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On vector variational-type inequalities for fuzzy mappings

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ABSTRACT

In this paper, we introduce vector variational-type inequalities for fuzzy mappings on Hausdorff topological vector spaces and obtain an existence theorem of solutions to the inequalities.

1. Introduction and preliminaries

In 1989, Chang *et al.* [6] introduced the concept of variational inequalities for fuzzy mappings in locally convex Hausdorff topological vector spaces and investigated the existence theorems for some kinds of variational inequalities for fuzzy mappings, which were the fuzzy extensions of some variational inequality problems in [11, 25, 28, 30]. Recently, Chang [3] proved the coincidence theorems for fuzzy mappings and some existence theorems for more general variational inequalities for fuzzy mappings. Lee *et al.* [13, 14] obtained some existence theorems of certain variational inequalities for fuzzy mappings. Especially, in [13] they followed the approach of Chang and Zhu [6] and used the results of Kim and Tan [9]. Noor [23] suggested an iterative scheme for finding the approximate solutions for a variational inequality for fuzzy mappings, and proved that this approximate solution converges strongly to the exact solution for his inequality. Lee *et al.* [17] formulated a strongly quasivariational inequality for fuzzy mappings, which is a more generalized form than that of Noor [23], used the projection method to suggest an iterative algorithm for finding the approximate solutions for their inequality, and proved that this approximate solution converges strongly to the exact solution for the inequality. Lee *et al.* [19] considered vector variational inequalities for fuzzy mappings, which were the fuzzy extensions of vector variational inequalities studied by Chen and Yang [7], and obtained some existence theorems of solutions for their inequalities for fuzzy mappings. By using the Fan-Glicksberg-Kakutani fixed point theorem and the scalarization method of Luc *et al.* [21, 22], Chang *et al.* [4] obtained several kinds of existence theorems for vector quasivariational inequalities of type (1) on locally convex Hausdorff topological vector spaces. They [5] also proved some other existence theorems for vector quasivariational inequalities of types (1) and (2) on Hausdorff topological vector spaces, using the Fan-Browder fixed point theorem, the selection theorem of Yannelis-Prabhakar [29] and the scalarization method of Luc *et al.* [21, 22]. Recently Park *et al.* [24] considered the fuzzy extension of their existence theorem for a general vector-valued variational inequality considered in the noncompact setting and Lee *et al.* [18] introduced two kinds of vector variational inequalities for fuzzy mappings; Their first inequality is a more general form than the vector variational inequality for fuzzy mappings, which was studied by Lee *et al.* [19]. And they proved the existence theorems of solutions for their inequalities. Their results can be regarded as fuzzy extensions of corresponding ones by Lee *et al.* [20] and Siddiqi *et al.* [27]. Most recently the authors [12] considered vector variational-like inequalities, which was introduced by Ansari [1], for fuzzy mappings on Hausdorff topological vector spaces.

On the other hand, Behera and Panda [2] considered a scalar variational-type inequality for single-valued mappings on Hausdorff topological vector spaces, which is a generalized form of a variational-like inequality. Recently, Lee *et al.* [16] considered vector variational-type inequality for set-valued mappings on Hausdorff topological vector spaces.

Our motivation of this paper is to consider vector variational-type inequalities for fuzzy mappings on Hausdorff topological vector spaces. In proof of our main theorem, we use Fan's geometrical lemma [8], which has been applied to variational inequality problems, complementarity problems, game theory and so on.

Let E be a nonempty subset of a vector space X and D be a nonempty set. A function F from D into the collection $\mathcal{F}(E)$ of all fuzzy sets on E is called a fuzzy mapping. If $F : D \to \mathcal{F}(E)$ is a fuzzy mapping, then $F(x)$, $x \in D$ (denoted by F_x in the sequel) is a fuzzy set in $\mathcal{F}(E)$ and $F_x(y)$, $y \in E$ is the degree of membership of y in F_x.

The fuzzy mapping $F : D \to \mathcal{F}(E)$ is said to be convex if E is a convex subset of X and for any $x \in D$,

$y, z \in E$ and $\lambda \in [0, 1]$,

$$F_x(\lambda y + (1-\lambda)z) \geq \min\{F_x(y), F_x(z)\}.$$

Let $A \in \mathcal{F}(E)$ and $\alpha \in (0, 1]$. Then the set

$$(A)_\alpha = \{x \in E : A(x) \geq \alpha\}$$

is called an α-cut set of A.

Now we give some definitions and preliminary results needed in the later section.
The following geometrical lemma is essential.

Lemma 1.1 [8] Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let A be a subset of $K \times K$ satisfying the following conditions;
(i) for each $x \in K$, $(x, x) \in A$,
(ii) for each fixed $x \in K$, the set $A_x = \{y \in K : (x, y) \in A\}$ is closed in K,
(iii) for each fixed $y \in K$, the set $A^y = \{x \in K : (x, y) \notin A\}$ is convex in K.
Then there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$.

Definition 1.1 [15] Let X and Y be two topological spaces, and $T : X \to 2^Y$ be a multifunction. We say that
(1) T is upper semi-continuous (briefly, u.s.c.) at $x_0 \in X$ if for any open set N containing $T(x_0)$, there exists a neighbourhood M of x_0 such that $T(M) \subset N$. T is u.s.c. if T is u.s.c. at every $x_0 \in X$.
(2) T is closed at $x \in X$ if for any net $\{x_\alpha\}$ in X such that $x_\alpha \to x$ and for any net $\{y_\alpha\}$ in Y such that $y_\alpha \to y$ and $y_\alpha \in T(x_\alpha)$ for any α, we have $y \in T(x)$.
(3) T has a closed graph if the graph of T, $Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Definition 1.2 [18] Let X and Y be two topological spaces, and $F : X \to \mathcal{F}(Y)$ be a fuzzy mapping. We say that F is a fuzzy mapping with closed fuzzy set-values if $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function.

Lemma 1.2 If A is a closed subset of a topological space X, then the characteristic function $\mathcal{X}_A$ of A is an u.s.c. real-valued function.

Lemma 1.3 [3] Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, E be a nonempty closed convex subset of a real Hausdorff topological vector space Y and $\alpha : X \to (0, 1]$ be a lower semi-continuous function. Let $F : K \to \mathcal{F}(E)$ be a fuzzy mapping with $(F_x)_{\alpha(x)} \neq \emptyset$ for any $x \in X$. Let $\tilde{F} : K \to 2^E$ be a multifunction defined by $\tilde{F}(x) = (F_x)_{\alpha(x)}$. If F is a convex fuzzy mapping with closed fuzzy set-values, then $\tilde{F}$ is a closed multifunction with nonempty convex values.

2. Vector variational-type inequalities for fuzzy mappings

Let X and Y be two Hausdorff topological vector spaces, and $L(X, Y)$ be the space of all linear continuous operators from X into Y. Let K be a nonempty compact convex subset of X and $\{C(x) : x \in K\}$ be a family of convex cones in Y such that for any $x \in K$, $C(x) \neq Y$ and $intC(x) \neq \emptyset$, where int denotes the interior. Let $F : X \to \mathcal{F}(L(X, Y))$ be a fuzzy mapping on $L(X, Y)$ and $\alpha : X \to (0, 1]$ be a function. We define a partial order $\leq_{C(x)}$ in Y with the convex cone $C(x)$ as; for $y_1, y_2 \in Y$,

$$y_1 \leq_{C(x)} y_2 \text{ if and only if } y_2 - y_1 \in C(x).$$

Definition 2.1 [10] A mapping $f : K \to Y$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

that is,

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x).$$

In this paper, first we consider the following vector variational-type inequality for fuzzy mappings:

(FVVTI) find $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in (F_{x_0})_{\alpha(x_0)}$ such that

$$\langle s_0, \theta(y, x_0)\rangle + \eta(x_0, y) \notin -intC(x_0),$$

where $\langle s, x\rangle$ is the evaluation of $s \in L(X, Y)$ at $x \in X$.

Next we consider the following vector variational-like inequality for fuzzy mappings:
(FVVLI) find $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in (F_{x_0})_{\alpha(x_0)}$ such that

$$\langle s_0, \theta(y, x_0)\rangle \notin -intC(x_0).$$

And then, we check the following vector variational inequality for fuzzy mappings:
(FVVI) find $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in (F_{x_0})_{\alpha(x_0)}$ such that

$$\langle s_0, y - x_0\rangle \notin -intC(x_0).$$

It is obvious that (FVVLI) is a generalized form of (FVVI), and that (FVVLI) is a special case of (FVVTI).

Now we consider our main result for (FVVTI), where bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous for simplicity.

Theorem 2.1 Let K be a nonempty compact convex subset of X. Let $F : K \in \mathcal{F}(L(X, Y))$ be a fuzzy mapping with closed fuzzy set-values. Let a multifunction $W : K \to 2^Y$, defined by $W(x) = Y\backslash\{-intC(x)\}$, $x \in K$, have a closed graph, and $\theta : K \times K \to X$ and $\eta : K \times K \to Y$ be mappings. Suppose further that

(1) there exists a lower semi-continuous function $\alpha : X \to (0, 1]$ such that for any $x \to K$, the cut set $(F_x)_{\alpha(x)}$ is nonempty and $\bigcup_{x \in K}(F_x)_{\alpha(x)}$ is contained in some compact subset of $L(X, Y)$,

(2) for any $x \in K$, there exists $s \in (F_x)_{\alpha(x)}$ such that

$<s, \theta(x, x)> + \eta(x, x) \notin -intC(x)$,

(3) the operator

$x \mapsto <s, \theta(x, y)> + \eta(y, x)$

of K into Y is convex for any $y \in K$ and any $s \in (F_y)_{\alpha(y)}$, and

(4) the mappings

$y \mapsto \theta(\cdot, y)$ and $y \mapsto \eta(y, \cdot)$

are continuous.

Then (FVVTI) is solvable.

Proof Define a multifunction $\tilde{F} : K \to 2^{L(X,Y)}$ by $\tilde{F}(x) = (F_x)_{\alpha(x)}$. It follows from Lemma 1.3 and the condition (1) that $\tilde{F}$ is a nonempty closed multifunction such that $\tilde{F}(K)$ is contained in some compact subset of $L(X, Y)$. Let

$A := \{(x, y) \in K \times K :$ there exists $s \in \tilde{F}(y)$ such that $<s, \theta(x, y)> + \eta(y, x) \notin -intC(y)\}$.

Now we will show that (i), (ii) and (iii) of Lemma 1.1 are satisfied. From the definition of A and the condition (2), we can deduce that $(x, x) \in A$.

Next we will show that for each fixed $x \in K$,

$A_x := \{y \in K : (x, y) \in A\}$
$= \{y \in K :$ there exists $s \in \tilde{F}(y)$ such that $<s, \theta(x, y)> + \eta(y, x) \notin -intC(y)\}$

is closed in K. Indeed, let $\{y_\lambda\}$ be a net in A_x such that $y_l \to y_0 \in K$. Since $y_\lambda \in A_x$,

there exists $s_\lambda \in \tilde{F}(y_\lambda)$ such that $< s_\lambda, \theta(x, y_\lambda) > + \eta(y_\lambda, x) \in W(y_\lambda)$.

By the condition (1), without loss of generality, we can assume that there exists $s_0 \in L(X, Y)$ such that $s_\lambda \to s_0$. From Lemma 1.3, $\tilde{F}$ is a nonempty closed multifunction, hence $s_0 \in \tilde{F}(y_0)$. Since W has a closed graph, by the condition (4),

there exists $s_0 \in \tilde{F}(y_0)$ such that $<s_0, \theta(x, y_0)> + \eta(y_0, x) \in W(y_0)$.

Hence $y_0 \in A_x$ and thus A_x is closed in K.

It remains to show that for each fixed $y \in K$,

$A^y : \{x \in K : (x, y) \notin A\}$
$= \{x \in K :$ for any $s \in \tilde{F}(y)$, $<s, \theta(x, y)> + \eta(y, x) \in -intC(y)\}$

is convex. In fact, let $x_1, x_2 \in A_y$ and $t \in [0, 1]$, by the condition (3), we have for any $y \in K$ and any $s \in \tilde{F}(y)$,

$$<s, \theta(tx_1 + (1-t)x_2, y)> + \eta(y, tx_1 + (1-t)x_2) \leq_{C(y)} t[<s, \theta(x_1, y)> + \eta(y, x_1)] + (1-t)[<s, \theta(x_2, y)> + \eta(y, x_2)].$$

Thus

$$t[<s, \theta(x_1, y)> + \eta(y, x_1)] + (1-t)[<s, \theta(x_2, y)> + \eta(y, x_2)] - [<s, \theta(tx_1 + (1-t)x_2, y)> + \eta(y, tx_1 + (1-t)x_2)] \in C(y).$$

Since $<s, \theta(x_1, y)> + \eta(y, x_1) \in -intC(y)$ and $<s, \theta(x_2, y)> + \eta(y, x_2) \in -intC(y)$, we have

$<s, \theta(tx_1 + (1-t)x_2, y)> + \eta(y, tx_1 + (1-t)x_2) \in -intC(y)$.

Thus $tx_1 + (1-t)x_2 \in A^y$ and hence A^y is convex.

By Lemma 1.1, there exists an $x_0 \in K$ such that $K \times x_0\} \subset A$. This implies that there exists an $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in \tilde{F}(x_0) = (F_{x_0})_{\alpha(x_0)}$ such that

$<s_0, \theta(y, x_0)> + \eta(x_0, y) \notin -intC(x_0)$.

This completes the proof.

Considering a zero mapping η in Theorem 2.1, we can obtain the existence theorem of solutions to the following vector variational-like inequality for fuzzy mappings.

Theorem 2.2 [12] Let K be a nonempty compact convex subset of X. Let $F : K \to \mathcal{F}(L(X, Y))$ be a fuzzy mapping with closed fuzzy set-values. Let a multifunction $W : K \to 2^Y$, defined by $W(x) = Y\backslash\{-intC(x)\}$, $x \in K$, have a closed graph, and $\theta : K \times K \to X$ be a continuous mapping. Suppose that

(1) there exists a lower semi-continuous function $\alpha : X \to (0, 1]$ such that for any $x \in K$, the cut set $(F_x)_{\alpha(x)}$ is nonempty and $\bigcup_{x \in K}(F_x)_{\alpha(x)}$ is contained in some compact subset of $L(X, Y)$,

(2) for any $x \in K$, there exists $s \in (F_x)_{\alpha(x)}$ such that

$<s, \theta(x, x)> \notin -intC(x)$, and

(3) the operator

$x \mapsto <s, \theta(x, y)>$

of K into Y is convex for any $y \in K$ and any $s \in (F_y)_{\alpha(y)}$.

Then (FVVLI) is solvable.

From Theorem 2.2, we can obtain the following theorem for multi-valued mappings as a corollary.

Theorem 2.3 [12] Let K be a nonempty compact convex subset of X. Let $T : K \to 2^{L(X,Y)}$ be a continuous mapping with nonempty closed set-values and $\theta : K \times K \to X$ be continuous mapping. Let a multifunction $W : K \to 2^Y$, defined by $W(x) = Y \setminus \{-int(x)\}$, $x \in K$, have a closed graph. Suppose that

(1) there exists $s \in T(x)$ such that

$<s, \theta(x, x)> \notin -intC(x)$ for any $x \in K$, and

(2) the operator

$x \mapsto <s, \theta(x, y)>$

of K into Y is convex for any $y \in K$ and $s \in T(y)$.

Then there exists an $x_0 \in K$ such that for any $y \in K$ there exists $s_0 \in T(x_0)$ such that

$<s_0, \theta(y, x_0)> \notin -intC(x_0)$.

Proof Define a fuzzy mapping $F : K \to \mathcal{F}(L(X, Y))$ by $F_x = \mathcal{X}_{T(x)}$, the characteristic function of $T(x)$ for $x \in K$, then from Lemma 1.2, F is a fuzzy mapping with closed fuzzy set-values. On the other hand, $(F_x)_1 = (\mathcal{X}_{T(x)})_1 = T(x)$ is nonempty and $\bigcup_{x \in K}(F_x)_1 = \bigcup_{x \in K} T(x) = T(K)$ is compact in $L(X, Y)$. By Theorem 2.2, we can obtain the conclusion of Theorem 2.3.

Now we can also obtain the following theorem as a corollary.

Theorem 2.4 [26] Let K be a nonempty compact convex subset of X. Let $T : K \to L(X, Y)$ and $\theta : K \times K \to X$ be continuous mappings. Let a multifunction $W : K \to 2^Y$, defined by $W(x) = Y \setminus \{-intC(x)\}$, $x \in K$, have a closed graph. Suppose that

(1) for any $x \in K$,

$<T(x), \theta(x, x)> \notin -intC(x)$, and

(2) the operator

$x \mapsto <T(y), \theta(x, y)>$

of K into Y is convex for any $y \in K$.

Then there exists an $x_0 \in K$ such that

$<T(x_0), \theta(y, x_0)> \notin -intC(x_0)$,

for any $y \in K$.

Remark Putting $\theta(x, y) = x - y$ in Theorem 2.2, we can also show the existence of solutions to (FVVI).

References

[1] Q. H. Ansari, On generalized variational-like inequalities, *Ann. Sci.* Québec 19, 131-137, 1995.

[2] A. Behera and G. K. Panda Generalized variational-type inequality in Hausdorff topological vector spaces Indian *J. pure appl. Math.* 28(3), 343-349, 1997.

[3] S. S. Chang Coincidence theorems and variational inequalities for fuzzy mappings, *Fuzzy Sets and Systems*, 61, 359-368, 1994.

[4] S. S. Chang, G. M. Lee and B. S. Lee, Vector quasivariational inequalities for fuzzy mappings (I), *Fuzzy Sets and Systems*, 87, 307-315, 1997.

[5] S. S. Chang, G. M. Lee and B. S. Lee, Vector quasivariational inequalities for fuzzy mappings (II), *Fuzzy Sets and Systems*, 102, 333-344, 1999.

[6] S. S. Chang and Y. G. Zhu, On variational inequalities for fuzzy mappings, *Fuzzy Sets and Systems*, 32, 359-367, 1989.

[7] G. Y. Chen and X. Q. Yang, Vector complementarity problems and its equivalences with weak minimal element in ordered spaces, *J. Math. Anal. Appl.* 153, 136-158, 1990.

[8] K. Fan, A generalization of Tychonoff's fixed point theorems, *Math. Ann.* 142, 305-310, 1961.

[9] W. K. Kim and K. K. Tan, A variational inequality in non-compact sets and its applications, *Bull. Austral. Soc.* 46, 139-148, 1992.

[10] D. Kuroiwa,Convexity for set-valued maps, *Appl. Math. Lett.* 9, 97-101, 1996.

[11] M. Lassonde, On the use of KKM multifunctions in fixed point theory and related topics, *J. Math. Anal. Appl.* 97, 151-201, 1983.

[12] Byung-Soo Lee and Doo-Young Jung, A fuzzy extension of Siddiqi *et al.*'s results for vector variational-like inequalities, submitted.

[13] B. S. Lee, G. M. Lee, S. J. Cho and D. S. Kim, A variational inequality for fuzzy mappings, Proceedings of Fifth International Fuzzy Systems Association World Congress, Seoul, 326-329, 1993.

[14] B. S. Lee, G. M. Lee and D. S. Kim, Generalized vector-valued variational inequalities and fuzzy extensions, *J. Korean Math. Soc.* 33, 609-624, 1996.

[15] B. S. Lee, G. M. Lee and D. S. Kim, Generalized vector variational-like inequalities on locally convex Hausdorff topological vector spaces, Indian *J. pure appl. Math.* 28, 33-41, 1997.

[16] Byung-Soo Lee and Suk-Jin Lee, Vector variational-type inequalities for set-valued mappings, *Appl. Math. Lett*(to appear).

[17] G. M. Lee, D. S. Kim and B. S. Lee, Strongly quasivariational inequalities for fuzzy mappings, *Fuzzy Sets and Systems*, 78, 381-386, 1996.

[18] G. M. Lee, D. S. Kim and B. S. Lee, Vector variational inequalities for fuzzy mappings, *Nonlinear Analysis Forum*, 4, 119-129, 1999.

[19] G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, Generalized vector variational inequality and fuzzy extension, *Appl. Math. Lett.* 6, 47-51, 1993.

[20] G. M. Lee, B. S. Lee, D. S. Kim and G. Y. Chen, On vector variational inequalities for multifunctions, Indian *J. pure appl. Math.* 28, 633-639, 1997.

[21] D. T. Lu, Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems, vol 319, Springer Berlin, 1989.
[22] D. T. Luc and C. Vargas, A saddle point theorem for set-valued maps, Nonlinear Analysis T.M.A. 18, 1-7, 1992.
[23] N. A. Noor, Variational inequalities for fuzzy mappings (I), *Fuzzy Sets and Systems*, 55, 309-312, 1993.
[24] S. Park, B. S. Lee and G. M. Lee, A general vector-valued variational inequality and its fuzzy extension, Internat. *J. Math & Math. Sci.* 21(4), 637-642, 1998.
[25] M. H. Shih and K. K. Tan, Generalized quasi-variational inequalities in locally convex topological spaces, *J. Math. Anal. Appl.* 108, 333-343, 1985.
[26] A. H. Siddiqi, Q. H. Ansari and R. Ahmad, On vector variational-like inequalities, Indian *J. pure appl. Math.* 28(8), 1009-1016, 1997.
[27] A. H. Siddiqi, Q. H. Ansari and A. Khaliq, On vector variational inequalities, *J. Optim. Th. Appl.* 84, 171-180, 1995.
[28] W. Takahashi, Nonlinear variational inequalities and fixed point theorems, *J. Math. Soc. Japan*, 28, 142-150, 1976.
[29] N. C. Yannelis and N. D. Prabhakar, Existence of minimal elements and equilibria in linear topological spaces, *J. Math. Econom.* 12, 233-245, 1983.
[30] C. L. Yen, A minimax inequality and its applications to variational inequalities, *Pacific J. Math.* 97, 142-150, 1981.

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