Journal of the Korean Statistical Society (2000), **29**: *3, pp 351–361*

The Cusum of Squares Test for Variance Changes in Infinite Order Autoregressive Models

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Abstract

This paper considers the problem of testing a variance change in infinite order autoregressive models. A cusum of squares test based on the residuals from an AR(q) model is constructed analogous to Inclán and Tiao (1994)'s test statistic, where q is a sequence of positive integers diverging to ∞. It is shown that under regularity conditions the limiting distribution of the test statistic is the sup of a standard Brownian bridge. Simulation results are given to illustrate the performance of the test.

Keywords: Cusum of squares test, testing a variance change, infinite order autoregressive models, a Brownian bridge.

1. Introduction

The problems concerning variance changes for random observations have attracted considerable attention from many researchers. For example, Hsu (1977) investigated a test for variance changes in iid samples when the points of change are unknown. Wichern, Miller, and Hsu (1976) considered a procedure for estimating unknown parameters in AR(1) models with a sudden change of variance at an unknown time. Abraham and Wei (1984) studied the same problem within the framework of Bayesian statistics. Baufays and Rasson (1985) proposed an iterative algorithm for obtaining maximum likelihood estimates of change points and variances for given autoregressive parameters, and conversely those of the autoregressive parameters for given change points and variances. Davis, Huang, and Yao (1995) considered a test for the changes of the parameters and the order of an autoregressive model, namely, a change from an AR(p_0) model with white noise variance σ_0^2 to an AR(p_1) model with white noise variance σ_1^2. They studied the asymptotic behavior of the Gaussian likelihood ratio statistic. Tsay (1988) proposed a method to detect outliers, level shifts, and variance changes

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in ARMA models. He used the least squares techniques and residual variance ratios for treating them in a unified manner. Meanwhile, Tang and MacNeil (1993) demonstrated that serial correlation can produce striking effects in the distribution of a change point statistic, and provided a precise adjustment that accounts for the serial correlation. For a general review of change point problems, see Csörgő and Horváth (1998).

Recently, Inclán and Tiao (1994) proposed the cusum of squares test for variance changes in iid normal r.v.'s as a centered version of the cusum of squares test of Brown, Durbin and Evans (1975), who considers the problem of testing the constancy of the coefficients in regression models. Inclán and Tiao's test is based on the following statistic:

$$IT_n = \max_{1\le k\le n} (n/2)^{1/2} \left| \frac{\sum_{j=1}^{k} \varepsilon_j^2}{\sum_{j=1}^{n} \varepsilon_j^2} - \frac{k}{n} \right|. \tag{1.1}$$

A large value of IT_n rejects the null hypothesis that no variance changes occur. The critical values are obtained asymptotically using the fact that IT_n has the same limiting distribution as $\sup_{0\le t\le 1} |B^o(t)|$, where B^o denotes a standard Brownian bridge. As well as testing variance changes, they discussed a way to estimate the positions of variance changes, using the so called D_k plot. The D_k plot is demonstrated to detect variance changes more drastically than the plot of the ordinary cusum of squares statistic. The method is based on the fact that the point where the signs of the slopes of the plot change is likely to be a change point with high probability. See also Kim, Cho and Lee who considered the same problem in GARCH (1,1) processes.

In this article, we intend to demonstrate the validity of Inclán and Tiao's test for autoregressive models since little study of the usefulness of the test in autoregressive models was made in their article. To this end, we adopt the infinite order autoregressive process as our basic model since we focus on detecting variance changes rather than the order change of autoregressive models. In fact, Lee and Park (1999) considered the same problem in the infinite order moving average context. They showed that Inclán and Tiao's test works adequately only when the data is not highly correlated. Since in general, the approach of using the AR(∞) models discards the effects caused by correlation, here we take account of the AR(∞) model.

In Section 2, we construct an Inclán-Tiao type statistic based on the residuals which are obtained by fitting a long AR(q) model to data, where q depends upon n and should be properly chosen. It is shown that the test statistic behaves asymptotically the same as the sup of a standard Brownian bridge provided q meets some regularity conditions. Simulation results are reported in Section 3, where it is shown that our test works properly even for highly correlated data. In actual practice, one may argue whether our method performs as well as the one based on the AR(q) models, where the order q is chosen through a model selection criterion such as the AIC (Akaike's information criterion). Those two methods are compared in our simulation study. Related results are also provided in Section 3.

2. Cusum of squares test

In this section, we propose a cusum of squares test in AR(∞) models by analogy with Inclán and Tiao's test statistic. Consider the AR(∞) model satisfying the difference equation

$$X_t - \sum_{j=1}^{\infty} \beta_j X_{t-j} = \varepsilon_t, \tag{2.1}$$

where the ε_t are iid r.v.'s with mean zero, unknown variance σ^2 and finite fourth moment, and the function $A(z) = 1 - \sum_{j=1}^{\infty} \beta_j z^j$ is analytic on an open neighborhood of the closed unit disk D in the complex plane and has no zeroes on D. This assumption implies that the coefficients β_j are geometrically bounded, i.e.,

$$|\beta_j| \le C\rho^j, \quad C > 0, 0 < \rho < 1. \tag{2.2}$$

The model (2.1) covers a broad class of stationary processes including causal and invertible ARMA(p, q) models (cf. Brockwell and Davis, 1990).

In order to construct the cusum of squares, we fit an AR(q) model to the data, where $q = q_n$ is a sequence of positive integers that are no more than n and diverges to ∞. Assume that $X_1, \ldots, X_n$ are observed. Solving the equation:

$$\frac{d}{d\beta_j} \sum_{t=q+1}^{n} (X_t - \sum_{j=1}^{q} \beta_j X_{t-j})^2 = 0, \quad j = 1, \ldots, q,$$

we estimate $\underset{\sim}{\beta}_n = (\beta_1, \cdots, \beta_q)'$ by $\hat{\underset{\sim}{\beta}}_n = (\hat{\beta}_1, \cdots, \hat{\beta}_q)'$, where

$$\hat{\underset{\sim}{\beta}}_n = (\sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1} \underset{\sim}{X}'_{t-1})^{-1} \sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1} X_t, \tag{2.3}$$

where $\underset{\sim}{X}_t = (X_t, \cdots, X_{t-q+1})'$. Define the residual

$$\hat{\varepsilon}_t = X_t - \hat{\underset{\sim}{\beta}}_n' \underset{\sim}{X}_{t-1}.$$

Let $\tau^2 = Var(\varepsilon_1^2)$ and let

$$\hat{\sigma}^2 = \frac{1}{n-q} \sum_{t=q+1}^{n} \hat{\varepsilon}_t^2 \quad \text{and} \quad \hat{\tau}^2 = \frac{1}{n-q} \sum_{t=q+1}^{n} \hat{\varepsilon}_t^4 - \hat{\sigma}^2.$$

Before we state the main theorem of this section, we introduce a lemma.

Lemma. Suppose that the q satisfy

$$n^{-1/2} q^2 \log n \to 0 \quad \text{and} \quad n^{7/4} q r^q \to 0 \quad \text{for all} \quad r \in (0,1). \tag{2.4}$$

Then,

$$\| \hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n \|^2 = O_P(q/n). \tag{2.5}$$

Remark. The first expression in (2.4) says that q should not be so large that the AR model is overfitted. On the other hand, the second implies that q should not be so small that the approximation is meaningless. A typical q satisfying (2.2) is $c(\log n)^d$ with $c, d > 0$, or un^v with $u > 0, 0 < v < 1/4$.

Proof. From (2.3), we can write that

$$\begin{aligned}
\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n &= \Big(\sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\underset{\sim}{X}_{t-1}'\Big)^{-1} \sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\Big(\varepsilon_t + \sum_{j=q+1}^{\infty} \beta_j X_{t-j}\Big) \\
&= \Big(\sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\underset{\sim}{X}_{t-1}'\Big)^{-1} \sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\varepsilon_t \\
&+ \Big(\sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\underset{\sim}{X}_{t-1}'\Big)^{-1} \sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1} \sum_{j=q+1}^{\infty} \beta_j X_{t-j} \\
&= L_1 + L_2, \quad \text{say,}
\end{aligned}$$

By Lemma 3.3 of Lee and Wei (1999), we have that

$$\| L_1 \|^2 = O_P(n^{-1} q). \tag{2.6}$$

Meanwhile, L_2 can be rewritten as follows

$$L_2 = \Big(\frac{1}{n-q} \sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\underset{\sim}{X}_{t-1}'\Big)^{-1} \Big(\frac{1}{n-q} \sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1} \sum_{j=q+1}^{\infty} \beta_j X_{t-j}\Big).$$

Note that

$$\|\Big(\frac{1}{n-q}\sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1}\underset{\sim}{X}_{t-1}'\Big)^{-1}\| = O_P(1) \tag{2.7}$$

due to Lemma 3.1 of Lee and Wei (1999). Further,

$$\|\sum_{t=q+1}^{n} \underset{\sim}{X}_{t-1} \sum_{j=q+1}^{\infty} \beta_j X_{t-j}\|^2 = \sum_{i=1}^{q}\sum_{t=q+1}^{n} X_{t-i}^2 (\sum_{j=q+1}^{\infty} \beta_j X_{t-j})^2,$$

and hence

$$\begin{aligned} E\|\sum_{i=1}^{q}\sum_{t=q+1}^{n} X_{t-i}^2(\sum_{j=q+1}^{\infty}\beta_j X_{t-j})^2\|^2 &\le \sum_{i=1}^{q}\sum_{t=q+1}^{n} EX_{t-i}^2(\sum_{j=q+1}^{\infty}\beta_j X_{t-j})^2 \\ &\le \sum_{i=1}^{q}\sum_{t=q+1}^{n} E^{1/2}X_{t-i}^4\{E^{1/4}(\sum_{j=q+1}^{\infty}\beta_j X_{t-j})^4\}^2 \\ &= O(nq(\sum_{j=q+1}^{\infty}|\beta_j|)^2) = O(nq\rho^{2q}), \end{aligned}$$

where we have used Minkowski's inequality and (2.2). This with (2.7) yields $\|L_2\|^2 = O_P(q\rho^{2q}/n)$. Therefore, in view of (2.6), we have $\| \hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n \|^2 = O_P(n^{-1}q)$. □

The following is the main result of this section.

Theorem. Suppose that (2.2) and (2.4) are satisfied.

$$T_n := \max_{q+1\le k\le n} \frac{(n-q)^{1/2}\hat{\sigma}^2}{\hat{\tau}}\left|\frac{\sum_{t=q+1}^{k}\hat{\varepsilon}_t^2}{\sum_{t=q+1}^{n}\hat{\varepsilon}_t^2} - \frac{k-q}{n-q}\right| \xrightarrow{d} \sup_{0\le u\le 1}|B^o(u)|, \tag{2.8}$$

where B^o denotes a standard Brownian bridge.

Proof. It suffices to show that

$$\max_{q+1\le k\le n}|\frac{1}{\sqrt{n}}\sum_{t=q+1}^{k}(\hat{\varepsilon}_t^2 - \varepsilon_t^2)| \to 0. \tag{2.9}$$

Write that

$$\begin{aligned} \hat{\varepsilon}_t^2 &= \varepsilon_t^2 + (\sum_{j=q+1}^{\infty}\beta_j X_{t-j})^2 + \{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n)'\underset{\sim}{X}_{t-1}\}^2 \\ &+ 2\varepsilon_t\sum_{j=q+1}^{\infty}\beta_j X_{t-j} + 2\varepsilon_t(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n)'\underset{\sim}{X}_{t-1} + 2(\sum_{j=q+1}^{\infty}\beta_j X_{t-j})\{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n)'\underset{\sim}{X}_{t-1}\} \\ &= \varepsilon_t^2 + \sum_{i=1}^{5} I_{ti}, \quad \text{say}. \end{aligned}$$

First, note that by Minkowski's inequality,

$$E\{\frac{1}{\sqrt{n}}\max_{q+1\le k\le n}|\sum_{t=q+1}^{k} I_{t1}|\} \le \frac{1}{\sqrt{n}}\sum_{t=q+1}^{n}(\sum_{j=q+1}^{\infty}|\beta_j|E^{1/2}X_{t-j}^2)^2,$$

which together with (2.2) yields that

$$\frac{1}{\sqrt{n}}\max_{q+1\le k\le n}|\sum_{t=q+1}^{n} I_{t1}| = O_P(\sqrt{n}\rho^{2q}) = o_P(1). \tag{2.10}$$

Second, in view of the Lemma, we have

$$\begin{aligned}\max_{q+1\le k\le n}|\frac{1}{\sqrt{n}}\sum_{t=q+1}^{k} I_{t2}| &\le \| \hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n \|^2 \frac{1}{\sqrt{n}}\sum_{t=q+1}^{n} \| \underset{\sim}{X}_{t-1} \|^2 \\ &= O_P(q^2/\sqrt{n}) = o_P(1). \end{aligned}\tag{2.11}$$

Third, it follows from Doob's maximal inequality that

$$\frac{1}{\sqrt{n}}\max_{q+1\le k\le n}|\sum_{t=q+1}^{k} I_{t3}| = \frac{1}{\sqrt{n}}(\sum_{j=q+1}^{\infty}|\beta_j|)\max_{q+1\le k\le n}|\sum_{t=q+1}^{k}\varepsilon_t X_{t-j}| = O_P(\rho^q). \tag{2.12}$$

Again using Doob's maximal inequality and the Lemma, we have

$$\begin{aligned}\frac{1}{\sqrt{n}}\max_{q+1\le k\le n}|\sum_{t=q+1}^{k} I_{t4}| &\le \| \hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}_n \| \frac{1}{\sqrt{n}}\max_{q+1\le k\le n}\| \sum_{t=q+1}^{k}\varepsilon_t \underset{\sim}{X}_{t-1} \| \\ &= O_P(q^{3/2}/\sqrt{n}) = o_P(1). \end{aligned}\tag{2.13}$$

Finally, using the Schwarz inequality, (2.10) and (2.11), we have

$$\frac{1}{\sqrt{n}}\max_{q+1\le k\le n}|\sum_{t=q+1}^{k} I_{t5}| = O_P(q\rho^q) = o_P(1). \tag{2.14}$$

Combining (2.10)-(2.14), we have (2.9). This completes the proof. □

The Theorem shows that our test statistic behaves asymptotically the same as the one based on true errors provided the q are properly chosen. The critical values corresponding to given significance levels are available from an existing table (cf. Inclán and Tiao, 1994, page 914). For example, we reject the null hypothesis if $T_n > 1.358$ when the significance level is 0.05.

3. Simulation results and discussions

In this section, we evaluate the performance of the test T_n in the Theorem through a simulation study. In order to generate data that follows an AR(∞) model, we consider the ARMA(1,1) model $X_j = \phi X_{j-1} + \varepsilon_j + \theta\varepsilon_{j-1}$, where the ε_j are iid standard normal random variables. We do this because it can be rewritten as an AR(∞) model if $|\theta| < 1$. The data for our simulation study is generated from the above model with $\phi = 0.1, 0.5$ and 0.8 and $\theta = 0.5$. In each simulation 100 initial observations are discarded to remove initialization effects. Throughout this simulation study, we used $q = 2[n^{1/5}]$ as the order of the long AR model. The empirical sizes are produced with 100, 200, 300 and 500 observations. The figures in the 'size' section of Table 3.1 indicate the ratio of the number of rejections of the null hypothesis out of 2000 repetitions. They indicate that the sizes are not much affected by the correlation of the data, which is not true for the test based on the observations themselves.

In order to examine the power, we consider the alternative hypothesis

$$\begin{aligned} H_1 \quad : \quad & \varepsilon_j \sim \mathcal{N}(0,1),\ j = 1, \ldots, [n/2], \\ & \varepsilon_j \sim \mathcal{N}(0,\Delta),\ j = [n/2]+1, \ldots, n, \end{aligned}$$

where n denotes the sample size and $[n/2]$ is the point where the variance change occurs. For $\Delta = 1.5, 2.0, 3.0$, $n = 100, 200, 300, 500$, and the ϕ's and θ are as mentioned above, the number of rejections of the null hypothesis are calculated out of 2000 repetitions. As might be guessed, it can be seen that the powers increase as either Δ or n increases. Compared to the case of an iid normal sample (cf. Inclán and Tiao, 1994), the powers are somewhat low. However, the powers improve remarkably for fairly large samples. It can be observed that the powers are not affected much by the value of ϕ. This enables us to conclude that the test performs analogously regardless of whether the data is strongly correlated or not.

As mentioned in the Introduction, one may argue whether the test performs better if one selects an autoregressive model whose order is determined through a model selection criterion. Taking account of this aspect, we performed a simulation with autoregressive models whose orders are determined by the AIC. We picked it up since it is asymptotically efficient in the sense of Shibata (1980). The sizes and powers are produced and summarized in Table 3.2. There, we can observe that the powers are no more than those in Table 3.1, although the gaps are slight. Although not reported here, we have had a similar experience with the BIC (Bayesian information criterion). The results suggest that the gains, if any, obtained by model selection criteria are not so much as one might expect.

So far we have seen that our method using Inclán and Tiao's approach works properly in autoregressive time series models, as long as the data is suitably fitted by a long AR model. Particularly, we could see that it is not necessary to go through the step of selecting the order of the autoregressive model. Although we do not discuss the details, the D_k plot as used in Inclán and Tiao can be applied to the detection of multiple variance changes. It is believed that most of the properties of the Inclán and Tiao's statistic in the iid sample are also enjoyed in our setting.

Table 3.1. The empirical sizes and powers of T_n; $q = 3$ for $n = 100, 200, 300$ and $q = 4$ for $n = 500$.

		ϕ		
	n	.1	.5	.8
size	100	.026	.034	.026
	200	.037	.034	.029
	300	.033	.033	.038
	500	.038	.040	.036
Δ=1.5	100	.119	.117	.115
	200	.336	.332	.335
	300	.523	.500	.519
	500	.787	.788	.793
Δ=2.0	100	.369	.358	.343
	200	.783	.773	.784
	300	.956	.949	.944
	500	.998	.999	.998
Δ=3.0	100	.739	.768	.752
	200	.994	.998	.993
	300	1.000	1.000	.999
	500	1.000	1.000	1.000

Table 3.2. The empirical sizes and powers of T_n based on the AIC method; $q = 5, 6, 7, 8$ for n =100, 200, 300, 500, respectively.

	n	ϕ .1	.5	.8
size	100	.026	.029	.023
	200	.035	.032	.026
	300	.026	.030	.029
	500	.043	.037	.038
Δ=1.5	100	.118	.120	.111
	200	.327	.313	.319
	300	.512	.495	.498
Δ=2.0	100	.357	.336	.318
	200	.772	.753	.772
	300	.951	.942	.941
	500	.998	.999	.998
Δ=3.0	100	.721	.742	.725
	200	.992	.992	.991
	300	1.000	1.000	.999
	500	1.000	1.000	1.000

Acknowledgements. The authors wish to acknowledge the support of the Korea Research Foundation (1998-015-D00047) made in the year of 1998.

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