The Korean Communications
in Statistics Vol. 7, No. 1, 2000
pp. 285-290

A Note on the Covariance Matrix of Order Statistics of Standard Normal Observations

Hak-myung Lee[1)]

Abstract

We noted a property of a stationary distribution on the matrix C, which is the covariance matrix of order statistics of standard normal distribution. That is, the sup norm of the powers of C is $\mathbf{ee}'$ divided by its dimension. The matrix C can be taken as a transition probability matrix in an acyclic Markov chain.

1. Introduction

The covariance structure of symmetrically dependent ordered observations has been developed by Olkin and Viana (1995) and Viana and Olkin (1997) for visual acuity studies. For example, if $\mathbf{Y}$ is permutation-symmetric p-variate normal with a common variance σ^2 and a common correlation γ, then the covariance matrix of order statistics $\boldsymbol{Y}$ is

$$\mathrm{Cov}(\boldsymbol{Y}) = \sigma^2[\gamma\, \mathbf{ee}' + (1-\gamma)C],$$

where $\mathbf{e} = (1, \ldots, 1)$ has p components and C is the covariance matrix of the order statistics of p independent standard normal random variables. In this paper, we note that the matrix C has a property of a stationary distribution.

Suppose that the components of $\mathbf{U}' = (U_1, \ldots, U_p)$ are independent and identically distributed, and let $\boldsymbol{U}$ be the order statistics associated with $\mathbf{U}$. Then, the moments, covariances, and variances of order statistics can be obtained from

$$E[U^k_{(i)}] = \frac{p!}{(i-1)!(p-i)!} \int u^k f(u)[F(u)]^{i-1}[1-F(u)]^{p-i} du,$$

$$E[U_{(i)}U_{(j)}] = \frac{p!}{(i-1)!(j-i-1)!(p-j)!} \int u_i u_j f(u_i) f(u_j) [F(u_i)]^{i-1}[1-F(u_j)]^{p-j}[F(u_j)-F(u_i)]^{j-i-1} du_i du_j,$$

where F indicates the distribution of U and f is its density function. Thus we have the

1) Biostatistician, PPD Development, 4009 Banister Lane, Austin, TX 78704, USA

covariance matrix Cov(U) of order statistics. However, numerical integration is generally needed. When the density function is symmetric about zero, both computation and tabulation are reduced by the fact that (e.g., David, 1981)

$$E(U_{(r)}) = -E(U_{(p-r+1)}), \tag{1}$$

$$\text{cov}(U_{(r)}, U_{(s)}) = \text{cov}(U_{(p-r+1)}, U_{(p-s+1)}). \tag{2}$$

The theory of order statistics of i.i.d variables is well-known. An extensive reference is David (1981).

In the normal N(0, 1), the mean vector $\mathbf{c}$ and the covariance matrix C of order statistics are given, for example, in Beyer (1991). The symmetry results (1) and (2) hold, and in David (1981) we have $\mathbf{e}'\mathbf{c} = 0$ and C is a stochastic matrix, that is $C\mathbf{e} = \mathbf{e}$ (the sum of the elements of each row is 1), where $\mathbf{e} = (1, \dots, 1)'$. For example,

for p = 2,

$$\mathbf{c}' = [-0.56419,\ 0.56419],$$

$$C_2 = \begin{bmatrix} 0.68169 & 0.31831 \\ & 0.68169 \end{bmatrix},$$

for p = 3,

$$\mathbf{c}' = [\ -0.84628\ \ 0.00000\ \ 0.84628],$$

$$C_3 = \begin{bmatrix} 0.55947 & 0.27566 & 0.16487 \\ & 0.44867 & 0.27566 \\ & & 0.55947 \end{bmatrix}.$$

Note that since the matrix C is symmetric and stochastic, it is automatically doubly stochastic, that is $\mathbf{e}'C = \mathbf{e}'$. Some simple results of multiplications of C and a permutation symmetric matrix having the form of $\Sigma = b\mathbf{e}\mathbf{e}' + (a-b)\mathbf{I}$ are discussed in Section 2 and some characteristics on C are also noted. In Section 3, we show that the matrix C has a property of a stationary distribution and it can be a transition probability matrix in a stochastic process.

Now, we conclude the section with that the 'doubly stochastic' in Cov(U) characterizes the normal distribution. The following result is due to Govindarajulu (1966).

Proposition 1 (Govindarajulu, 1966) Let $U_1, \dots, U_p$ be i.i.d random variables with zero means and variance σ^2. Then Cov(U) is doubly stochastic for all p if and only if the distribution of U is normal.

2. Some Characteristics on C

Let $\mathbf{U} = [U_1, \ldots, U_p]$ be distributed with $N(\mathbf{0}, \mathbf{I})$. Then, the matrix $C = \mathrm{Cov}(\mathbf{U})$ has positive entries. To prove, it is sufficient to show that for arbitrary $U_{(r)}, U_{(s)}$, $r < s$, without loss of generality, $\mathrm{cov}(U_{(r)}, U_{(s)}) > 0$. It follows from Tukey (1958) that the distribution of $U_{(r)}$ given $U_{(s)}$ shows complete positive regression on $U_{(s)}$; that is, for $-\infty < u' < u'' < \infty$,

$$\Pr[U_{(r)} \le u | U_{(s)} = u'] = \Pr[\frac{\Phi(U_{(r)})}{\Phi(U_{(s)})} \le \frac{\Phi(u)}{\Phi(u')} | U_{(s)} = u']$$

$$= \Pr[B \le \frac{\Phi(u)}{\Phi(u')}]$$

$$> \Pr[B \le \frac{\Phi(u)}{\Phi(u'')}] = \Pr[U_{(r)} \le u \mid U_{(s)} = u''],$$

where B is a beta distribution $B(\alpha = r, \beta = s - r)$, and $\Phi(\cdot)$ is a standard normal cumulative function. Therefore, $\mathrm{cov}(U_{(r)}, U_{(s)}) > 0$. Kim and David (1990)'s idea can be also applied to show positive entries of C. They studied on stochastic behavior of differences of order statistics and inequalities for covariances of order statistics. With Kim and David (1990), for fixed $r \le p$, 1) for $r \le s$, $\mathrm{cov}(U_{(r)}, U_{(s)})$ is monotone decreasing as s is monotone increasing, 2) as a similar results, for $s \le r$, $\mathrm{cov}(U_{(r)}, U_{(s)})$ is monotone decreasing as s is monotone decreasing.

With $\mathbf{U} \sim N(\mathbf{0}, \mathbf{I})$, then the matrix $C = \mathrm{Cov}(\mathbf{U})$ is positive definite as well. It is enough to show that for each nonzero $\mathbf{x}$ in R^p, $\mathbf{x}' C \mathbf{x}' > 0$.

$$\mathbf{x}' C \mathbf{x}' = \sum_i^p C_{ii} x_i^2 + 2 \sum\sum_{i<j} C_{ij} x_i x_j$$

$$= \sum_i^p x_i^2 - \sum\sum_{i<j} C_{ij} (x_i - x_j)^2$$

$$\ge \sum_i x_i^2 (1 - \sum_{j>i} C_{ij}) > 0,$$

where x_i is the i-th element of $\mathbf{x}$ and C_{ij} is the (i, j)-th cell element of C. It results from that C is doubly symmetric and stochastic.

Therefore, it says that the matrix C is symmetric, stochastic and positive definite with positive entries. With these characteristics, we may be able to significantly reduce the work required in the multiplications or the inversion of a certain matrix and C. For example, with a $p \times p$ matrix $\Sigma = b\,\mathbf{ee}' + (a - b)\,\mathbf{I}$, $a \ne b$ and $a \ne -(p-1)b$, then since Σ and C are nonsingular, we have

$$(\Sigma C)^{-1}\mathbf{e} = \frac{1}{a+b(p-1)}\mathbf{e}.$$

Likewise, when a nonsingular matrix is permutation symmetric, we note the following facts.

Result 1. Let Σ be an $p\times p$ matrix in the form of $\Sigma = b\,\mathbf{ee}' + (a-b)\,\mathbf{I}$, where $a\neq b$ and $a\neq -(p-1)b$. Then,

$$1)\ \Sigma^{-1}(C-\mathbf{I}) = \frac{1}{(a-b)}(C-\mathbf{I})$$

$$2)\ \Sigma^{-1}(C-\mathbf{I})\Sigma^{-1} = \frac{1}{(a-b)^2}(C-\mathbf{I})$$

$$3)\ (b\,\mathbf{ee}' + (a-b)\ C)^{-1}\mathbf{e} = \frac{1}{a+b(p-1)}\mathbf{e}$$

$$4)\ ((a-b)\,\mathbf{I} + b\ C)^{-1}\mathbf{e} = \frac{1}{a}\mathbf{e},\ a\neq 0,\ b\neq 0.$$

It helps to get the covariance structures of order statistics and induced order statistics. For details see Olkin and Viana (1995) and Lee and Viana (1999).

As an extension, the matrix $b\,\mathbf{ee}' + (a-b)\,\mathbf{A}$, $A\in\mathbf{G}$, where $\mathbf{G}$ is the set of all nonsingular $p\times p$ real doubly stochastic matrices, consists a class of patterned matrices. The study on this multiplicative class of patterned matrices is given in Viana (1996).

3. A Stationary Property

The following result is a property of a stationary distribution on the matrix C.

Proposition 2. If $\mathbf{A}$ is symmetric and stochastic with positive entries, then

$$\mathbf{A}^{\infty} = \lim_{n\to\infty}\mathbf{A}^{n} = \frac{1}{p}\mathbf{ee}'. \tag{3}$$

Proof. Since the maximal eigenvalue is always included between the largest and the smallest row sums (e.g., Gantmacher 1959, p.83), and all the row sums are 1 in a stochastic matrix, the matrix $\mathbf{A}$ has the maximal eigenvalue 1. Since $\mathbf{A}$ has positive entries, by Perron's theorem (see Gantmacher 1959, p.53), the module of all the other eigenvalues λ_i, $i=2,\dots,p$, is strictly less than the maximal eigenvalue 1, so that

$$1 > |\lambda_i|, \ i = 2, \ldots, p. \tag{4}$$

Since $\mathbf{A}$ is symmetric and stochastic, $\mathbf{A}$ and $\mathbf{ee}'$ commute and hence are simultaneously diagonalizable. Let $\mathbf{P}$ be an orthogonal matrix that diagonalizes $\mathbf{A}$ and $\mathbf{ee}'$, so that

$$\mathbf{D}^n = \begin{bmatrix} 1 & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_p^n \end{bmatrix} = \mathbf{P}'\mathbf{A}^n\mathbf{P},$$

$$\lim_{n\to\infty} \mathbf{D}^n = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} = \frac{1}{p}\mathbf{P}'(\mathbf{ee})'\mathbf{P},$$

and therefore (using the sup norm)

$$\mathbf{A}^\infty = \lim_{n\to\infty} \mathbf{A}^n = \lim_{n\to\infty} \mathbf{PD}^n\mathbf{P}' = \frac{1}{p}\mathbf{PP}'(\mathbf{ee}')\mathbf{PP}' = \frac{1}{p}\mathbf{ee}',$$

concluding the proof.

□

In the proof, the expression (4) can become $1 > \lambda_i > 0, \ i = 2, \ldots, p$ for C since the matrix C is positive definite. The matrix C and the matrix $\frac{1}{p}\mathbf{ee}'$ satisfy the conditions of Proposition 2. However the identity matrix $\mathbf{I}$ does not, because its entries are non-negative even though it is symmetric and stochastic. From the example, we know that

$$C_2 = \begin{bmatrix} 0.6817 & 0.3183 \\ 0.3183 & 0.6817 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0.5595 & 0.2757 & 0.1648 \\ 0.2757 & 0.4486 & 0.2757 \\ 0.1648 & 0.2757 & 0.5595 \end{bmatrix}.$$

It then follows that

$$C_2^8 = \begin{bmatrix} 0.50015 & 0.49985 \\ 0.49985 & 0.50015 \end{bmatrix} \approx \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C_3^8 = \begin{bmatrix} 0.33363 & 0.33333 & 0.33304 \\ 0.33333 & 0.33333 & 0.33333 \\ 0.33304 & 0.33333 & 0.33363 \end{bmatrix} \approx \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

From the Proposition 2, the matrix C can be taken as a transition probability matrix in an acyclic Markov chain. Starting with a state distribution π_0, after k steps the state distribution π_k is given by

$$\pi_k = C^k \pi_0 .$$

In the limit,

$$\pi_\infty = \frac{ee'}{p} \pi_0 = (\frac{e' \pi_0}{p}) e = \frac{1}{p} e ,$$

which is uniform. This holds for any initial state distribution π_0.

The Proposition 2 is also a corresponding fact that every matrix with non-negative entries can be represented as the limit of a sequence of irreducible matrices with positive entries $\mathbf{A}_n$ (Gantmacher 1959, p.66).

$$\mathbf{A} = \lim_{n \to \infty} \mathbf{A}_n \ (\mathbf{A}_n > 0, \ n = 1, 2, ...).$$

For the matrix C, $\mathbf{A} = \frac{1}{p} \mathbf{ee}'$ and $\mathbf{A}_n = C^n$.

References

[1] Beyer, W. H. (1991). *Standard Probability and Statistics Tables and Formulae*, CRC Press, Boca ranton.

[2] David, H. A. (1981). *Order Statistics*, Wiley, New York.

[3] Gantmacher, F. R. (1959). *The Theory of Matrices*, Chelsea Pub. co., New York.

[4] Govindatajulu, Z. (1966). Characterization of normal and generalized truncated normal distributions using order statistic, *Ann Math. Statist.* **37**, 1011-1015.

[5] Kim, S. H. and David, H. A. (1990). On the dependence structures of order statistics and concomitants of order statistics, *J. Statist. Plann. Infer.* **24**, 363-368.

[6] Lee, H. M. and Viana, M. (1999). The joint covariance structure of ordered symmetrically dependent observations and their concomitants of order, *Stat. Prob. Letters*. **43**, 411-414.

[7] Olkin, I. and Viana, M. (1995). Correlation analysis of extreme observations from a multivariate normal distribution, *J. Amer. Statist. Assn.* **90**, 1373-1379.

[8] Tukey, J. (1958). A problem of berkson, and minimum variance orderly estimations, *Ann. Math. statist.* **29**, 588-592.

[9] Viana, M. (1996). On a class of patterned matrix, Unpublished paper.

[10] Viana, M. and Olkin, I. (1997). Correlation analysis of ordered observation from a block-equicorrelated multvariate normal distribution., in S. Panchapakesan and N. Balakrihnan, eds, *Advances in Statistical Decision Theory and Application*, Birkhauser, Boston, MA.