{r(1-\theta)}{\theta}, E(X_n^{PN}) = \lambda \\ \textit{and } \gamma^{NB}(h) &= Cov(X_n^{NB}, X_{n\pm h}^{NB}) = \alpha^{|h|}\frac{r(1-\theta)}{\theta^2}, \\ \gamma^{PN}(h) &= Cov(X_n^{PN}, X_{n\pm h}^{PN}) = \alpha^{|h|}\lambda. \end{aligned} \tag{1.4}$$

From (1.4), we obtain the nonnegative autocorrelation function

$$\rho^{NB}(\pm h) = \rho^{PN}(\pm h) = \alpha^{|h|} \tag{1.5}$$

for the respective INAR(1) processes defined in (1.1) or (1.2). Note that $E(X_n^{NB}) \to E(X_n^{PN})$ and $\gamma^{NB}(h) \to \gamma^{PN}(h)$ as $r(1-\theta) \to \lambda$ and $\theta \to 1$. This can be easily inferred from the fact that $X^{NB} \xrightarrow{d} X^{PN}$ as $r(1-\theta) \to \lambda$ and $\theta \to 1$.

This paper is summarized into the following 2 sections. In Section 2 we first calculate the moments $E\prod_{i=1}^{k} X_{t_i}^{NB}$ for $k = 2, 3, 4$ and then show that, for each k, $E\prod_{i=1}^{k}(X_{t_i}^{NB}) \to E\prod_{i=1}^{k}(X_{t_i}^{PN})$ as $r(1-\theta) \to \lambda$ and $\theta \to 1$. Using these results, we also derive the variances of the sample mean and sample auto-covariances for the negative binomial and Poisson INAR(1) processes.The sample auto-covariances used in this section 2 are normed by the respective population means instead of their own sample means. Section 3 shows that the sample mean and sample auto-covariances deviated by the sample mean jointly converge to a normal vector when X_n follows a negative binomial distribution. Then we show that the asymptotic results shown with a negative binomial sequence can be applied to the case of a Poisson INAR(1) process by letting $r(1-\theta) \to \lambda$ and $\theta \to 1$. The proofs of all results in section 2 and 3 are given in Appendix.

2. Relationship between Poisson and Negative Binomial INAR(1) processes

Suppose that X is a random variable taking values in $N^+ = \{0, 1, \ldots\}$ with probability generating function $P(s) = E(s^X)$. Then the distribution of X belongs to discrete self-decomposable class if

$$P(s) = P(1 - \alpha + \alpha + s)P_\alpha(s), \;\; |s| \le 1, \;\; \alpha \in (0, 1) \tag{2.1}$$

where $P_\alpha(s)$ is also a p.g.f.. It is easy to see that $NB(r,\theta)$ and $Poisson(\lambda)$ INAR(1) processes defined in (1.1) or (1.2) satisfy (2.1). This implies that $NB(r,\theta)$ and Poisson (λ) belong to the discrete self-decomposable class and produce the following:

Lemma 2.1. *W_n^{NB} can be represented by*

$$W_n^{NB} = \begin{cases} 0, & \textit{with probability } \alpha^r \\ NB(j,\theta), & \textit{with probability } \binom{r}{r-j}\alpha^{r-j}(1-\alpha)^j, \; 1 \le j \le r. \end{cases} \tag{2.2}$$

And also $\alpha * X^{NB} \xrightarrow{d} \alpha * X^{PN}$ *and* $W^{NB} \xrightarrow{d} W^{PN}$ *as* $r(1-\theta) \to \lambda$ *and* $\theta \to 1$*. Moreover, the kth moment of* W^{NB} *converges to that of* W^{PN} *for* $k = 1, 2, 3, 4$ *as* $r(1-\theta) \to \lambda$ *and* $\theta \to 1$*.*

Let $\mu_1 = E(X^{NB})$ and $\mu_2 = E(X^{PN})$ throughout this paper and define

$$\begin{aligned}\tilde{\gamma}^{NB}(h) &= \frac{1}{n}\sum_{t=1}^{n}(X_t^{NB}-\mu_1)(X_{t+h}^{NB}-\mu_1) \text{ and} \\ \tilde{\gamma}^{PN}(h) &= \frac{1}{n}\sum_{t=1}^{n}(X_t^{PN}-\mu_2)(X_{t+h}^{PN}-\mu_2)\end{aligned}$$

where $\bar{X}_n^{NB} = \frac{1}{n}\sum_{t=1}^{n} X_t^{NB}$ and $\bar{X}_n^{PN} = \frac{1}{n}\sum_{t=1}^{n} X_t^{PN}$. Then the following result is given by

Lemma 2.2. *Suppose that* $\{X_n^{NB}\}$ *and* $\{X_n^{PN}\}$ *follow INAR(1) processes given in (1.1) or (1.2) . Then*

(i) $\lim_{n\to\infty} nVar(\bar{X}_n^{NB}) = \frac{r\bar{\theta}}{\theta^2}\frac{1+\alpha}{1-\alpha}$,

(ii)

$$\lim_{n\to\infty} nCov(\bar{X}_n^{NB}, \tilde{\gamma}^{NB}(h)) \tag{2.3}$$

$$= \begin{cases} \frac{r\bar{\theta}}{\theta^3(1-\alpha^2)}[(2\theta(1+\alpha)+\bar{\theta}(1+\theta)(2+\alpha)], & \text{if } h=0, \\ \frac{2r\bar{\theta}}{\theta^2}\frac{\alpha}{1-\alpha} + \frac{r\bar{\theta}^2(1+\theta)\alpha}{\theta^3(1-\alpha)}(\frac{2}{1+\alpha}+\alpha) & \text{if } h=1, \\ \frac{r(1-\theta)}{\theta^2}\alpha^h(\frac{1+\alpha}{1-\alpha}+h) + \frac{r\bar{\theta}^2(1+\theta)\alpha^h}{\theta^3(1-\alpha)} & \\ \times[\frac{2}{1+\alpha}+\alpha(1-\alpha^{h-2})+\alpha^{2h}] & \text{if } h \geq 2 \end{cases}$$

(iii)

$$\begin{aligned}\lim_{\substack{r(1-\theta)\to\lambda \\ \theta\to 1}} \lim_{n\to\infty} nVar(\bar{X}_n^{NB}) &= \lim_{n\to\infty} nVar(\bar{X}_n^{PN}) \\ \lim_{\substack{r(1-\theta)\to\lambda \\ \theta\to 1}} \lim_{n\to\infty} nCov(\bar{X}_n^{NB}, \tilde{\gamma}^{NB}(h)) &= \lim_{n\to\infty} nCov(\bar{X}_n^{PN}, \tilde{\gamma}^{PN}(h)).\end{aligned}$$

Next, we derive $nCov(\tilde{\gamma}^{NB}(h_1), \tilde{\gamma}^{NB}(h_2))$ which will be used for obtaining the covariances of sample auto-covariance functions.

Lemma 2.3. *For $h_1 \geq h_2$*

$$
\begin{aligned}
&(1/b_1)\lim_{n\to\infty} nCov(\tilde{\gamma}^{NB}(h_1), \tilde{\gamma}^{NB}(h_2)) \\
&= \begin{cases}
b_2 + (1+\alpha)^2 + \{2b_2 - 2\alpha - b_3 + 4\alpha^{h_2}(1+\alpha)^2 \\
\qquad -2\alpha^{2h_2+1} + r(1-\alpha^2)\big((1+h_1-h_2)\alpha^{-h_2} \\
\qquad +(1+h_1+h_2)\alpha^{h_2}\big) + 2\alpha^2 r(\alpha^{-h_2}+\alpha^{h_2})\}\bar{\theta} \\
\qquad +\{b_2 - (1+\alpha^2) + 2b_3 + 2\alpha^{h_2}(\alpha^2 - 4\alpha + 1)\}\bar{\theta}^2, & \text{if } h_2 \geq 1, \\
h_1(1-\alpha^2) + (1+\alpha)^2 + \{4(1+\alpha)^2 - 4\alpha^{h_1+1} \\
\qquad -2h_1(1-\alpha^2) + 2r(h_1(1-\alpha^2) + (1+\alpha^2))\}\bar{\theta} \\
\qquad +\{h_1(1-\alpha^2) + 4\alpha^{h_1+1} + 1 - 6\alpha + \alpha^2\}\bar{\theta}^2, & \text{if } h_2 = 0,
\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 &= \frac{\alpha^{h_1}}{1-\alpha^2}\frac{r\bar{\theta}}{\theta^4} \\
b_2 &= (1-\alpha^2)(h_1-h_2) \\
b_3 &= 2\{\alpha^{h_1-h_2}(1+\alpha) - \alpha^{2h_1-2h_2}(1-\alpha^{2h_2}) - \alpha^{h_1+h_2}(1-\alpha)\}.
\end{aligned}
$$

And

$$
\lim_{n\to\infty} nCov(\tilde{\gamma}^{PN}(h_1), \tilde{\gamma}^{PN}(h_2)) = \lim_{\substack{r(1-\theta)\to\lambda \\ \theta\to 1}} \lim_{n\to\infty} nCov(\tilde{\gamma}^{NB}(h_1), \tilde{\gamma}^{NB}(h_2)).
$$

3. Asymptotic distributions

In this section, we establish asymptotic results for the sample mean and sample auto-covariance functions when the sequence $\{X_n\}$ follows the $NB(r,\theta)$ INAR(1) process described in (1.1) or (1.2). Then we show that the asymptotic results pertaining to the INAR(1) process with a negative binomial marginals are reduced to a special case when the sequence $\{X_n\}$ is the INAR(1) process with Poisson marginals. Theorem 3.1 and 3.2 below are the the results for negative binomial marginals and Corollary 3.1 is for Poisson marginals. Let

$$
V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}' & V_{22} \end{pmatrix}
$$

be the $(h+1) \times (h+1)$ dimensional matrix with

$$\begin{aligned} V_{11} &= \lim_{n\to\infty} nVar(\bar{X}_n^{NB}), \\ V_{12}' &= \lim_{n\to\infty} nCov(\bar{X}_n^{NB}, \tilde{\gamma}^{NB}(h_1)), \text{and} \\ V_{22} &= \lim_{n\to\infty} nCov(\tilde{\gamma}^{NB}(h_1), \tilde{\gamma}^{NB}(h_2)), \quad h_1, h_2 = 0, \cdots, h. \end{aligned}$$

Then we have

Theorem 3.1. *Suppose that $X_n^{NB} = \alpha * X_{n-1}^{NB} + W_n^{NB}$, where X_n^{NB} is according to $NB(r,\theta)$. Then*

$$\sqrt{n}\begin{pmatrix} \bar{X}_n^{NB} \\ \tilde{\gamma}^{NB}(0) \\ \tilde{\gamma}^{NB}(1) \\ \vdots \\ \tilde{\gamma}^{NB}(h) \end{pmatrix} \sim AN\left(\begin{pmatrix} r\bar{\theta}/\theta \\ \gamma^{NB}(0) \\ \gamma^{NB}(1) \\ \vdots \\ \gamma^{NB}(h) \end{pmatrix}, V\right).$$

Now, let $\hat{\gamma}^{NB}(p) = \frac{1}{n}\sum_{t=1}^{n-p}(X_t^{NB} - \bar{X}_n^{NB})(X_{t+p}^{NB} - \bar{X}_n^{NB})$, $0 \le p \le h$. Then we have the following:

Theorem 3.2. *Under the same model as in Theorem 3.1, we have*

$$\sqrt{n}\begin{pmatrix} \bar{X}_n^{NB} \\ \hat{\gamma}^{NB}(0) \\ \hat{\gamma}^{NB}(1) \\ \vdots \\ \hat{\gamma}^{NB}(h) \end{pmatrix} \sim AN\left(\begin{pmatrix} r\bar{\theta}/\theta \\ \gamma^{NB}(0) \\ \gamma^{NB}(1) \\ \vdots \\ \gamma^{NB}(h) \end{pmatrix}, V\right).$$

For a Poisson INAR(1) process, define $\hat{\gamma}^{PN}(p) = \frac{1}{n}\sum_{t=1}^{n-p}(X_t^{PN} - \bar{X}_n^{PN})(X_{t+p}^{PN} - \bar{X}_n^{PN})$. Then we have the following corollary by Lemma 2.2, Lemma 2.3 and the exactly same arguments as in Theorems 3.1 and 3.2.

Corollary 3.1. *Let X_n^{PN} be a Poisson INAR(1) process defined as (1.1) or (1.2)* *with* $X_n^{PN} \sim Poisson(\lambda)$. *Then,*

$$\sqrt{n}\begin{pmatrix} \bar{X}_n^{PN} \\ \hat{\gamma}^{PN}(0) \\ \hat{\gamma}^{PN}(1) \\ \vdots \\ \hat{\gamma}^{PN}(h) \end{pmatrix} \sim AN\left(\begin{pmatrix} \lambda \\ \gamma^{PN}(0) \\ \gamma^{PN}(1) \\ \vdots \\ \gamma^{PN}(h) \end{pmatrix}, V^{PN}\right)$$

where $\lambda = \lim_{\substack{r(1-\theta)\to\lambda \\ \theta\to 1}} E(X^{NB})$, $\gamma^{PN}(h) = \lim_{\substack{r(1-\theta)\to\lambda \\ \theta\to 1}} \gamma^{NB}(h)$, *and* $V^{PN} = \lim_{\substack{r(1-\theta)\to\lambda \\ \theta\to 1}} V$.

Theorem 3.2 and Corollary 3.1 can be applied to obtain asymptotic distributions for natural estimators of parameters involved in model (1.1). More precisely, let f_1, f_2 and f_3 be a function of $\mu_1, \gamma^{NB}(0)$ and $\gamma^{NB}(1)$ such a way that $f_1 = r/(\mu_1 + r)$, $f_2 = \mu_1^2/(\gamma^{NB}(0) - \mu_1)$, and $f_3 = \gamma^{NB}(1)/\gamma^{NB}(0)$. Then, since $\mu_1 = r\bar{\theta}/\theta$, $\gamma^{NB}(0) = r\bar{\theta}/\theta^2$, and $\gamma^{NB}(1) = \alpha r\bar{\theta}/\theta^2$, $f_1(\cdot) = \theta$, $f_2(\cdot) = r$, and $f_3(\cdot) = \alpha$. Hence $\hat{f}_1$,$\hat{f}_2$ and $\hat{f}_3$ are estimators of θ, r, and α, respectively in which $\hat{f}_i = f_i(\bar{X}_n^{NB}, \hat{\gamma}^{NB}(0), \hat{\gamma}^{NB}(1))$, $i = 1, 2, 3$. Similarly, for a Poisson INAR(1) process, define $g_1 = \mu_2$ and $g_2 = \gamma^{PN}(0)/\gamma^{PN}(1)$. Then $\hat{g}_1 = \bar{X}_n^{PN}$ and $\hat{g}_2 = \hat{\gamma}^{PN}(0)/\hat{\gamma}^{PN}(1)$ are natural moment estimators of λ and α, respectively. It is an easy exercise to obtain asymptotic distribution of $\hat{f}_1, \hat{f}_2$ and $\hat{f}_3$ by Theorem 3.2, and $\hat{g}_1, \hat{g}_2$ by Corollary 3.1 since all functions of f_1, f_2 and f_3, and g_1 and g_2 are continuous.

Appendix

Proof of Lemma 2.1. Under model (1.1), we have $E(s^{X_n^{NB}}) = \left(\frac{\theta}{1-\bar{\theta}s}\right)$ and $E(s^{\alpha * X_{n-1}^{NB}}) = \left(\frac{\theta}{1-\bar{\theta}(1-\alpha+\alpha s)}\right)^r \equiv P_*^{NB}(s)$ where $\bar{\theta} = 1-\theta$. Thus by independence of $\alpha * X_{n-1}^{NB}$ and W_n^{NB} ,

$$E(s^{W_n^{NB}}) = \left[\frac{1-\bar{\theta}(1-\alpha+\alpha s)}{1-\bar{\theta}}\right]^r \equiv P_W^{NB}(s).$$

On the other hand, under (2.2),

$$E(s^{W_n^{NB}}) \;=\; \alpha^r + \sum_{j=1}^{r} \binom{r}{r-j}\alpha^{r-j}(1-\alpha)^j \left(\frac{\theta}{1-\bar{\theta}r}\right)^j = P_W^{NB}(s)$$

This shows the first claim.

The second claim is immediate since $P_*^{NB}(s) \to P_*^{PN}(s)$ and $P_W^{NB}(s) \to P_W^{PN}(s)$ as $r(1-\theta) \to \lambda$ and $\theta \to 1$. Finally, from (2.2), the followings are easily obtained:

$$\begin{aligned}
E(W_n^{NB}) &= \frac{r\bar{\theta}}{\theta}(1-\alpha), \\
E(W_n^{NB})^2 &= \frac{r\bar{\theta}}{\theta^2}(1-\alpha)(1+\bar{\theta}\alpha+\bar{\theta}r(1-\alpha)), \\
E(W_n^{NB})^3 &= \frac{r\bar{\theta}}{\theta^3}(1-\alpha)(1+\bar{\theta}+3\alpha\bar{\theta}-\alpha(1-2\alpha)\bar{\theta}^2+3r\bar{\theta}(1-\alpha)(1+\bar{\theta}\alpha) \\
&\qquad +(r\bar{\theta}(1-\alpha))^2), \qquad \text{and} \\
E(W_n^{NB})^4 &= \frac{r\bar{\theta}(1-\alpha)}{\theta^4}(1+4\bar{\theta}+\bar{\theta}^2+7\alpha\bar{\theta}+\alpha\bar{\theta}^3-2\alpha\bar{\theta}^2+12\alpha^2\bar{\theta}^2 \\
&\qquad -6\alpha^2\bar{\theta}^3+6\alpha^3\bar{\theta}^3) \\
&\quad +\frac{r^2\bar{\theta}^2(1-\alpha)^2}{\theta^4}(7+4\bar{\theta}+18\alpha\bar{\theta}+7\alpha\bar{\theta}^2-4\bar{\theta}^2(1-\alpha)\alpha) \\
&\quad +6\frac{r^3\bar{\theta}^3(1-\alpha)^3}{\theta^4}(1+\alpha\bar{\theta})+\frac{\bar{\theta}^2}{\theta^4}(r(1-\alpha))^4. \qquad \text{(A.1)}
\end{aligned}$$

Hence, the result holds by letting $r(1-\theta) = \lambda$ and $\theta = 1$ in (A.1) since $W_n^{PN} \sim Poisson((1-\alpha)\lambda)$. This completes the proof.

In order to prove Lemma 2.2, we need to calculate the moments of $E\prod_{i=1}^k X_{t_i}^{NB}$ and $E\prod_{i=1}^k X_{t_i}^{PN}$ for $k = 2, 3, 4$. The moments are necessary to derive limiting distribution of sample mean and sample autocovariance measured from the process defined in (1.1) or (1.2). Throughout this Appendix, denote $\mu_1 = E(X^{NB})$ and $\mu_2 = E(X^{PN})$.

Lemma A.1. Let X_t^{NB}, $t \geq 1$ be the process defined in (1.1) or (1.2) with a $NB(r,\theta)$ marginal. Then, for $\xi \geq h_1$, $h_1, h_2 \geq 0$, as $r(1-\theta) \to \lambda$ and $\theta \to 1$,

$$\begin{aligned}
&E(X_t^{NB}X_{t+h_1}^{NB}) \to E(X_t^{PN}X_{t+h_1}^{PN}), \\
&E(X_t^{NB}X_{t+h_1}^{NB}X_{t+h_1+h_2}^{NB}) \to E(X_t^{PN}X_{t+h_1}^{PN}X_{t+h_1+h_2}^{PN}), \text{ and} \\
&E((X_t^{NB}-\mu_1)(X_{t+h_1}^{NB}-\mu_1)(X_{t+\xi}^{NB}-\mu_1)(X_{t+\xi+h_2}^{NB}-\mu_1) \to \\
&E((X_t^{PN}-\mu_2)(X_{t+h_1}^{PN}-\mu_2)(X_{t+\xi}^{PN}-\mu_2)(X_{t+\xi+h_2}^{PN}-\mu_2).
\end{aligned}$$

Explicit forms of the above moments are given in the following proof.

Proof. For $k = 2$, the claim is easily obtained by observing that

$\gamma^{NB}(h_1) \to \gamma^{PN}(h_1)$ and $E(X_t^{NB}) \to E(X_t^{PN})$. Consider $k = 3$. Observe that

$$
\begin{aligned}
&E(X_t X_{t+h_1} X_{t+h_2}) \\
&= E(\sum_{i=0}^{\infty} \alpha^i * W_{t-i} \sum_{j=-h}^{\infty} \alpha^{h_1+j} * W_{t-j} \sum_{k=-h_1-h_2}^{\infty} \alpha^{h_1+h_2+k} * W_{t-k}) \\
&= E(\sum_{i=0}^{\infty} \alpha^i * W_{t-i}\alpha^{h_1+i} * W_{t-i}\alpha^{h_1+h_2+i} * W_{t-i}) \\
&+ E(\sum_{i=0}^{\infty} \alpha^i * W_{t-i}\alpha^{h_1+i} * W_{t-i} \sum_{\substack{k=-h_1-h_2 \\ k\neq i}}^{\infty} \alpha^{h_1+h_2+k} * W_{t-k}) \\
&+ E(\sum_{i=0}^{\infty} \alpha^i * W_{t-i}\alpha^{h_1+h_2+i} * W_{t-i} \sum_{\substack{j=-h_1 \\ j\neq i}}^{\infty} \alpha^{h_1+j} * W_{t-j}) \\
&+ E(\sum_{j=-h_1}^{\infty} \alpha^{h_1+j} * W_{t-j}\alpha^{h_1+h_2+j} * W_{t-j} \sum_{\substack{i=0 \\ i\neq j}}^{\infty} \alpha^i * W_{t-i}) \\
&+ E(\sum_{i\neq j\neq k} \alpha^i * W_{t-i}\alpha^{h_1+j} * W_{t-j}\alpha^{h_1+h_2+j} * W_{t-j}). \qquad \text{(A.2)}
\end{aligned}
$$

The first term in (A.2) (i.e., the case of $i = j = k$) is by (1.3)

$$
\begin{aligned}
&E(W)[\sum_{i=0}^{\infty} \alpha^{h_1+h_2+i}] \\
&+ E(W^2 - W)[\sum_{i=0}^{\infty} \alpha^{i+h_1}\alpha^{i+h_1+h_2} + \sum_{i=0}^{\infty} \alpha^i\alpha^{i+h_1} + \sum_{i=0}^{\infty} \alpha^i\alpha^{i+h_1+h_2}] \\
&+ E[W(W-1)(W-2)\sum_{i=0}^{\infty} \alpha^i\alpha^{i+h_1}\alpha^{i+h_1+h_2}].
\end{aligned}
$$

By (1.3) again, the second term in (A.2) (i.e., the case of $(i = j) \neq k$) is given by

$$
\begin{aligned}
&E[\sum_{i=0}^{\infty} \alpha^i * W_{t-i}\alpha^{h_1+i} * W_{t-i} \sum_{\substack{k=-h_1-h_2 \\ k\neq i}}^{\infty} \alpha^{h_1+h_2+k} * W_{t-k}] \\
&= \sum_{i=0}^{\infty}[E(\alpha^i * W_{t-i}\alpha^{h_1+j} * W_{t-i}]E(\sum_{\substack{k=-h_1-h_2 \\ k\neq i}}^{\infty} \alpha^{h_1+h_2+k} * W_{t-k}) \\
&= \sum_{i=0}^{\infty}[E(W)\alpha^{h_1+i} + E(W^2 - W)\alpha^i\alpha^{h_1}][\sum_{\substack{j=0 \\ j\neq h_1+h_2+i}}^{\infty} \alpha^j E(W)]
\end{aligned}
$$

The 3rd and 4th terms are also calculated by the similar method as in the second term. Finally, the 5th term(i.e, $i \neq j \neq k$) is

$$
\begin{aligned}
E & \Big(\sum_{i\neq j\neq k} \alpha^i * W_{t-i}\alpha^{h_1+j} * W_{t-j}\alpha^{h_1+h_2+j} * W_{t-j}\Big) \\
= & \sum_{i=0}^{\infty} E\alpha^i * W_{t-i} \sum_{j=-h}^{\infty} E\alpha^{h_1+j} * W_{t-j} \sum_{k=-h_1-h_2}^{\infty} E\alpha^{h_1+h_2+k} * W_{t-k} \\
- & \sum_{i=0}^{\infty} E(\alpha^i * W_{t-i}\alpha^{h_1+i} * W_{t-i}) \sum_{\substack{k=-h_1-h_2 \\ k\neq i}}^{\infty} E(\alpha^{h_1+h_2+k} * W_{t-k}) \\
- & \sum_{i=0}^{\infty} E(\alpha^i * W_{t-i}\alpha^{h_1+h_2-i} * W_{t-i}) \sum_{\substack{j=-h_1 \\ j\neq i}}^{\infty} E(\alpha^{h_1+j} * W_{t-j}) \\
- & \sum_{i=0}^{\infty} E(\alpha^i * W_{t-i}) \sum_{j=-h_1}^{\infty} E(\alpha^{h_1+j} * W_{t-j}\alpha^{h_1+h_2+j} * W_{t-j}) \\
& + \quad 2E(\sum_{i=0}^{\infty} \alpha^i * W_{t-i}\alpha^{h_1+i} * W_{t-i}\alpha^{h_1+h_2+i} * W_{t-i})
\end{aligned}
$$

Combining all calculation above yields

$$
\begin{aligned}
& E(X_t^{NB} X_{t+h_1}^{NB} X_{t+h_1+h_2}^{NB}) \\
= & \ (\frac{r\bar{\theta}}{\theta})^3 + E(W_t^{NB})\alpha^{\xi+h}\frac{1}{1-\alpha} + \frac{r\bar{\theta}}{\theta}\frac{E(W_t^{NB})}{1-\alpha}(\alpha^{\xi} + \alpha^{\xi+h} + \alpha^h) \\
+ & \ \frac{Var(W_t^{NB}) - E(W_t^{NB})}{1-\alpha^2}(\alpha^{\xi+h} + 2\alpha^{2\xi+h}) \\
+ & \ \frac{r\bar{\theta}}{\theta}\frac{Var(W_t^{NB}) - E(W_t^{NB})}{1-\alpha^2}(\alpha^{\xi} + \alpha^{\xi+h} + \alpha^h) \\
+ & \ \Big[E(W_t^{NB}(W_t^{NB}-1)(W_t^{NB}-2)) - 3E(W_t^{NB})E(W_t^{NB^2} - W_t^{NB}) \\
+ & \ 2E(W_t^{NB})^3\Big] \times \frac{\alpha^{2\xi+h}}{1-\alpha^3}. \qquad (A.3)
\end{aligned}
$$

Now, consider $k = 4$. By the similar approach as in $k = 3$, we arrive at the following result after a long computation:

$$
\begin{aligned}
&E(X_t^{NB}-\mu_1)(X_{t+h_1}^{NB}-\mu_1)(X_{t+\xi}^{NB}-\mu_1)(X_{t+\xi+h_2}^{NB}-\mu_1)\\
&=E(W_t^{NB})\frac{\alpha^{\xi+h_2}}{1-\alpha}+\frac{A_1(\theta,\alpha)}{1-\alpha^2}(\alpha^{\xi+h_2}+2\alpha^{h_1+\xi+h_2}+4\alpha^{2\xi+h_2})\\
&+\frac{A_2(\theta,\alpha)}{1-\alpha^3}(\alpha^{h_1+\xi+h_2}+2\alpha^{2\xi+h_2}+3\alpha^{h_1+2\xi+h_2})+\gamma(h_1)\gamma(h_2)\\
&+\gamma(\xi)\gamma(\xi-h_1+h_2)+\gamma(\xi+h_1)\gamma(\xi-h_2)+\frac{A_3(\theta,\alpha)}{1-\alpha^4}\alpha^{h_1+2\xi+h_2} \qquad \text{(A.4)}
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(\theta,\alpha) &= Var(W)-E(W)=\frac{r\bar{\theta}^2}{\theta^2}(1-\alpha^2),\\
A_2(\theta,\alpha) &= E(W^3-3W^2+2W)+2E^3(W)-3E(W)E(W^2-W)\\
&= \frac{r\bar{\theta}^3}{\theta^3}(1-\alpha^3), \quad \text{and}\\
A_3(\theta,\alpha) &= E(W^4-6W^3+11W^2-6W)-3(E(W^2-W))^2-6E^4(W)\\
&\quad -4E(W)E(W^3-3W^2+2W)+12E^2(W)E(W^2-W)\\
&= 6\frac{r\bar{\theta}^4}{\theta^4}(1-\alpha^4).
\end{aligned}
$$

where $W=W^{NB}$ and the third equality in each of A_1,A_2, and A_3 are computed from (A.1). This completes the proof by Lemma 2.1 since the moments given in above depend on only the moments of $E(W^{NB})^k, k=1,2,3,4$.

Proof of Lemma 2.2.

$$
\begin{aligned}
E(\bar{X}_n^{NB})^2 &= \frac{1}{n^2}E\bigg(\sum_{t=1}^{n}X_t^{NB}\sum_{s=1}^{n}X_s^{NB}\bigg)\\
&= \frac{1}{n^2}\bigg(nE(X_t^{NB})^2+2\sum_{\xi=1}^{n-1}(n-\xi)E(X_t^{NB}X_{t+\xi}^{NB})\bigg)\\
&= \frac{1}{n^2}\bigg[n^2(r\bar{\theta})^2+n\frac{r\bar{\theta}}{\theta^2}+2\sum_{\xi=1}^{n-1}(n-\xi)\frac{r\bar{\theta}}{\theta^2}\alpha^{\xi}\bigg]. \qquad \text{(A.5)}
\end{aligned}
$$

(1.4) is used for the last equality. Thus

$$
Var(\bar{X}_n^2)=\frac{1}{n}\left(\frac{r\bar{\theta}}{\theta^2}+2\sum_{\xi=1}^{n-1}\left(1-\frac{\xi}{n}\right)\frac{r\bar{\theta}}{\theta^2}\alpha^{\xi}\right)
$$

This shows (i).
For (ii), observe that for $h \geq 0$,

$$
\begin{aligned}
&E\Big(\frac{1}{n^2}\sum_{t=1}^{n} X_t^{NB}\sum_{s=1}^{n}(X_s^{NB}-\mu_1)(X_{s+h}^{NB}-\mu_1)\Big)\\
&= \frac{1}{n}\sum_{\xi=0}^{n-1}\Big(1-\frac{\xi}{n}\Big)E\left(X_t^{NB}(X_{t+\xi}^{NB}-\mu_1)(X_{t+\xi+h}^{NB}-\mu_1)\right)\\
&+ \frac{1}{n}\sum_{\xi=1}^{h-1}\Big(1-\frac{\xi}{n}\Big)E\left((X_t^{NB}-\mu_1)X_{t+\xi}^{NB}(X_{t+h}^{NB}-\mu_1)\right)\\
&+ \frac{1}{n}\sum_{\xi=h}^{n-1}\Big(1-\frac{\xi}{n}\Big)E\left((X_t^{NB}-\mu_1)(X_{t+h}^{NB}-\mu_1)X_{t+\xi}^{NB}\right). \qquad (A.6)
\end{aligned}
$$

By (1.4) and (A.2), (A.5) can be rewritten as

$$
\begin{aligned}
\frac{r\bar{\theta}}{\theta}\gamma^{NB}(h) &+ \frac{1}{n}\sum_{\xi=0}^{n-1}\Big(1-\frac{\xi}{n}\Big)\Big\{\frac{r\bar{\theta}}{\theta}\alpha^{\xi+h}+\frac{r\bar{\theta}^2}{\theta^2}(\alpha^{\xi+h}+2\alpha^{2\xi+h})+\frac{r\bar{\theta}^3}{\theta^3}\alpha^{2\xi+h}\Big\}\\
&+ \frac{1}{n}\sum_{\xi=1}^{n-1}\Big(1-\frac{\xi}{n}\Big)\Big\{\frac{r\bar{\theta}}{\theta}\alpha^{h}+\frac{r\bar{\theta}^2}{\theta^2}(\alpha^{h}+2\alpha^{\xi+h})+\frac{r\bar{\theta}^3}{\theta^3}\alpha^{\xi+h}\Big\}\\
&+ \frac{1}{n}\sum_{\xi=h}^{n-1}\Big(1-\frac{\xi}{n}\Big)\Big\{\frac{r\bar{\theta}}{\theta}\alpha^{\xi}+\frac{r\bar{\theta}^2}{\theta^2}(\alpha^{\xi}+2\alpha^{\xi+h})+\frac{r\bar{\theta}^3}{\theta^3}\alpha^{\xi+h}\Big\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lim_{n\to\infty} nCov(\bar{X}_n^{NB},\tilde{\gamma}^{NB}(h))\\
&= \sum_{\xi=0}^{\infty}\Big(\frac{r\bar{\theta}}{\theta^2}\alpha^{\xi+h}+2\frac{r\bar{\theta}^2}{\theta^3}(1+\theta)\alpha^{2\xi+h}\Big)+\sum_{\xi=1}^{h-1}\Big(\frac{r\bar{\theta}}{\theta^2}\alpha^{h}+\frac{r\bar{\theta}^2}{\theta^3}(1+\theta)\alpha^{\xi+h}\Big)\\
&+\sum_{\xi=h}^{\infty}\Big(\frac{r\bar{\theta}}{\theta^2}+\frac{r\bar{\theta}^2}{\theta^3}(1+\theta)\alpha^{\xi+h}\Big).
\end{aligned}
$$

The calculation of this summations produces (ii). Finally, Lemma A.1 together with (A.4) and (A.5) shows (iii).

Proof of Lemma 2.3. It can be shown that

$$E\Big(\frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}(X_t-\mu)(X_{t+h_1}-\mu)(X_s-\mu)(X_{s+h_2}-\mu)\Big)$$

$$\begin{aligned}
= \ & \frac{1}{n^2}\Big\{\sum_{\xi=0}^{h_1-h2-1}(n-\xi)E(X_t-\mu)(X_{t+\xi}-\mu)(X_{t+\xi+h_2}-\mu)(X_{t+h_1}-\mu) \\
&+\sum_{\xi=h_1-h_2}^{h_1-1}(n-\xi)E(X_t-\mu)(X_{t+\xi}-\mu)(X_{t+h_1}-\mu)(X_{t+\xi+h_2}-\mu) \\
&+\sum_{\xi=h_1}^{n-1}(n-\xi)E(X_t-\mu)(X_{t+h_1}-\mu)(X_{t+\xi}-\mu)(X_{t+\xi+h_2}-\mu) \\
&+\sum_{\xi=1}^{h_2-1}(n-\xi)E(X_t-\mu)(X_{t+\xi}-\mu)(X_{t+h_2}-\mu)(X_{t+\xi+h_1}-\mu) \\
&+\sum_{\xi=h_2}^{n-1}(n-\xi)E(X_t-\mu)(X_{t+h_2}-\mu)(X_{t+\xi}-\mu)(X_{t+\xi+h_1}-\mu)\Big\} \qquad (A.7)
\end{aligned}$$

where $X_t = X_t^{NB}$. Using (1.4),(A.2) and (A.3) after properly changing the order of X_t, $X_{t+\xi}$, $X_{t+\xi+h_2}$, (A.6) can be written in the form

$$\begin{aligned}
&\gamma(h_1)\gamma(h_2)+\frac{1}{n^2}\Big\{\sum_{\xi=0}^{h_1-h_2-1}(n-\xi)(a_1\alpha^{h_1}+a_2\alpha^{\xi+h_1}+2a_2\alpha^{\xi+h_1+h_2} \\
+\ &\gamma(\xi)\gamma(h_1-\xi-h_2)+\gamma(h_1-h_2)\gamma(\xi+h_2)) \\
+\ &\sum_{\xi=h_1-h_2}^{h_1-1}(n-\xi)(a_1\alpha^{\xi+h_2}+a_2\alpha^{2\xi+h_1}+2a_2\alpha^{\xi+h_1+h_2} \\
+\ &\gamma(\xi+h_2)\gamma(h_1-\xi)+\gamma(\xi)\gamma(\xi-h_1-h_2)) \\
+\ &\sum_{\xi=h_1}^{n-1}(n-\xi)(a_1\alpha^{\xi+h_2}+a_2\alpha^{\xi+h_1+h_2}+2a_2\alpha^{2\xi+h_2} \\
+\ &\gamma(\xi)\gamma(\xi-h_1-h_2)+\gamma(\xi+h_2)\gamma(\xi-h_1)) \\
+\ &\sum_{\xi=1}^{h_2-1}(n-\xi)(a_1\alpha^{\xi+h_1}+a_2\alpha^{2\xi+h_1}+2a_2\alpha^{\xi+h_1+h_2}+\gamma(\xi)\gamma(\xi-h_2+h_1) \\
+\ &\gamma(\xi+h_1)\gamma(h_2-\xi))+\sum_{\xi=h_2}^{n-1}(n-\xi)(a_1\alpha^{\xi+h_1}+a_2\alpha^{\xi+h_1+h_2}+2a_2\alpha^{2\xi+h_1}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \gamma(\xi)\gamma(\xi - h_2 + h_1) + \gamma(\xi + h_1)\gamma(\xi - h_2)) \\
&+ \quad 12\frac{r\bar{\theta}^3}{\theta^4}\sum_{\xi=0}^{n-1}(n-\xi)\alpha^{2\xi+h_1+h_2} - 6n\frac{r\bar{\theta}^3}{\theta^4}\alpha^{h_1+h_2}\Big\},
\end{aligned}
$$

where $\gamma(\cdot) = \gamma^{NB}(\cdot)$, $a_1 = \frac{r\bar{\theta}}{\theta^2}$ and $a_2 = \frac{r\bar{\theta}^2}{\theta^3}$. Subtracting $\gamma^{NB}(h_1)\gamma^{NB}(h_2)$ from (A.6) and letting them $n \to \infty$ after multiplying n, we have the results. The second claim is obvious by (A.6) and Lemma A.1.

Proof of Theorem 3.1. First, define a sequence of $(h+2)$ random vectors by

$$\mathbf{Z}_t' = (X_t^*, X_t^* X_t^*, X_t^* X_{t+1}^*, \cdots, X_t^* X_{t+h}^*)$$

where $X_t^* = \sum_{i=0}^{m} \alpha^i * W_{t-i}^{NB}$. Since $\mathbf{Z}_t$ is a strictly stationary $(m+h)$ dependent sequence, one can show by m-dependent C.L.T. and Cramer-Wold device that

$$\sqrt{n}\begin{pmatrix} \bar{X}_n^* \\ \tilde{\gamma}_m(0) \\ \tilde{\gamma}_m(1) \\ \vdots \\ \tilde{\gamma}_m(h) \end{pmatrix} \xrightarrow{\text{d}} N\left(\begin{pmatrix} \mu_m \\ \gamma_m(0) \\ \gamma_m(1) \\ \vdots \\ \gamma_m(h) \end{pmatrix}, V_m\right)$$

where $\bar{X}_n^* = \frac{1}{n}\sum_{t=1}^n X_t^*, \tilde{\gamma}_m(p) = \frac{1}{n}\sum_{t=1}^n (X_t^* - \mu_m)(X_{t+p}^* - \mu_m), \mu_m = \frac{r\bar{\theta}}{\theta}(1 - \alpha^{m+1})$, $\gamma_m(p)$ and V_m such that $\gamma_m(p) \to \gamma^{NB}(p)$ and $V_m \to V$ as $m \to \infty$. Note that

$$
\begin{aligned}
Var(\sqrt{n}(\bar{X}_n^{NB} - \bar{X}_n^*)) &= nVar\left(\frac{1}{n}\sum_{j=1}^{n}\sum_{i=m+1}^{\infty} \alpha^i * W_{j-i}^{NB}\right) \\
&= \left(\frac{r\bar{\theta}}{\theta}\alpha^{m+1} + \frac{r\bar{\theta}^2}{\theta}\alpha^{2(m+1)}\right)\left(1 + 2\frac{\alpha(1-\alpha^{n-1})}{1-\alpha}\right) \\
&\quad + O(n^{-1}).
\end{aligned}
$$

Thus we have, for $\varepsilon > 0$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup P\left[\sqrt{n}\left|\bar{X}_n^* - \bar{X}_n^{NB} - \mu_m + \frac{r\bar{\theta}}{\theta}\right| > \varepsilon\right] = 0.$$

Similarly, it can be shown by Lemma 2.4 that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup P\left[\sqrt{n}\left|\tilde{\gamma}_m(p) - \tilde{\gamma}^{NB}(p) - \gamma_m(p) + \gamma^{NB}(p)\right| > \varepsilon\right] = 0.$$

This establishes the claim by an application of Proposition 6.3.9 in Brockwell and Davis(1987) since $\mu_m \to r\bar{\theta}/\theta$ and $V_m \to V$ as $m \to \infty$.

Proof of Theorem 3.2. A simple algebra gives, for $0 \le p \le h$,

$$\sqrt{n}\big(\tilde{\gamma}^{NB}(p) - \hat{\gamma}^{NB}(p)\big) = \frac{1}{\sqrt{n}} \sum_{t=n-p+1}^{n} (X_t^{NB} - \mu_1)(X_{t+p}^{NB} - \mu_1)$$

$$+\sqrt{n}(\bar{X}_n^{NB} - \mu_1)\left[\frac{1}{n}\sum_{t=1}^{n-p} X_t^{NB} + \frac{1}{n}\sum_{t=1}^{n-p} X_{t+p}^{NB} - (1-\frac{p}{n})(\bar{X}_n^{NB} + \mu_1)\right]. \qquad (A.7)$$

The first term in (A.7) is $o_p(1)$, since

$$E\left|\sum_{t=n-p+1}^{n} (X_t^{NB} - \mu_1)(X_{t+p}^{NB} - \mu_1)\right| \le p\,\gamma^{NB}(0) + 4p\mu_1^2.$$

The second term in (A.7) is also $o_p(1)$, since $\sqrt{n}(\bar{X}_n^{NB} - \mu_1) = O_p(1)$ and $\bar{X}_n^{NB}$ converges to μ_1 in probability as $n \to \infty$ by Theorem 3.1. This completes the proof.

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