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Jackknife Estimates for Parameter Changes in the Weibull Distribution

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Abstract

We shall propose several estimators for the shape and scale parameters in the Weibull distribution based upon the complete or truncated samples when both parameters are functions of a known exposure level and study properties for proposed several estimators

1. Introduction

In 1930, a Swedish engincer, Wallori Weibull, began to apply the "weakest link" theory to strength of materials using a Gumbel extreme - value distribution which came to be called the Weibull distribution because of its wide application in reliability engineering and statistical inference (see Bain & Engelhart(1987), Saunders & Mann(1985), and Saunders & Woo(1989)). Here we shall consider an application of the Weibull distribution to the strength of materials when its shape and scale parameters are functions of a known exposure level t, which often occurs in the engineering and physical phenomena.

The purpose of this work is to estimate the shape and scale parameters in the Weibull distribution when both parameters change a function of an environment dosage, say t.

First, we shall give this estimation problems for the Weibull model with a known shape parameter on the base of the complete samples by the maximum likelihood and jackknife methods or the truncated samples. Next, estimators for the case when both the shape and scale parameters in the Weibull distribution are unknown will be found by the maximum likelihood method on the base of the complete samples. However, the expectations and variances of these estimators can't be given.

2. The Weibull Distribution

We shall consider the Weibull distribution with the p.d..f.

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$$f(x;\alpha(t),\beta(t))=\begin{cases}\dfrac{\alpha(t)}{\beta(t)}\left\{\dfrac{x}{\beta(t)}\right\}^{\alpha(t)-1}\exp\left\{-\left\{\dfrac{x}{\beta(t)}\right\}^{\alpha(t)}\right\}, & x>0\\ 0, \qquad \text{elsewhere}, & \end{cases}$$

where $\alpha(t)$ and $\beta(t)$ are positive for $t>0$, denoted by $X\sim WEI(\beta(t),\alpha(t))$.

We shall estimate the parameters when $\alpha(t)$ and $\beta(t)$ are polynomials of t ;

$$\alpha(t)=a_0+a_1t+\cdots+a_rt^r, \quad \text{and}$$

$$\beta(t)=b_0+b_1t+\cdots+b_rt^r, \quad t>0 \text{ and } a_i>0,\ b_i>0, \text{ for all } i=0,1,\cdots,r.$$

2.1 The shape parameter is known

2.1.A The complete samples

Assume $X_{1i},\cdots,X_{n_jj}$ be a simple random samples taken from $X_j\sim WEI(\beta(t),\alpha(t))$, $j=1,\cdots r+1$, and $X_1,\cdots,X_{r+1}$ are independent, $t_i\neq t_k$ for $i\neq k$. We assume that the shape parameter $\alpha(t)$ is known constant α_0 .

The likelihood functions are given by, for $j=1,\cdots,r+1$,

$$L(b_0,b_1,\cdots,b_r\mid t_j)=\alpha_0^{n_j}\cdot\left\{\sum_{k=0}^{r}b_k\cdot t_j^k\right\}^{-\alpha_0\cdot n_j}\cdot\prod_{i=1}^{n_j}x_{ij}^{\alpha_0-1}\cdot\exp\left\{-\sum_{i=1}^{n_j}\left(x_{ij}/\sum_{k=0}^{r}b_k\cdot t_j^k\right)^{\alpha_0}\right\}$$

from which we find $\partial\log L((b_0,b_1,\cdots,b_r\mid t_j)/\partial b_i$, $i=0,1,\cdots,r$ and $j=1,\cdots,r+1$. Setting the derivatives equal to zero, with simplication yield the following equations :

$$\sum_{k=0}^{r}b_k t_j^k=\left\{\sum_{i=0}^{n_j}x_{ij}^{\alpha_0}/n_j\right\}^{1/\alpha_0}, \quad j=1,\cdots,r+1.$$

Hence, from these linear equations, the MLE's $\hat{b}_j^{(1)}$ for b_j, $j=0,1,\cdots,r$, are

$$\hat{b}_j^{(1)}=\frac{\det\left[t_i^0,\cdots,t_i^{j-1},\left(\sum_{m=1}^{n_i}X_{mi}^{\alpha_0}/n_i\right)^{1/\alpha_0},t_i^{j+1},\cdots,t_i^r\right]}{\det\left[t_i^0,\cdots,t_i^r\right]},$$

$$\text{where } \det\left[t_i^0,\cdots,t_i^r\right]=\begin{vmatrix}1 & t_1 & t_1^2 & \cdots & t_1^r\\ 1 & t_2 & t_2^2 & \cdots & t_2^r\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 1 & t_{r+1} & t_{r+1}^2 & \cdots & t_{r+1}^r\end{vmatrix}.$$

The expectations and variances of these MLE's $\hat{b}_j^{(1)}$ for b_j can be obtained by

$$E(\hat{b}_j^{(1)}) = \frac{\sum_{k=0}^{r} b_k \cdot \det[t_i^0, \cdots, t_i^{j-1}, (1/n_i)^{1/\alpha_0} \cdot t_i^k \cdot \Gamma(n_i + 1/\alpha_0)/\Gamma(n_i), t_i^{j+1}, \cdots, t_i^r]}{\det[t_i^0, \cdots, t_i^r]},$$

and

$$VAR(\hat{b}_j^{(1)}) = \frac{1}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \sum_{k=1}^{r+1} (1/n_k)^{2/\alpha_0} \beta^2(t_k) \{\Gamma(n_k + 2/\alpha_0)/\Gamma(n_k) - \Gamma^2(n_k + 1/\alpha_0)/\Gamma^2(n_k)\} \cdot \det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k},$$

where $\det[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}$ is a minor determinant eliminated k-row and j-column in the determinant, $\det[t_i^0, \cdots, t_i^r]$, and $\Gamma(\cdot)$ is the gamma function.

Therefore, by taking limits for these expressions, the MLE's $\hat{b}_j^{(1)}$, $j = 0, 1, \cdots, r$, are asymptotically unbiased and MSE-consistent.

By definition of the jackknife technique, we can obtain the jackknife estimators of $\hat{b}_j^{(1)}$, for every $j = 0, 1, \cdots, r$,

$$J(\hat{b}_j^{(1)}) = \frac{1}{\det[t_i^0, \cdots, t_i^r]} \det[t_i^0, \cdots, t_i^{j-1}, (n. \cdot n_i + n. - n_i)\left(\sum_{m=1}^{n_i} X_{mi}^{\alpha_0} / n_i\right)^{1/\alpha_0} - (n. - 1) \sum_{k=1}^{n_i} \left\{\left(\sum_{m=1}^{n_i} X_{mi}^{\alpha_0} - X_{ki}^{\alpha_0}\right) / (n_i - 1)\right\}^{1/\alpha_0}, t_i^{j+1}, \cdots, t_i^r]$$

where $n. = n_1 + n_2 + \cdots + n_{r+1}$.

The expectations of these jackknife estimators $J(\hat{b}_j^{(1)})$ can be obtained by

$$E(J(\hat{b}_j^{(1)})) = \frac{1}{n. \cdot \det[t_i^0, \cdots, t_i^r]} \sum_{k=0}^{r} b_k \cdot \det[t_i^0, \cdots, t_i^{j-1}, t_i^k \cdot \{(n. \cdot n_i + n. - n_i)(n_i)^{-1/\alpha_0} \Gamma(n_i + 1/\alpha_0)/\Gamma(n_i) - n_i(n. - 1)(n_i - 1)^{-1/\alpha_0} \Gamma(n_i + 1/\alpha_0 - 1)/\Gamma(n_i - 1), t_i^{j+1}, \cdots, t_i^r]$$

Therefore, as using limit formula in Gradshteyn et al(1980), $\lim n^{b-a} \Gamma(n+a)/\Gamma(n+b) = 1$, the jackknife estimators $J(\hat{b}_j^{(1)})$, $j = 0, \cdots, r$, are asymptotically unbiased estimators for b_j.

2.1.B The truncated sample

For given $t_i \neq t_k$ for every $i \neq k$. $1, 2, \cdots, r+1$, let $X_{1i}, \cdots, X_{k_j j}, \cdots, X_{n_j j}$ be the truncated random samples taken from $X_j \sim WEI(\beta(t), \alpha_0)$ and $X_1, \cdots, X_{r+1}$ be independent, where $X_{1i}, \cdots, X_{k_j j}$ are dead items or items of failures and $X_{k_j+1 j}, \cdots, X_{n_j j}$ are alive items or runouts, $j = 1, \cdots, r+1$.

The likelihood functions are given by, for $j = 1, \cdots r+1$,

$$L(b_0, b_1, \cdots, b_r \mid t_j) = \alpha_0^{k_j} \cdot \left\{ \sum_{m=0}^{r} b_m \cdot t_j^m \right\}^{-\alpha_0 \cdot k_j} \cdot \prod_{i=1}^{k_j} x_{ij}^{\alpha_0 - 1} \cdot \exp\left\{ - \sum_{i=1}^{n_j} \left(x_{ij} / \sum_{m=0}^{r} b_k \cdot t_j^m \right)^{\alpha_0} \right\}$$

from which we find $\partial \log L((b_0, b_1, \cdots, b_r \mid t_j) / \partial b_i$, $i = 0, 1, \cdots, r$ and $j = 1, \cdots, r+1$, Setting the derivatives equal to zero, with simplication yield the following equations :

$$\sum_{m=0}^{r} b_m t_j^m = \left\{ \sum_{i=0}^{n_j} x_{ij}^{\alpha_0} / k_j \right\}^{1/\alpha_0}, \quad j = 1, \cdots, r+1.$$

Hence, the MLE's $\widehat{b}_j^{(2)}$ for b_j, $j = 0, 1, \cdots, r$, are

$$\widehat{b}_j^{(2)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left(\sum_{m=1}^{n_i} X_{mi}^{\alpha_0} / k_i \right)^{1/\alpha_0}, t_i^{j+1}, \cdots, t_i^r \right]}{\det\left[t_i^0, \cdots, t_i^r \right]}.$$

If we assume $k_j - 1$ follows a poisson distribution with mean λ_j, $j = 1, \cdots, r+1$, and k_j's are independent, then the expectations and variances of these MLE's $\widehat{b}_j^{(2)}$ for b_j can be obtained by

$$E(\widehat{b}_j^{(2)}) = \frac{\sum_{m=0}^{r} b_m \cdot \det\left[t_i^0, \cdots, t_i^{j-1}, t_i^m A(\lambda_i; 1/\alpha_0) \exp(-\lambda_i) \cdot \Gamma(n_i + 1/\alpha_0)/\Gamma(n_i), t_i^{j+1}, \cdots, t_i^r \right]}{\det\left[t_i^0, \cdots, t_i^r \right]},$$

and

$$\begin{aligned} VAR(\widehat{b}_j^{(2)}) = & \frac{1}{\det^2\left[t_i^0, \cdots, t_i^r \right]} \\ & \cdot \sum_{m=1}^{r+1} \beta^2(t_m) \exp(-\lambda_m) \det^2\left[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r \right]_{i \neq m} \\ & \{ A(\lambda_m; 2/\alpha_0) \Gamma(n_m + 2/\alpha_0)/\Gamma(n_m) - A^2(\lambda_m; 1/\alpha_0) \Gamma^2(n_k + 1/\alpha_0)/\Gamma^2(n_k) \}, \end{aligned}$$

where $A(x; a) = \sum_{y=0}^{\infty} x^y / (y! \cdot (y+1)^a)$, $x > 0$.

From this expectations and variances, the truncated MLE's $\widehat{b}_j^{(2)}$ for b_j are asymptotically unbiased and MSE-consistent if $n_j = \lambda_j$ for every j.

Table 1 & 2 show numerical values of biases for the MLE, jackknife estimator, and the truncated MLE for b_j's when n_1=5(5)25, n_2=5(5)25 $r=1$ and $a_0 = 1/2$ and 2. Throughout the numerical values of Table 1 & 2, the jackknife technique is more useful in the bias reduction than the MLE and truncated MLE.

2.2 The shape parameter is unknown

Here the estimators for the case when both the shape and scale parameters in the Weibull distribution are unknown will be found by maximum likelihood method on the base of the complete samples.

The likelihood functions are given by, for $j=1,\cdots,r+1$,

$$L(b_0, b_1, \cdots, b_r, a_0, a_1, \cdots, a_r \mid t_j) = \left\{ \left\{ \sum_{k=0}^{r} a_k \cdot t_j^k \right\} \left\{ \sum_{k=0}^{r} b_k \cdot t_j^k \right\}^{-\sum_{k=0}^{r} a_k \cdot t_j^k} \right\}^{n_j}$$
$$\prod_{i=1}^{n_j} x_{ij}^{\sum_{k=0}^{r} a_k \cdot t_j^k - 1} \cdot \exp\left\{ - \sum_{i=1}^{n_j} \left(x_{ij} / \sum_{m=0}^{r} b_k \cdot t_j^m \right)^{\sum_{k=0}^{r} a_k \cdot t_j^k} \right\},$$

from which we find $\partial \log L(b_0, \cdots, b_r, a_0, \cdots, a_r \mid t_j)/\partial b_i$, $\partial \log L(b_0, \cdots, b_r, a_0, \cdots, a_r \mid t_j)/\partial a_i$, $i=0,1,\cdots,r$ and $j=1,\cdots,r+1$.
Setting the partial derivatives equal to zero, with simplification yield the following equations (see Saunders & Mann(1985)) :

$$f_j(a_0, \cdots, a_r) = \frac{1}{\sum_{k=0}^{r} a_k \cdot t_j^k} - \frac{\sum_{i=1}^{n_j} X_{ij}^{\sum_{k=0}^{r} a_k \cdot t_j^k} \log X_{ij}}{\sum_{i=1}^{n_j} X_{ij}^{\sum_{k=0}^{r} a_k \cdot t_j^k}} + \sum_{i=1}^{n_j} \frac{\log X_{ij}}{n_j} = 0,$$

$$\sum_{k=0}^{r} b_k \cdot t_j^k = \left\{ \sum_{i=1}^{n_j} \frac{X_{ij}}{n_j} \right\}^{1/\sum_{k=0}^{r} a_k \cdot t_j^k}, \quad j=1,\cdots,r+1.$$

Let $a_0 + a_1 t_j + \cdots + a_r t_j^r = A_j$, $j=1,\cdots,r+1$. Then $f_j(A_j) = 0$.

It has been shown in Bain & Engelhart(1987) that the roots $\widehat{A}_1, \widehat{A}_2, \cdots, \widehat{A_{r+1}}$ are unique solutions of these equations. The Newton procedures for solving the equations $f_1(\widehat{A}_1) = f_2(\widehat{A}_2) = \cdots = f_{r+1}(\widehat{A_{r+1}}) = 0$ are to determine successive approximations (see Stummel & Hamer1980) ; $j=1,\cdots,r+1$.

$$\widehat{A}_j^{(t)}, \text{ where } \widehat{A}_j^{(t+1)} = \widehat{A}_j^{(t)} - f_j(\widehat{A}_j^{(t)})/f_j'(\widehat{A}_j^{(t)}), \quad t=0,1,2,\cdots.$$

Then the MLE's $\widehat{a}_j^{(3)}$ and $\widehat{b}_j^{(3)}$ for a_j and b_j, $j=0,\cdots,r$, are

$$\widehat{a}_j^{(3)}=\frac{\det[t_i^0,\cdots,t_i^{j-1},\widehat{A}_j,t_i^{j+1},\cdots,t_i^r]}{\det[t_i^0,\cdots,t_i^r]},$$

$$\widehat{b}_j^{(3)}=\frac{\det\left[t_i^0,\cdots,t_i^{j-1},\left(\sum_{m=1}^{n_i}X_{mi}^{\sum_{k=0}^{r}a_k^{(3)}t_i^k}/n_i\right)^{1/\sum_{k=0}^{r}a_k^{(3)}t_i^k},t_i^{j+1},\cdots,t_i^r\right]}{\det[t_i^0,\cdots,t_i^r]}.$$

Next we shall consider estimation of the shape and scale parameters in the Weibull model on the base of the truncated samples in (2.1.B).
The likelihood functions can be obtained by, $j=1,2,\cdots,r+1$,

$$\begin{aligned}\log L(b_0,b_1,\cdots,b_r,a_0,a_1,\cdots,a_r\mid t_j)=\ & k_j\cdot\log\left\{\sum_{k=0}^{r}a_k\cdot t_j^k\right\}-k_j\cdot\left\{\sum_{k=0}^{r}a_k\cdot t_j^k\right\}\\ &\cdot\log\left\{\sum_{k=0}^{r}b_k\cdot t_j^k\right\}+\left\{\sum_{k=0}^{r}a_k\cdot t_j^k-1\right\}\cdot\left\{\sum_{i=1}^{k_j}\log x_{ij}\right\}\\ &-\sum_{i=1}^{n_i}\left\{x_{ij}/\sum_{m=0}^{r}b_k\cdot t_j^m\right\}^{\sum_{k=0}^{r}a_k\cdot t_j^k},\end{aligned}$$

from which we find $\partial\log L(b_0,\cdots,b_r,a_0,\cdots,a_r\mid t_j)/\partial b_i$, $\partial\log L(b_0,\cdots,b_r,a_0,\cdots,a_r\mid t_j)/\partial a_i$, $i=0,1,\cdots,r$ and $j=1,\cdots,r+1$.
Setting the partial derivatives equal to zero, with simplication yield the following equations(see Saunders & Mann(1985)) :

$$g_j(a_0,\cdots,a_r)=\frac{1}{\sum_{k=0}^{r}a_k\cdot t_j^k}-\frac{\sum_{i=1}^{n_i}X_{ij}^{\sum_{k=0}^{r}a_k\cdot t_j^k}\log X_{ij}}{\sum_{i=1}^{n_i}X_{ij}^{\sum_{k=0}^{r}a_k\cdot t_j^k}}+\sum_{i=1}^{n_i}\frac{\log X_{ij}}{k_j}=0,$$

$$\sum_{k=0}^{r}b_k\cdot t_j^k=\left\{\sum_{i=1}^{n_i}\frac{X_{ij}}{k_j}\right\}^{1/\sum_{k=0}^{r}a_k\cdot t_j^k},\quad j=1,\cdots,r+1.$$

Let $a_0+a_1t_j+\cdots+a_rt_j^r=B_j$, $j=1,\cdots,r+1$. By the same method of the complete case in (2.2), we can obtain unique solutions $\widehat{B}_j$'s of $g_j(\widehat{B}_j)=0$, $j=1,\cdots,r+1$. Then the MLE's $\widehat{a}_j^{(4)}$ and $\widehat{b}_j^{(4)}$ for a_j and b_j, respectively, $j=0,\cdots,r$, are

$$\widehat{a}_j^{(3)}=\frac{\det[t_i^0,\cdots,t_i^{j-1},\widehat{B}_j,t_i^{j+1},\cdots,t_i^r]}{\det[t_i^0,\cdots,t_i^r]},$$

and

$$\widehat{b}_j^{(3)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left(\sum_{m=1}^{n_i} X_{mi}^{\sum_{k=0}^{r} a_k^{(1)} t_i^k} / k_i\right)^{1/\sum_{k=0}^{r} a_k^{(4)} t_i^k}, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]}.$$

These estimators can be considered applications of the Weibull distribution as numerical results by use of the computer method.

2.3 An example

The random data of size 30(35) generated by the Weibull distribution with the shape parameter 3.53(1.93) and the scale parameter 1.12(1.30) when the given time is 1(2) are the following ;

time $t=1$ and WEI(3.53, 1.12) ;

0.28377	1.04073	1.09710	1.13564	1.24619	1.55693	0.46712	1.36307	0.83292	1.17392
1.01126	0.93986	1.57909	1.06920	0.97780	1.18065	0.66854	1.17879	0.93925	1.33254
0.99796	1.25192	1.62951	0.95682	0.69183	1.47483	1.57735	1.45404	1.97065	0.72120

time $t=2$ and WEI(1.93, 1.30) ;

0.10553	1.13667	1.25180	1.33341	1.58034	2.37459	0.26261	1.86191	0.75632	1.41676
1.07849	0.94331	2.43677	1.19420	1.01412	1.43164	0.50591	1.42752	0.94219	1.78635
1.05269	1.59366	2.58095	0.97468	0.53862	2.15058	2.43186	2.09547	3.65402	0.58118
0.79479	0.59421	1.00052	1.01866	0.36339					

Then the MLE's of parameters, a_0, a_1, b_0 and b_1 in the Weibull model can be numerically evaluated based on the random data by equation (2.2.1) and (2.2.2) through Newton and Raphson method as follows ;

$$\widehat{a_0} = 5.10739, \quad \widehat{a_1} = -1.64408, \quad \widehat{b_0} = 0.90430, \quad \widehat{b_1} = 0.13074$$

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Table 1.1 Biases of estimators for scale in the Weibull distribution with a known shape (1/2) ($n_1=\lambda_1$, $n_2=\lambda_2$, $b_0=1$, $b_1=2$, $t=1$, $t_2=2$).

size		PA	BIAS		
n_1	n_2		MLE	JE	TMLE
5	5	b_0	0.20000	-0.02500	0.53620
		b_1	0.40000	-0.05000	0.10724
	10	b_0	0.70000	-0.14148	2.14352
		b_1	-0.10000	0.08148	-0.53478
	15	b_0	0.86667	-0.21905	2.46763
		b_1	-0.26667	0.10655	-0.85902
	20	b_0	0.95000	-0.23738	2.67344
		b_1	-0.35000	0.11738	-1.06485
	25	b_0	0.99999	-0.24862	2.79016
		b_1	-0.39999	0.12363	-1.18125
10	5	b_0	-0.40000	0.14444	1.39257
		b_1	0.70000	-0.15555	2.03688
	10	b_0	0.10000	-0.00555	0.21475
		b_1	0.20000	-0.01111	0.42949
	15	b_0	0.26667	-0.03047	0.53890
		b_1	0.03333	0.01047	0.10533
	20	b_0	0.35000	-0.04007	0.74469
		b_1	-0.04999	0.01784	-0.10045
	25	b_0	0.39999	-0.04525	0.86142
		b_1	-0.09998	0.02144	-0.21718

size		PA	BIAS		
n_1	n_2		MLE	JE	TMLE
15	5	b_0	-0.60000	0.18036	-1.78159
		b_1	0.80000	-0.18393	2.23131
	10	b_0	-0.09999	0.02190	-0.17424
		b_1	0.30000	-0.02762	0.62399
	15	b_0	0.06666	-0.00238	0.14991
		b_1	0.13333	-0.00476	0.29983
	20	b_0	0.15000	-0.01070	0.35570
		b_1	0.05000	0.00253	-0.02268
	25	b_0	0.19999	-0.01474	0.47240
		b_1	0.01001	0.00581	-0.02268
20	5	b_0	-0.70000	0.19685	-2.02842
		b_1	0.85000	-0.19843	2.35473
	10	b_0	-0.20000	0.03178	-0.42118
		b_1	0.35000	-0.03441	0.74746
	15	b_0	-0.03333	0.00685	-0.09702
		b_1	0.18333	-0.01023	0.42330
	20	b_0	0.05000	-0.00131	0.10876
		b_1	0.10000	-0.00262	0.21752
	25	b_0	0.09999	-0.00507	0.22549
		b_1	0.05001	0.00069	0.10078
25	5	b_0	-0.75999	0.20668	-2.16854
		b_1	0.87999	-0.20751	2.42485
	10	b_0	-0.25999	0.03684	-0.56127
		b_1	0.37999	-0.03826	0.81750
	15	b_0	-0.09332	0.01114	-0.23711
		b_1	0.21333	-0.01301	0.49334
	20	b_0	-0.00998	0.00287	-0.03132
		b_1	0.12999	-0.00509	0.28756
	25	b_0	0.04000	-0.00083	0.08641
		b_1	0.08000	-0.00166	0.17082

Table 1.2 Biases of estimators for scale in the Weibull distribution with a known shape (2) ($n_1 = \lambda_1$, $n_2 = \lambda_2$, $b_0 = 1$, $b_1 = 2$, $t = 1$, $t_2 = 2$).

size		PA	BIAS		
n_1	n_2		MLE	JE	TMLE
5	5	b_0	-0.02465	0.00252	0.11168
		b_1	-0.04929	0.00505	0.22336
	10	b_0	-0.08581	0.01940	0.49360
		b_1	0.01187	-0.00880	-0.15856
	15	b_0	-0.10642	0.02368	0.57806
		b_1	0.32468	-0.01157	-0.24302
	20	b_0	-0.11675	0.02580	0.61165
		b_1	0.04280	-0.01279	-0.27662
	25	b_0	-0.12296	0.02713	0.63356
		b_1	0.04901	-0.01351	-0.29852
10	5	b_0	0.04875	-0.01551	-0.34662
		b_1	-0.08601	0.01659	0.45251
	10	b_0	-0.01241	0.00058	0.03529
		b_1	-0.02483	0.00117	0.07059
	15	b_0	-0.03301	0.00336	0.11976
		b_1	-0.00423	-0.00119	-0.01387
	20	b_0	-0.04334	0.00446	0.15335
		b_1	0.00609	-0.00202	-0.04746
	25	b_0	-0.04956	0.00504	0.17525
		b_1	0.01231	-0.00240	-0.06936
15	5	b_0	0.07347	-0.01954	-0.44798
		b_1	-0.09835	0.01986	0.50319
	10	b_0	0.01230	-0.00244	-0.01660
		b_1	-0.03719	0.00303	0.12127
	15	b_0	-0.00829	0.00025	0.01840
		b_1	-0.01659	0.00051	0.03680
	20	b_0	-0.01862	0.00118	0.05199
		b_1	-0.00625	-0.00029	0.00321
	25	b_0	-0.02484	0.00165	0.07390
		b_1	-0.00455	-0.00066	-0.01869

size		PA	BIAS		
n_1	n_2		MLE	JE	TMLE
20	5	b_0	0.08587	-0.02142	-0.48829
		b_1	-0.10456	0.02155	0.52334
	10	b_0	0.02470	-0.00356	-0.10637
		b_1	-0.04339	0.00382	0.14142
	15	b_0	0.00410	-0.00076	-0.02191
		b_1	-0.02279	0.00113	0.05696
	20	b_0	-0.00622	-0.00014	0.01168
		b_1	-0.01245	0.00028	0.02336
	25	b_0	-0.01244	0.00056	0.03358
		b_1	-0.00624	-0.00008	0.00146
25	5	b_0	0.09332	-0.02256	-0.51457
		b_1	-0.10829	0.02263	0.53648
	10	b_0	0.03215	-0.00412	-0.13266
		b_1	-0.04711	0.00426	0.15457
	15	b_0	0.01156	-0.00126	-0.04819
		b_1	-0.02652	0.00145	0.07010
	20	b_0	0.00122	-0.00032	-0.01459
		b_1	-0.01618	0.00056	0.03650
	25	b_0	-0.00498	0.00009	0.00730
		b_1	-0.00997	0.00018	0.01461