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Optimum Strategies in Red & Black

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Abstract

In a game called red and black, you can stake any amount s in your possession. Suppose your goal is 1 and your current fortune is f, with $0<f<1$. You win back your stake and as much more with probability p and lose your stake with probability, $q=1-p$. In this paper, we consider optimum strategies for this game with the value of p less than ½ where the house has the advantage over the player, and with the value of p greater than ½ where the player has the advantage over the house. The optimum strategy at any f when $p<½$ is to play boldly, which is to bet as much as you can. The optimum strategy when $p>½$ is to bet $f\cdot a$ with a, a sufficiently small number between 0 and 1.

Keywords : stochastic process, bold play, gambler's ruin

Ⅰ. Introduction

In a game, red and black, you can stake any amount s in your possession. Suppose your goal is 1 and your current fortune is f, $0<f<1$. You win back your stake and as much more with probability p and lose your stake with probability $q(=1-p)$. This problem was first considered by Coolidge (1909), but the optimum strategy when $p<½$ was presented by Dubins and Savage (1965). They showed that the bold play is optimal when $p<½$, and provided the basic idea of proving this theorem. In this paper we consider the optimum strategy when $p>½$. We also complete the proof of the theorem for optimum strategy when $p<½$, given by Dubins and Savage (1965).

We will consider an optimum strategy at any f when $p<½$ in section 2, and where $p>½$ in section 3. In section 4, we consider future topics to be studied related to this problem.

Ⅱ. Optimum Strategy with $p<½$

The strategy 1 is to bet a small amount each time. We can easily see that this is a bad s

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trategy. Consider a case, $p=.3$ and $f=.2$. Suppose you bet .01 each time so that you could play at least 20 times. Consider the bet .01 as 1, then the current fortune .2 will be 20 and the goal 1 will be 100. By the gambler's ruin probability(Parzen(1962), p233), the probability that a player with the current fortune 20 will go bankrupt can be written as follows.

The probability that f will go to 0 when $f=K=20$ and $N=100$ can be written as

$$\frac{(\frac{q}{p})^K-(\frac{q}{p})^N}{1-(\frac{q}{p})^N}$$

$$=\frac{(\frac{7}{3})^{20}-(\frac{7}{3})^{100}}{1-(\frac{7}{3})^{100}}=\frac{(\frac{7}{3})^{-80}-1}{(\frac{7}{3})^{-100}-1}\approx 1$$

The strategy 2 is to bet as much as you can. We will call this a bold strategy since you bet you entire fortune f or enough to reach 1 whichever is least. A bet function $S(f)$ under bold strategy can be written as

$$\begin{aligned} S(f) &= f & f &\leqq ½ \\ &= 1-f & f &\geqq ½ \end{aligned}$$

Theorem 1. The bold strategy at f is optimal for $p<½$

Proof : First, we will define a function $Q(f)$ to denote a probability of reaching 1 at any f between 0 and 1 under the bold strategy. $Q(f)$ is continuous, non-decreasing, and $Q(0)=0$, $Q(1)=1$, and $Q(½)=p$. Moreover, $Q(¼)=p\cdot Q(½)+q\cdot Q(0)=p^2$ and $Q(¾)=p\cdot Q(1)+q\cdot Q(½)=p+(1-p)\cdot p$

we can now derive $Q(f)$ more generally

i) $f\leqq ½$: *Bet* f

$$Q(f)=p\cdot Q(2f)+q\cdot Q(0)=p+Q(2f)$$

ii) $f\geqq ½$: *Bet* $1-f$

$$Q(f)=p\cdot Q(1)+q\cdot Q(2f-1)=p+q\cdot Q(2f-1)$$

In summary,

$$Q(f)=\begin{cases} pQ(2f), & f\leq ½ \quad (1)\\ p+qQ(2f-1), & f\geq ½ \quad (2)\end{cases}$$

We now consider a strategy such that you bet s at first and then play boldly. Then, the probability of reaching 1 under this strategy will be $p\cdot Q(f+s)+q\cdot Q(f-s)$. To prove theorem 1 is equivalent to show that

$$p\cdot Q(f+s)+q\cdot Q(f-s)\leq Q(f) \text{ for all } f \text{ and } s \qquad (3)$$

According to Dubins and Savage(1965), it suffices to establish (3) for binary rational values of f and s, that is, for numbers of the form $K\cdot 2^{-n}$ where K and n, non-negative integers and $K\cdot 2^{-n}\leq 1$ since $Q(f)$ is continuous. A number of this form, $K\cdot 2^{-n}$ will be said to be of order at most n. It is trivial to verify (3) if f and s are of order at most 0 or 1.

When $n=1, K\cdot 2^{-n}=\frac{K}{2}=0$ or $\frac{1}{2}$ or 1.

If $f=½$ and $s=½$,

then L.H.S. of (3)= $p\cdot Q(1)+q\cdot Q(0)=p$

R.H.S. of (3)= $Q(½)=p$. Thus, (3) holds.

Next, suppose that(3) has been established for all binary rationals f and s of order at most $n+1$.

Rewrite (3) as $Q(f)-p\cdot Q(f+s)-q\cdot Q(f-s)\geq 0$ (4)

Case 1. $f-s\geq ½$

Using (1) and (2), we can write

$$Q(f)=p+q\cdot Q(2f-1)$$
$$Q(f+s)=p+q\cdot Q(2f+2s-1)$$
$$Q(f-s)=p+q\cdot Q(2f-2s-1)$$

Therefore, (4) can be rewritten as

$$Q(f)-p\cdot Q(f+s)-q\cdot Q(f-s)$$
$$=p+q\cdot Q(2f-1)-p\cdot[p+q\cdot Q(2f+2s-1)]-q\cdot[p+q\cdot Q(2f-2s-1)]$$
$$=q\cdot[Q(2f-1)-p\cdot Q(2f-1+2s)-q\cdot Q(2f-1-2s)]$$

≥ 0 since $2f-1, 2f-1+2s$, and $2f-1-2s$ are of order at most n.

Case 2. $f-s \leqq \frac{1}{2} < f$

Using (1) and (2), we can write

$$Q(f)=p+q\cdot Q(2f-1)$$
$$Q(f+s)=p+q\cdot Q(2f+2s-1)$$
$$Q(f-s)=p\cdot Q(2f-2s)$$

We now claim that $f \leqq \frac{3}{4}$. If that is not the case, then $f > \frac{3}{4}$. And so $s \geq \frac{1}{4}$ since $f-s \leq \frac{1}{2}$. This will gives us $f+s>1$, which is contradiction.

As a result, $Q(2f-1)=p\cdot Q(4f-2)$.

Now, $Q(f)-p\cdot Q(f+s)-q\cdot Q(f-s)$

$$=p+q\cdot Q(2f-1)-p\cdot[p+q\cdot Q(2f+2s-1)]-q\cdot p\cdot Q(2f-2s)$$
$$=p+q\cdot p\cdot Q(4f-2)-p^2-p\cdot q\cdot Q(2f+2s-1)-q\cdot p\cdot Q(2f-2s)$$

But $Q(2f-\frac{1}{2})=p+q\cdot Q(4f-2)$ since $2f-\frac{1}{2}>\frac{1}{2}$.

$$=p+p\cdot[Q(2f-\tfrac{1}{2})-p]-p^2-p\cdot q\cdot Q(2f+2s-1)-q+p\cdot Q(2f-2s)$$
$$=p[Q(2f-\tfrac{1}{2})-q\cdot Q(2f-2s)+1-2p-q\cdot Q((2f+2s-1)]$$

But note that the following inequally holds ;

$$1-2p-q\cdot Q(2f+2s-1)\geqq -p\cdot Q(2f+2s-1),$$

which can be easily seen once it is rewritten as $(1-2p)[1-Q(2f+2s-1)]\geqq 0$ since $p<\frac{1}{2}$ and $Q(2f+2s-1)\leqq 1$.

Therefore,

$$Q(f)-p\cdot Q(f+s)-q\cdot Q(f-s)$$
$$=p\cdot[Q(2f-\tfrac{1}{2})-q\cdot Q(2f-2s)+1-2p-q\cdot Q(2f+2s-1)]$$
$$\geqq p\cdot[Q(2f-\tfrac{1}{2})-q\cdot Q(2f-2s)-p\cdot Q(2f+2s-1)]$$
$$=p\cdot[Q(2f-\tfrac{1}{2})-q\cdot Q(2f-\tfrac{1}{2}-(2s-\tfrac{1}{2}))-p\cdot Q(2f-\tfrac{1}{2}+2s-\tfrac{1}{2})]$$

$\geqq 0$ since $2f-\frac{1}{2}$, $2f-2s$, and $2f+2s-1$ are of order at most n.

This will complete the proof of theorem 1. The proofs of Case 3 ($f+s \leqq \frac{1}{2}$) and Case 4 ($f \leqq \frac{1}{2} < f+s$) are similar to those of Case 1 and 2 respectively and thus are omitted here. Q. E. D.

III. Optimum Strategy with $p>½$

Theorem 2. For $p>½$ there exists a strategy which wins with probability one. The strategy is to bet $f\cdot\alpha$ with $0<\alpha<1$. For α sufficiently small, f reaches 1 with probability one.

Proof : The current fortune f can reach $f+f\cdot\alpha$ with probability p and $f-f\cdot\alpha$ with probability $q(=1-p)$. Let us define the random variable z_i at the ith play which has $1+\alpha$ with probability p and $1-\alpha$ with probability q. Then the fortune f changes as follows :

$$f \to f\cdot z_1 \to f\cdot z_1\cdot z_2 \to \cdots \to f\cdot z_1\cdot z_2\cdots z_n$$

Let f_n denote $f\cdot z_1\cdot z_2\cdots z_n$. Then, $\log f_n = \log f + \log z_1 + \log z_2 + \cdots + \log z_n$.
Now, consider the expectation of $\log z$. $E(\log z) = p\cdot\log(1+\alpha) + q\cdot\log(1-\alpha)$.
Thus, $E(\log z)$ is a function of α, say $g(\alpha)$.
We can easily see that $g(0)=0$, $g'(\alpha) = \frac{p}{1+\alpha} - \frac{q}{1-\alpha}$, and $g'(0) = p-q>0$.
Therefore, $g(\alpha)>0$ for small $\alpha>0$. And $\log f_n$ goes to infinity almost surely, and so does f_n , which implies that the probability of f reaching 1 is equal to one.
Q. E. D.

IV. Conclusion

The optimum strategy is to play boldly, thus to bet as much as you can when $p<½$, and to bet the small portion of the current when $p>½$. We can consider the same kind of optimum strategies in the case where the bet is discrete. This problem can also be many other problems in stochastic processes.

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