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Final Smooth Fuzzy Topologies

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ABSTRACT

We will prove the existence of final smooth fuzzy topological spaces and final smooth fuzzy closure spaces. From this fact, we can define quotient spaces of their spaces.

1. Introduction

A.P. Sostak [14,15] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology [1]. It has been developed in many directions [2-9,12,13] Moreover, K.C. Chattopadhyay and S.K. Samanta [3] introduced smooth fuzzy closure spaces. In [8,12], it is proved the existence of initial smooth fuzzy topological spaces as a generalization of subspaces and product spaces of smooth fuzzy topological spaces.

In this paper, we will prove the existence of final smooth fuzzy topological spaces and final smooth fuzzy closure spaces. From this fact, we can define quotient spaces of their spaces. Furthermore, we investigate the relationship between final smooth fuzzy topological spaces and final smooth fuzzy closure spaces.

2. Preliminaries

In this paper, let X be a nonempty set, $I=[0, 1]$ and $I_0=(0, 1]$. Let I^X be the set of all fuzzy sets on X. $\tilde{0}$ and $\tilde{1}$ denote the constant fuzzy sets in X, taking values 0 and 1 respectively. A *fuzzy point* x_t for $t\in I_0$ is an element of I^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y=x, \\ 0, & \text{otherwise.} \end{cases}$$

Let $f: X\to Y$ be a function, $\mu\in I^X$ and $\nu\in I^Y$. We define the direct image, $f(\mu)$ and inverse image, $f^{-1}(\nu)$ of μ and ν under f

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x\in f^{-1}(y)\}, & \text{if } f^{-1}(y)\neq\emptyset, \\ 0, & \text{if } f^{-1}(y)=\emptyset, \end{cases}$$

and $f^{-1}(\nu)(x)=\nu(f(x))$, respectively.

Lemma 2.1 [6] If $f: X\to Y$, then we have the following properties for direct and inverse image of fuzzy sets under the mapping f: for ν, $\nu_i\in I^Y$ and μ, $\mu_i\in I^X$,

(1) $\nu\geq f(f^{-1}(\nu))$ with equality if f is surjective,
(2) $\mu\leq f^{-1}(f(\mu))$ with equality if f is injective,
(3) $f^{-1}(\tilde{1}-\nu)=\tilde{1}-f^{-1}(\nu)$,
(4) $f(\tilde{1}-\mu)=\tilde{1}-f(\mu)$ if f is bijective,
(5) $f^{-1}(\bigvee_{i\in\Gamma}\nu_i)=\bigvee_{i\in\Gamma}f^{-1}(\nu_i)$,
(6) $f^{-1}(\bigwedge_{i\in\Gamma}\nu_i)=\bigwedge_{i\in\Gamma}f^{-1}(\nu_i)$,
(7) $f(\bigvee_{i\in\Gamma}\mu_i)=\bigvee_{i\in\Gamma}f(\mu_i)$,
(8) $f(\bigwedge_{i\in\Gamma}\mu_i)\leq\bigwedge_{i\in\Gamma}f(\mu_i)$ with equality if f is injective.

Definition 2.2 [14] A function $\tau: I^X\to I$ is called a *fuzzy smooth topology* on X if it satisfies the following conditions:

(O1) $\tau(\tilde{0})=\tau(\tilde{1})=1$.
(O2) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1, \mu_2\in I^X$.
(O3) $\tau(\bigvee_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for any $\{\mu_i \mid i\in\Gamma\}\subseteq I^X$.

The pair (X, τ) is called a *smooth fuzzy topological space.*

Let τ_1 and τ_2 be smooth fuzzy topologies on X. We say that τ_1 is *finer* than τ_2 (τ_2 is *coarser* than τ_1) iff $\tau_2(\lambda)\leq\tau_1(\lambda)$, for all $\lambda\in I^X$.

Let (X, τ) be a smooth fuzzy topological space, then for each $r\in I$, $\tau_r=\{\mu\in I^X \mid \tau(\mu)\geq r\}$ is a Chang's fuzzy topology on X.

Definition 2.3 [3] A function $C: I^X\times I_0\to I^X$ is called a *smooth fuzzy closure operator* on X if it satisfies the following conditions: for each $\lambda, \mu\in I^X$, $r, s\in I_0$,

(C1) $C(\tilde{0}, r)=\tilde{0}$.
(C2) $\lambda\leq C(\lambda, r)$.
(C3) $C(\lambda\vee\mu, r)=C(\lambda, r)\vee C(\mu, r)$.
(C4) If $r\leq s$, then $C(\lambda, r)\leq C(\lambda, s)$.

The pair (X, C) is called *a smooth fuzzy closure space.*

A smooth fuzzy closure space (X, C) is called *topological* provided that

(C5) $C(C(\lambda, r), r) = C(\lambda, r)$, for all $\lambda \in I^X$, $r \in I_0$.

Let C_1 and C_2 be smooth fuzzy closure operators on X. We say that C_1 is *finer* than C_2(C_2 is *coarser* than C_1) iff $C_1(\lambda, r) \leq C_2(\lambda, r)$, for all $\lambda \in I^X$, $r \in I_0$.

Theorem 2.4 [3] Let (X, τ) be a smooth fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$, we define an operator $C_\tau : I^X \times I_0 \rightarrow I^X$ as follows:

$C_\tau(\lambda, r) = \wedge \{\mu \mid \mu \geq \lambda,\ \tau(\tilde{1} - \mu) \geq r\}$.

Then it satisfies the following properties: for each λ, $\mu \in I^X$, r, $s \in I_0$,

(1) $C_\tau(\tilde{0}, r) = \tilde{0}$.

(2) $\lambda \leq C_\tau(\lambda, r)$.

(3) $C_\tau(\lambda \vee \mu, r) = C_\tau(\lambda, r) \vee C_\tau(\mu, r)$.

(4) If $r \leq s$, then $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$.

(5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$,

(6) If $r = \vee \{s \in I_0 \mid C_\tau(\lambda, s) = \lambda\}$, then $C_\tau(\lambda, r) = \lambda$.

Theorem 2.5 [3] Let (X, C) be a smooth fuzzy closure space. Define the function $\tau_C : I^X \rightarrow I$ on X by

$\tau_C(\lambda) = \vee \{r \in I_0 \mid C(\tilde{1} - \lambda, r) = \tilde{1} - \lambda\}$.

Then:

(1) τ_C is a smooth fuzzy topology on X.

(2) We have $C = C_{\tau_C}$ iff (X, C) satisfies the following conditions:

(a) It is a topological smooth fuzzy closure space.

(b) If $r = \vee \{s \in I_0 \mid C(\lambda, s) = \lambda\}$, then $C(\lambda, r) = \lambda$.

Theorem 2.6 [9] Let (X, τ) be a smooth fuzzy topological space. Let (X, C_τ) be a smooth fuzzy closure space induced by (X, τ). Then τ_{C_τ} is a smooth fuzzy topology on X such that $\tau_{C_\tau} = \tau$.

Definition 2.7 Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a function. Then:

(1) f is called *smooth continuous* if $\tau_2(\lambda) \leq \tau_1(f^{-1}(\lambda))$ for all $\lambda \in I^Y$.

(2) f is called *smooth open* if $\tau_1(\mu) \leq \tau_2(f(\mu))$ for all $\mu \in I^X$.

(3) f is called *smooth closed* if $\tau_1(\tilde{1} - \mu) \leq \tau_2(\tilde{1} - f(\mu))$ for all $\mu \in I^X$.

Definition 2.8 [3] Let (X, C_1) and (Y, C_2) be smooth fuzzy closure spaces. A function $f : (X, C_1) \rightarrow (Y, C_2)$ is called a *C-map* if for all $\lambda \in I^X$, $r \in I_0$, $f(C_1(\lambda, r)) \leq C_2(f(\lambda), r)$.

Theorem 2.9 [9] Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. Then $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is a smooth continuous map iff $f : (X, C_{\tau_1}) \rightarrow (Y, C_{\tau_2})$ is a *C-map.*

3. Final smooth fuzzy topological spaces

Theorem 3.1 Let Y be a set and $\{(X_i, \tau_i)\}_{i \in \Lambda}$ be a family of smooth topological spaces. Let $f_i : X_i \rightarrow Y$ be a function for each $i \in \Lambda$. Define the function $\tau : I^Y \rightarrow I$ by

$\tau(\lambda) = \wedge_{i \in \Lambda} \tau_i(f_i^{-1}(\lambda))$.

Then:

(1) τ is the finest smooth fuzzy topology on Y for which each f_i is a smooth continuous map.

(2) $f : (Y, \tau) \rightarrow (Z, \tau_Z)$ is a smooth continuous map iff each $f \circ f_i : (X, \tau_i) \rightarrow (Z, \tau_Z)$ is a smooth continuous map.

Proof. (1) First, we will show that τ is the smooth fuzzy topology on Y.

(O1) It is easily proved from the definition of τ.

(O2) Suppose there exist μ, $\nu \in I^Y$ such that $\tau(\mu \wedge \nu) < \tau(\mu) \wedge \tau(\nu)$.

From the definition of τ, there exists $i \in \Lambda$ such that $\tau(\mu \wedge \nu) \leq \tau_i(f_i^{-1}(\mu \wedge \nu)) < \tau(\mu) \wedge \tau(\nu)$.

On the other hand, we have

$$\begin{aligned} \tau_i(f_i^{-1}(\mu \wedge \nu)) &= \tau_i(f_i^{-1}(\mu) \wedge (f_i^{-1}(\nu)) \\ &\geq \tau_i(f_i^{-1}(\mu)) \wedge \tau_i(f_i^{-1}(\nu)) \qquad \text{(by (O2))} \\ &\geq \tau(\mu) \wedge \tau(\nu). \end{aligned}$$

It is a contradiction. Hence $\tau(\mu \wedge \nu) \geq \tau(\mu) \wedge \tau(\nu)$ for all μ, $\nu \in I^Y$.

(O3) Suppose there exists a family $\{\mu_j \in I^Y \mid j \in \Gamma\}$ such that

$\tau(\vee_{j \in \Gamma} \mu_j) < \wedge_{j \in \Gamma} \tau(\mu_j)$.

From the definition of τ, there exists $i \in \Lambda$ such that $\tau(\vee_{j \in \Gamma} \mu_j) \leq \tau_i(f_i^{-1}(\vee_{j \in \Gamma} \mu_j)) < \wedge_{j \in \Gamma} \tau(\mu_j)$.

On the other hand, we have

$$\begin{aligned} \tau_i(f_i^{-1}(\vee_{j \in \Gamma} \mu_j) &= \tau_i(\vee_{j \in \Gamma} f_i^{-1}(\mu_j)) \\ &\geq \wedge_{j \in \Gamma} \tau_i(f_i^{-1}(\mu_j)) \qquad \text{(by (O3))} \\ &\geq \wedge_{j \in \Gamma} \tau(\mu_j). \end{aligned}$$

It is a contradiction. Hence $\tau(\vee_{j \in \Gamma} \mu_j) \geq \wedge_{j \in \Gamma} \tau(\mu_j)$, for any $\{\mu_j \mid j \in \Gamma\} \subseteq I^Y$.

Second, since $\tau(\lambda) \leq \tau_i(f_i^{-1}(\lambda))$ for each $i \in \Lambda$, each f_i is a smooth continuous map.

Finally, we will show that τ is the finest smooth fuzzy topology on Y for which each f_i is a smooth continuous map.

If $f_i : (X_i, \tau_i) \rightarrow (Y, \tau^*)$ is smooth continuous, we have $\tau^*(\lambda) \leq \tau_i(f_i^{-1}(\lambda))$, for each $i \in \Lambda$, $\lambda \in I^Y$. From the definition of τ, it follows $\tau^*(\lambda) \leq \tau(\lambda)$ for all $\lambda \in I^Y$.

(2) Necessity of the composition condition is clear since the composition of smooth continuous maps is a

smooth continuous map.

Conversely, since each $f \circ f_i : (X, \tau_i) \to (Z, \tau_Z)$ is a smooth continuous map, we have for each $\mu \in I^Z$,

$$\tau_Z(\mu) \leq \tau_i((f \circ f_i)^{-1}(\mu)) = \tau_i(f_i^{-1}(f^{-1}(\mu))).$$

From the definition of τ, it follows $\tau_Z(\mu) \leq \tau(f^{-1}(\mu))$ for all $\mu \in I^Z$. Hence $f : (Y, \tau) \to (Z, \tau_Z)$ is a smooth continuous map. □

From Theorem 3.1, we can define the following definition.

Definition 3.2 In Theorem 3.1, the structure τ is called the *final smooth fuzzy topology* on Y associated the families $\{(X_i, \tau_i)\}_{i\in\Lambda}$ and $(f_i)_{i\in\Gamma}$.

From Theorem 3.1, we can easily prove the following corollaries.

Corollary 3.3 Let $\{(X_i, \tau_i)\}_{i\in\Lambda}$ be a family of smooth fuzzy topological spaces, for different $i, j \in \Lambda$, X_i and X_j be disjoint, $X = \bigcup_{i\in\Lambda} X_i$. Let $E_i : X_i \to X$ be an identity function for each $i \in \Gamma$. Define the function $\tau : I^Y \to I$ by

$$\tau(\lambda) = \bigwedge_{i\in\Lambda} \tau_i(E_i^{-1}(\lambda)).$$

Then:

(1) τ is the finest smooth fuzzy topology on X for which each E_i is smooth continuous.

(2) $f : (X, \tau) \to (Z, \tau_Z)$ is a smooth continuous map iff each $f \circ E_i : (X, \tau_i) \to (Z, \tau_Z)$ is a smooth continuous map.

In above corollary, the pair (X, τ) is called the *sum smooth fuzzy topological space* of $\{(X_i, \tau_i)\}_{i\in\Lambda}$.

Corollary 3.4 Let Y be a set and (X, τ) be a smooth topological space. Let $f : X \to Y$ be a surjective function. Define the function $\tau_f : I^Y \to I$ by

$$\tau_f(\lambda) = \tau(f^{-1}(\lambda)).$$

Then:

(1) τ_f is the finest smooth fuzzy topology on X which f is a smooth continuous map.

(2) $g : (Y, \tau_f) \to (Z, \tau_Z)$ is smooth continuous iff $g \circ f : (X, \tau) \to (Z, \tau_Z)$ is smooth continuous.

From Corollary 3.4, we can define the following definition.

Definition 3.5 Let (X, τ) be a smooth fuzzy topological space and Y a set. Let $f : X \to Y$ be a surjective function. The final smooth fuzzy topology τ_f on Y associated the (X, τ) and f is called the *quotient smooth fuzzy topology* and the function f is called a *fuzzy quotient map.*

Theorem 3.6 Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. Let $f : (X, \tau_1) \to (Y, \tau_2)$ is a surjective smooth continuous map.

(1) If f is a smooth open map, then f is a fuzzy quotient map.

(2) If f is a smooth closed map, then f is a fuzzy quotient map.

Proof. (1) We only show that $\tau_2 = \tau_f$. From Corollary 3.4 (1), we have $\tau_2(\lambda) \leq \tau_f(\lambda)$ for all $\lambda \in I^Y$. Conversely, we have it from the followings:

$$\begin{aligned}\tau_f(\lambda) &= \tau_1(f^{-1}(\lambda)) \\ &\leq \tau_2(f(f^{-1}(\lambda))) && \text{(because } f \text{ is smooth open)} \\ &= \tau_2(\lambda). && \text{(because } f \text{ is surjective)}\end{aligned}$$

(2) It is similar to (1).

Theorem 3.7 Let Y be a set and $\{(X_i, C_i)\}_{i\in\Lambda}$ be a family of smooth closure spaces. Let $f_i : X_i \to Y$ be a surjective function for each $i \in \Lambda$. Define the function $C : I^Y \times I_0 \to I^Y$ by

$$C(\lambda, r) = \bigvee_{i\in\Lambda} f_i(C_i(f_i^{-1}(\lambda), r)).$$

Then:

(1) C is the finest smooth fuzzy closure operator on Y for which each f_i is a fuzzy C-map.

(2) $f : (Y, C) \to (Z, C_Z)$ is a C-map iff each $f \circ f_i : (X, C_i) \to (Z, C_Z)$ is a C-map.

Proof. (1) First, we will show that C is the smooth fuzzy closure operator on Y.

(C1), (C3) and (C4) are easily proved from the definition of C.

(C2) We proved it as follows:

$$\begin{aligned}C(\lambda, r) &= \bigvee_{i\in\Lambda} f_i(C_i(f_i^{-1}(\lambda), r)) \\ &\geq f_i(C_i(f_i^{-1}(\lambda), r)) \\ &\geq f_i(f_i^{-1}(\lambda)) && \text{(by (C2))} \\ &= \lambda. && \text{(because } f_i \text{ is surjective)}\end{aligned}$$

Second, we have

$$\begin{aligned}C(f_i(\lambda), r) &= \bigvee_{i\in\Lambda} f_i(C_i(f_i^{-1}(f_i(\lambda)), r)) \\ &\geq f_i(C_i(f_i^{-1}(f_i(\lambda)), r)) \\ &\geq f_i(C_i(\lambda, r)). && \text{(by Lemma 2.1(2))}\end{aligned}$$

Hence $f_i(X_i, C_i) \to (Y, C)$ is a C-map.

Finally, we will show that C is the finest smooth fuzzy closure operator on Y for which each f_i is a C-map.

If $f_i : (X_i, C_i) \to (Y, C^*)$ is a C-map for each $i \in \Lambda$, then we have, for each $\lambda_i \in I^{X_i}$ and $r \in I_0$,

$$f_i(C_i(\lambda_i, r)) \leq C^*(f_i(\lambda_i), r). \qquad \text{(A)}$$

It follows that

$$C(\lambda, r) = \bigvee_{i\in\Lambda} f_i(C_i(f_i^{-1}(\lambda), r))$$
$$\leq \bigvee_{i\in\Lambda} C^*(f_i(f_i^{-1}(\lambda)), r)) \quad \text{(by (A))}$$
$$= C^*(\lambda, r). \quad \text{(by Lemma 2.1(1))}$$

(2) Necessity of the composition condition is clear since the composition of C-maps is a C-map.

Conversely, since each $f \circ f_i(X_i, C_i) \rightarrow (Z, C_Z)$ is a C-map, then we have

$$(f \circ f_i)(C_i(\lambda_i, r)) \leq C_Z(f \circ f_i(\lambda_i), r). \quad \text{(B)}$$

It follows that

$$f(C(\lambda, r)) = f(\bigvee_{i\in\Lambda} f_i(C_i(f_i^{-1}(\lambda), r)))$$
$$= \bigvee_{i\in\Lambda} f(f_i(C_i(f_i^{-1}(\lambda), r))) \quad \text{(by Lemma 2.1(7))}$$
$$\leq \bigvee_{i\in\Lambda} C_Z(f(f_i(f_i^{-1}(\lambda))), r) \quad \text{(by (B))}$$
$$= C_Z(f(\lambda), r). \quad \text{(by Lemma 2.1(1))}$$

From Theorem 3.7, we can define the following definition.

Definition 3.8 The structure C is called the *final smooth fuzzy closure operator* on Y associated the families $\{(X_i, C_i)\}_{i\in\Lambda}$ and $(f_i)_{i\in\Gamma}$.

Corollary 3.9 Let $\{(X_i, C_i)\}_{i\in\Lambda}$ be a family of smooth fuzzy closure spaces, for different $i, j\in\Lambda$, X_i and X_j be disjoint, $X=\bigcup_{i\in\Lambda} X_i$. Let $E_i : X_i\rightarrow X$ be an identity function for each $i\in\Gamma$. Define the function $C : I^X\times I_0\rightarrow I^X$ by

$$C(\lambda, r) = \bigvee_{i\in\Lambda} E_i(C_i(E_i^{-1}(\lambda), r)).$$

Then:

(1) C is the finest smooth fuzzy closure operator on X for which each E_i is a fuzzy C-map.

(2) $f : (Y, C)\rightarrow(Z, C_Z)$ is a C-map iff each $f \circ E_i : (X, C_i)\rightarrow(Z, C_Z)$ is a C-map.

In above corollary, the pair (X, C) is called the *sum smooth fuzzy closure space* of $\{(X, C_i)\}_{i\in\Lambda}$.

Definition 3.10 Let (X, C) be a smooth fuzzy closure space and Y a set. Let $f : X\rightarrow Y$ be a surjective function. Define the function $C_f : I^Y\times I_0\rightarrow I^Y$ by

$$C_f(\lambda, r) = f(C(f^{-1}(\lambda), r)).$$

The (Y, C_f) induced by f is called the *fuzzy quotient space* of (X, C) and the function f is called a *fuzzy quotient map*.

Theorem 3.11 Let Y be a set and $\{(X_i, \tau_i)\}_{i\in\Lambda}$ be a family of smooth topological spaces. Let $f : X_i\rightarrow Y$ be a surjective function for each $i\in\Lambda$ and $\{(X_i, \tau_i)\}_{i\in\Lambda}$ a family of smooth closure spaces induced by $\{(X_i, \tau_i)\}_{i\in\Lambda}$.

Define the function τ on Y as Theorem 3.1. Define the function $C : I^Y\times I_0\rightarrow I^Y$ by

$$C(\lambda, r)= \bigvee_{i\in\Lambda} f_i(C_{\tau_i}(f_i^{-1}(\lambda), r)).$$

Then:

(1) C is finer than C_τ induced by τ.

(2) $\tau_C=\tau$.

Proof. (1) Since $f_i : (X, \tau_i)\rightarrow(Y, \tau)$ is a smooth continuous for each $i\in\Gamma$, by Theorem 2.9, $f_i : (X, C_{\tau_i})\rightarrow(Y, C_\tau)$ is a C-map for each $i\in\Gamma$. From Theorem 3.7(1), C is finer than C_τ.

(2) First, we will show that for each $i\in\Lambda$, $f_i : (X_i, \tau_i)\rightarrow(Y, \tau_C)$ is fuzzy continuous. Suppose there exists $\lambda \in I^Y$ such that

$$\tau_C(\lambda)>\tau_i(f_i^{-1}(\lambda)).$$

Then there exists $r_0\in I_0$ with $C(\tilde{1} - \lambda, r_0)=\tilde{1} - \lambda$ such that

$$\tau_C(\lambda)\geq r_0>\tau_i(f_i^{-1}(\lambda)).$$

On the other hand, we have

$$\tilde{1} - \lambda = C(\tilde{1} - \lambda, r_0)$$
$$= \bigvee_{i\in\Lambda} f_i(C_{\tau_i}(f_i^{-1}(\tilde{1} - \lambda), r_0))$$
$$\geq f_i(C_{\tau_i}(\tilde{1} - f_i^{-1}(\lambda), r_0)).$$

It implies

$$\tilde{1} - f_i^{-1}(\lambda) = f_i^{-1}(\tilde{1} - \lambda)$$
$$\geq f_i^{-1}(f_i(C_{\tau_i}(\tilde{1} - f_i^{-1}(\lambda), r_0)))$$
$$\geq C_{\tau_i}(\tilde{1} - f_i^{-1}(\lambda), r_0). \quad \text{(by Lemma 2.1 (2))}$$

From (C2) of Theorem 2.4, we have $C_{\tau_i}(\tilde{1} - f_i^{-1}(\lambda), r_0)=\tilde{1} - f_i^{-1}(\lambda)$. Since $\tau_{C_{\tau_i}}=\tau_i$ from Theorem 2.6, we have

$$\tau_i(f_i^{-1}(\lambda))\geq r_0.$$

It is a contradiction.

Hence $f_i : (X_i, \tau_i)\rightarrow(Y, \tau_C)$ is smooth continuous.

Next, since τ is the final smooth fuzzy topology on Y, by Theorem 3.1(1), we have $\tau_C(\lambda)\leq\tau(\lambda)$ for all $\lambda\in I^Y$.

Conversely, since $\tau_{C_\tau}=\tau$ from Theorem 2.6, we only show that $\tau_{C_\tau}(\lambda)\leq\tau_C(\lambda)$ for all $\lambda\in I^Y$.

Suppose there exists $\lambda\in I^Y$ such that $\tau_{C_\tau}(\lambda)>\tau_C(\lambda)$.

Then there exists $r_0\in I_0$ with $C_\tau(\tilde{1} - \lambda, r_0)=\tilde{1} - \lambda$ such that

$$\tau_{C_\tau}(\lambda)\geq r_0>\tau_C(\lambda).$$

On the other hand, we have

$$\tilde{1} - \lambda = C_\tau(\tilde{1} - \lambda, r_0)$$
$$\geq C(\tilde{1} - \lambda, r_0). \quad \text{(by (1))}$$

Hence $C(\tilde{1} - \lambda, r_0)=\tilde{1} - \lambda$ from Definition 2.3(C2).

So, $\tau_C(\lambda) \geq r_0$. It is a contradiction. □

Example 1. Let $X=\{a, b\}$, $Y=\{x\}$ be sets. Define $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, \text{ if } \lambda=\tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, \text{ if } \lambda=\tilde{1}-a_{0.5} \text{ or } \tilde{1}-b_{0.7}, \\ \frac{1}{2}, \text{ if } \lambda=\tilde{1}-(a_{0.5} \vee b_{0.7}), \\ 0, \text{ otherwise.} \end{cases}$$

From Theorem 2.4, we can obtain $C_\tau : I^X \times I_0 \to I^X$ as follows:

$$C_\tau(\lambda, r) = \begin{cases} \tilde{0}, \text{ if } \lambda=\tilde{0},\ r \in I_0, \\ a_{0.5}, \text{ if } \tilde{0} \neq \lambda \leq a_{0.5},\ 0 < r \leq \frac{1}{2}, \\ b_{0.7}, \text{ if } \tilde{0} \neq \lambda \leq b_{0.7},\ 0 < r \leq \frac{1}{2}, \\ a_{0.5} \vee b_{0.7}, \text{ if } \lambda \leq a_{0.5} \vee b_{0.7}, \\ \quad \lambda \not\leq a_{0.5},\ \lambda \not\leq b_{0.7},\ 0 < r \leq \frac{1}{2}, \\ \tilde{1}, \text{ otherwise.} \end{cases}$$

From Corollary 3.4, we have a quotient space τ_f on Y of (X, τ) as follows:

$$\tau_f(v)=\tau(f^{-1}(v))= \begin{cases} 1, \text{ if } v=\tilde{0} \text{ or } \tilde{1}, \\ 0, \text{ otherwise.} \end{cases}$$

From Theorem 2.4, we have

$$C_{\tau_f}(v, r) = \begin{cases} \tilde{0}, \text{ if } v=\tilde{0} \text{ or } r \in I_0, \\ \tilde{1}, \text{ otherwise.} \end{cases}$$

Since $C_f(v, r)=f(C_\tau(f^{-1}(v), r))$ from Theorem 3.11, we have

$$C_f(v, r) = \begin{cases} \tilde{0}, \text{ if } v=\tilde{0},\ r \in I_0, \\ x_{0.7}, \text{ if } \tilde{0} \neq v \leq x_{0.5},\ 0 < r \leq \frac{1}{2}, \\ \tilde{1}, \text{ otherwise.} \end{cases}$$

Hence C_f is finer than C_{τ_f} and $C_{\tau_f} \neq C_f$. Moreover, C_τ is topological from Theorem 2.4. Since

$$x_{0.7} = C_f\left(x_{0.5}, \frac{1}{3}\right) \neq C_f\left(\left(x_{0.5}, \frac{1}{3}\right), \frac{1}{3}\right) = \tilde{1},$$

a fuzzy closure operator C_f is not topological. From Theorem 2.5, we have

$$\tau_{C_f}(v) = \begin{cases} 1, \text{ if } v=\tilde{0} \text{ or } \tilde{1}, \\ 0, \text{ otherwise.} \end{cases}$$

Hence $\tau_{C_f}=\tau_f$. □

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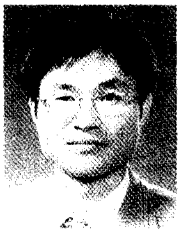

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