Robust Stability of a Two-Degrees-of-Freedom Servosystem with Structured and Unstructured Uncertainties

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A two-degrees-of-freedom servosystem for step-type reference signals has been proposed, in which the integral compensation is effective only when there is a modeling error or a disturbance input. This paper considers robust stability of the servosystem incorporating an observer against both structured and unstructured uncertainties of the plant. A condition is obtained as a linear matrix inequality, under which the servosystem is robustly stable independently of the gain of the integral compensator. This result implies that we can tune the gain to achieve a desirable transient response of the servosystem preserving robust stability. An example is presented to demonstrate that under the robust stability condition, the transient response can be improved by increasing the gain of the integral compensator.

Key Words : Two-Degree-of-Freedom Servosystem, Integral Compensation, Uncertainties, Robust Stability, Observer, Linear Matrix Inequality, Transient Response.

1. Introduction

In order to reject the steady-state tracking error to step-type reference signals, it is standard to introduce integral compensators in servosystems. However, if the mathematical model of the plant is exact and there is no disturbance to the plant, then the integral compensation is not necessary as implied by the Internal Model Principle (Francis and Wonham, 1975). From this point of view, a two-degrees-of-freedom (2DOF) servosystem has been proposed (Fujisaki and Ikeda, 1991, 1992, Hagiwara et al., 1991) in the context of LQ regulator theory, in which the integral compensation is effective only when there is a modeling error or a nonzero disturbance. And, to maintain the system performance when the plant parameters are uncertain or varying, the auto-tuning controller design problem has been considered (Cha et. al., 2000).

For the 2DOF servosystem, the authors have presented a robust stability condition for the case when the plant has structured uncertainty (Kim et al., 1995, Kobayashi et al., 1995). The present paper extends this condition to the case of both structured and unstructured uncertainties. We derive a robust stability condition in terms of a linear matrix inequality, which is independent of the gain of the integral compensator. This result implies that we can tune the gain to achieve a desirable transient response of the servosystem preserving robust stability. We present an example demonstrating that under the robust stability condition, the transient response can be improved by increasing the gain of the integral compensator.

2. Two-Degrees-of-Freedom Servosystem

Let us consider an uncertain plant described by

$$\dot{x}(t) = (A_0 + A_\delta)x(t) + (B_0 + B_\delta)u(t)$$
$$\tilde{y}(t) = (C_0 + C_\delta)x(t)$$
$$u_z(s) = N(s)u(s)$$

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$$u_w(s)=\tilde{\Delta}(s)u_z(s),\ \|\tilde{\Delta}(s)\|_\infty \le \gamma$$
$$y_\delta(s)=M(s)u_w(s)$$
$$y(t)=\tilde{y}(t)+y_\delta(t) \quad (1)$$

where $x(t)$, $u(t)$, and $y(t)$ are the state, control input, and controlled output, A_0, B_0, and C_0 represent the nominal plant, and A_δ, B_δ, and C_δ denote the structured uncertainties. The unstructured uncertainty of the plant is additive and written as $M(s)\tilde{\Delta}(s)N(s)$, where, $M(s)$, $N(s)$ are known stable transfer functions which have the following realizations

$$\dot{x}_m(t)=A_m x_m(t)+B_m u_w(t)$$
$$y_\delta(t)=C_m x_m(t)+D_m u_w(t)$$
$$\dot{x}_n(t)=A_n x_n(t)+B_n u(t)$$
$$\dot{u}_z(t)=C_n x_n(t)+D_n u(t) \quad (2)$$

where A_m, And A_n are stable matrices. We assume that the controlled output $y(t)$ is measurable. In Eq. (1), for simplicity, we omitted disturbances to the plant, which doesn't affect the stability analysis of the resultant servosystem.

We require the plant to track a step-type reference signal $r(t)$ with no error in the steady-state. For this, we assume that the pair (A_0, B_0) is stabilizable, (C_0, A_0) is detectable, and

$$\det\begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix} \neq 0 \quad (3)$$

In order to cope with uncertainties of the plant, we usually apply an integral compensator

$$\dot{w}(t)=e(t),\ e(t)=r(t)-y(t) \quad (4)$$

to the tracking error $e(t)$. To synthesize a 2DOF servosystem which is originally proposed using state feedback (Fujisaki and Ikeda, 1991, 1992, Hagiwara et al., 1991), we employ a full-order observer for the nominal plant,

$$\dot{\hat{x}}(t)=A_0\hat{x}(t)+B_0u(t)-L\{y(t) - C_0\hat{x}(t)\} \quad (5)$$

where L is chosen so that A_0+LC_0 is a stable matrix.

For the augmented system consisting of the plant, the integral compensator and the observer, the following control law has been proposed to obtain a 2DOF servosystem (Fujisaki and Ikeda, 1991, 1992, Hagiwara et al., 1991, 1994, Kim et al., 1995, Kobayashi et al., 1995) illustrated in Fig. 1. Where, F_0 is to be determined such that $A_0 + B_0F_0$ is a stable matrix, and

$$u(t)=F_0x(t)+H_0r(t)+v(t)$$
$$v(t)=G_0Wz(t)$$
$$z(t)=F_1\hat{x}(t)+w(t)-\{F_1\hat{x}(0)+w(0)\} \quad (6)$$

$$H_0=\{-C_0(A_0+B_0F_0)^{-1}B_0\}^{-1}$$
$$F_1=C_0(A_0+B_0F_0)^{-1} \quad (7)$$

The gain G_0 is chosen so that it satisfies

$$F_1B_0G_0+(F_1B_0G_0)^T<0 \quad (8)$$

and W is a positive definite matrix considered as a tuning parameter. This control law ensures stability of the servosystem if there is no uncertainty in the plant.

In the 2DOF servosystem illustrated in Fig. 1, $z(t)\equiv 0$ and $v(t)\equiv 0$ hold if there is no uncertainty in the plant, no disturbance to the plant, and no estimation error, that is, $\hat{x}(t)=x(t)$.

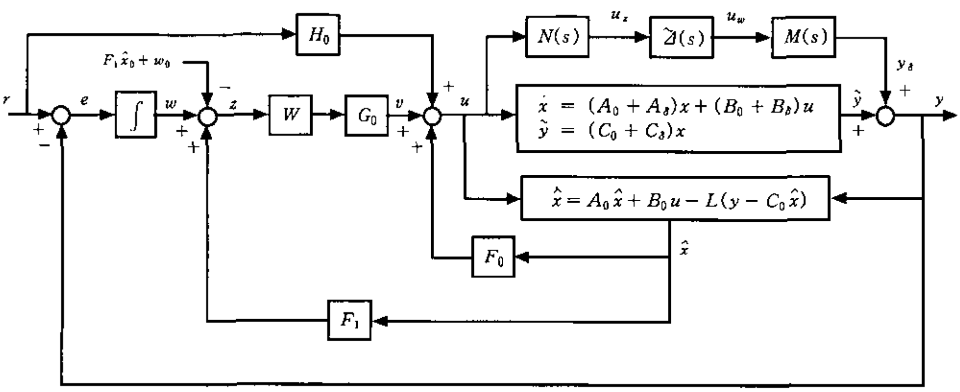


Fig. 1 A two-degree-of-freedom servosystem incorporating an observer

This implies that the nominal transient response of the servosystem to the reference signal $r(t)$ is determined only by the gain F_0. The effect of the integral compensator appears only when uncertainties of the plant, disturbances to the plant, or estimation errors exist. The integral compensator can be tuned by the gain W. In this sense, we call the control system of Fig. 1 a two-degrees-of-freedom servosystem.

3. Robust Stability Independent of Integral Compensator Gain

To examine the robust stability of the 2DOF servosystem, we set $r(t)=0$, $F_1\tilde{x}(0)+w(0)=0$ and consider the system with the input $u_w(t)$ and output $u_z(t)$,

$$\dot{\tilde{x}}(t)=\{\tilde{A}_0(W)+\tilde{A}_\delta(W)\}\tilde{x}(t)+\tilde{B}u_w(t)$$
$$u_z(t)=\tilde{C}(W)\tilde{x}(t) \tag{9}$$

where $\tilde{\Delta}(s)$ is removed,

$$\tilde{x}(t)=\begin{bmatrix} x_n(t) \\ x_m(t) \\ x(t) \\ \{\hat{x}(t)-x(t)\} \\ z(t) \end{bmatrix}$$

is chosen as the state,

$$\tilde{A}_0(W)=\begin{bmatrix} A_m & 0 & B_nF_0 & B_nF_0 & B_nG_0W \\ 0 & A_m & 0 & 0 & 0 \\ 0 & 0 & A_0+B_0F_0 & B_0F_0 & B_0G_0W \\ 0 & -LC_m & 0 & A_0+LC_0 & 0 \\ 0 & -(F_1L+I)C_m & 0 & (F_1L+I)C_0 & F_1B_0G_0W \end{bmatrix} \tag{10}$$

denotes the nominal part of the system matrix,

$$\tilde{A}_\delta(W)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\delta+B_\delta F_0 & B_\delta F_0 & B_\delta G_0W \\ 0 & 0 & -(A_\delta+LC_\delta+B_\delta F_0) & -B_\delta F_0 & -B_\delta G_0W \\ 0 & 0 & -(F_1L+I)C_\delta & 0 & 0 \end{bmatrix} \tag{11}$$

means the structured uncertain part, and

$$\tilde{B}=\begin{bmatrix} 0 \\ B_m \\ 0 \\ -LD_m \\ -(F_1L+I)D_m \end{bmatrix} \tag{12}$$

$$\tilde{C}(W)=[C_n\ 0\ D_nF_0\ D_nF_0\ D_nG_0W].$$

From the small gain theorem, the servosystem is stable if the system (9) is stable and its H_∞ norm from $u_w(s)$ to $u_z(s)$ is less than $1/\gamma$.

Applying the bounded real lemma expressed by a linear matrix inequality (LMI) (Gahinet and Apkarian, 1994) with a positive definite matrix

$$\tilde{P}(W)=\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & 0 \\ P_{12}^T & P_{22} & P_{23} & P_{24} & 0 \\ P_{13}^T & P_{23}^T & P_{33} & P_{34} & 0 \\ P_{14}^T & P_{24}^T & P_{34}^T & P_{44} & 0 \\ 0 & 0 & 0 & 0 & W \end{bmatrix} \tag{13}$$

we obtain a condition for the robust stability of the augmented system independent of W.

Theorem *For the structured uncertainties A_δ, B_δ, C_δ and the upper bound $\gamma>0$ of the unstructured uncertainty $\tilde{\Delta}(s)$, 2DOF servosystem shown in Fig. 1 is robustly stable independently of the tuning parameter W, if there exist a positive definite matrix $\tilde{P}(I)$ and a positive number μ such that the inequality*

$$\begin{bmatrix} \tilde{P}(I)\{\tilde{A}_0(I)+\tilde{A}_\delta(I)\}+\{\tilde{A}_0(I)+\tilde{A}_\delta(I)\}^T\tilde{P}(I) & \gamma\tilde{P}(I)\tilde{B} & \mu\tilde{C}^T(I) \\ \gamma\tilde{B}^T\tilde{P}(I) & -\mu I & 0 \\ \mu\tilde{C}(I) & 0 & -\mu I \end{bmatrix}<0 \tag{14}$$

holds.

Proof Equations (10) ~ (11) and (13) imply

$$\tilde{A}(W)=\tilde{A}(I)J(W)=[\tilde{A}_0(I)+\tilde{A}_\delta(I)]J(W)$$
$$\tilde{P}(W)=J(W)\tilde{P}(I) \tag{15}$$

where

$$J(W)=\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & W \end{bmatrix} \tag{16}$$

Using these relations and the condition (14), we obtain

$$\begin{bmatrix} \tilde{P}(W)\tilde{A}(W)+\tilde{A}^T(W)\tilde{P}(W) & \gamma\tilde{P}(W)\tilde{B} & \mu\tilde{C}(W)^T \\ \gamma\tilde{B}^T\tilde{P}(W) & -\mu I & 0 \\ \mu\tilde{C}(W) & 0 & -\mu I \end{bmatrix} = \begin{bmatrix} J(W) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \begin{bmatrix} \tilde{P}(I)\tilde{A}(I)+\tilde{A}^T(I)\tilde{P}(I) & \gamma\tilde{P}(I)\tilde{B} & \mu\tilde{C}(I)^T \\ \gamma\tilde{B}^T\tilde{P}(I) & -\mu I & 0 \\ \mu\tilde{C}(I) & 0 & -\mu I \end{bmatrix} \begin{bmatrix} J(W) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \tag{17}$$

for any $W>0$. This inequality implies that the system in Equation (9) is stable and its H_∞ norm from $u_w(s)$ to $u_z(s)$ is less than $1/\gamma$. Then, we conclude that the 2DOF servosystem of Fig. 1 is robustly stable independently of W.

This result implies a significantly useful characteristic of the 2DOF servosystem. If the plant uncertainties are in the allowable set defined by the condition (14), we can arbitrarily tune the positive definite gain W preserving robust stability. Note here that the objective of the integral compensation is to reject the tracking error in the steady state, and the effect of the integral compensation appears through W as seen in Fig. 1. Thus, we may make the transient response fast by increasing W. Actually, a numerical example in the next section shows that we can improve the transient response by increasing W.

Lemma *The robust stability condition described in the previous theorem is independent of gain W. It means that we can arbitrarily tune the positive definite gain W preserving robust stability. In other word, high-gain compensation is possible. This fact implies that for the 2DOF servosystem in Fig. 1, there is no zeros of the transfer function from the output to the input of gain W in the right half plan.*

Proof See the Appendix.

4. A Numerical Example

We present an example to illustrate the change of behaviors of the 2DOF servosystem shown in Fig. 1, increasing the tuning parameter W. Let the nominal matrices A_0, B_0, C_0 and the structured uncertainties A_δ, B_δ, C_δ of the plant be

$$A_0=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix},\ B_0=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C_0=[1\ 0\ 0]$$
$$A_\delta=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a_1 \\ a_2 & 0 & a_3 \end{bmatrix},\ B_\delta=\begin{bmatrix} 0 \\ b_1 \\ b_2 \end{bmatrix}$$
$$C_\delta=[0\ 0\ c_1] \tag{18}$$

where a_1, a_2, a_3, b_1, b_2, c_1 are structured uncertain elements. We assume that the unstructured uncertainty is defined by

$$M(s)=\frac{0.05}{s+10},\ N(s)=1 \tag{19}$$

and $\gamma=1$.

We consider a gain F_0 given by

$$F_0=[-6.16\ \ -6.29\ \ -2.19] \tag{20}$$

so that $A_0+B_0F_0$ is stable. Then, the gains H_0 and F_1 are calculated as

$$H_0=[3.16]$$
$$F_0=[-1.99\ \ -1.33\ \ -0.32]. \tag{21}$$

In addition, we choose an observer gain

$$L=\begin{bmatrix} -2.37 \\ -2.31 \\ -2.12 \end{bmatrix} \tag{22}$$

so that A_0+LC_0 is stable, and consider

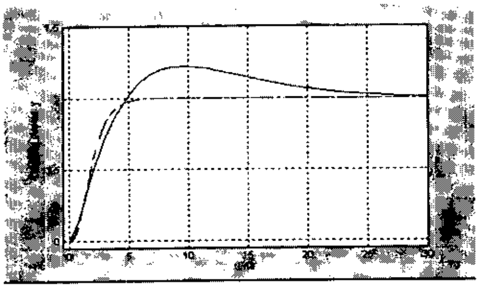
$$G_0=[0.32] \tag{23}$$

so that G_0 satisfies the condition (8).

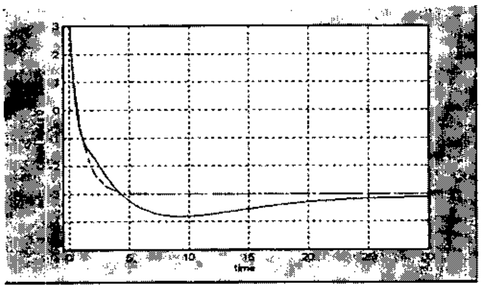
Now, suppose that the structured uncertain elements are

$$a_1=0.05,\ a_2=0.40,\ a_3=-0.50$$
$$b_1=0.05,\ b_2=-0.05,\ c_1=0.10. \tag{24}$$

Then, we see that the robust stability condition (14) holds if we choose P_{22}, P_{23}, P_{24}, P_{33}, P_{34} and

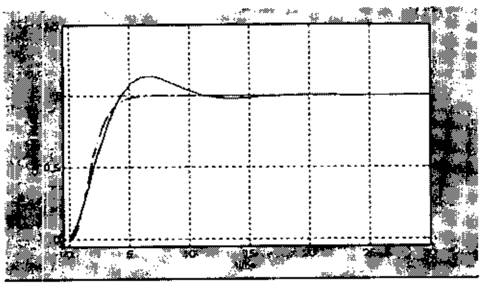


(a) Controlled outputs

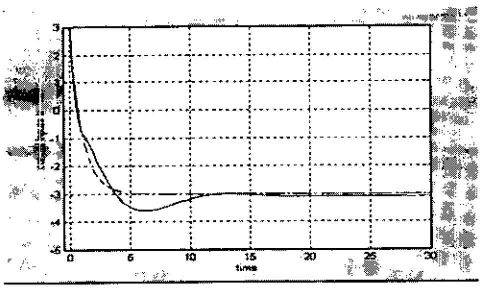


(b) Control inputs

Fig. 2 Step response($W=1$)



(a) Controlled outputs

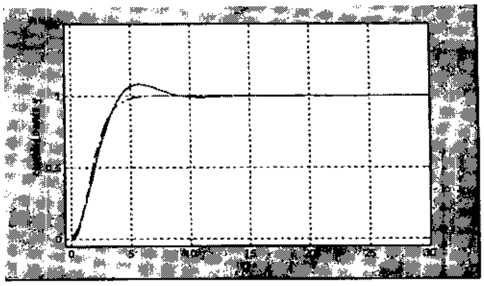


(b) Control inputs

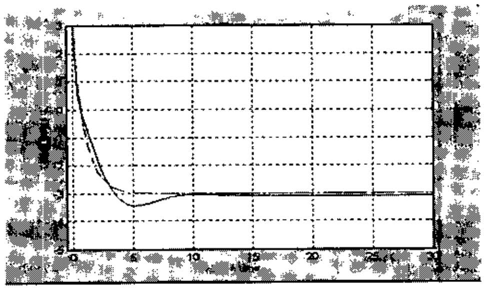
Fig. 3 Step response($W=10$)

P_{44} in $\tilde{P}(W)$ of Eq. (13) as

$$P_{22}=[1.51]$$



(a) Controlled outputs



(b) Control inputs

Fig. 4 Step response($W=100$)

$$P_{23}=[-0.02,\ 1.06,\ 1.21]$$
$$P_{24}=[-1.22\ 0.84\ -0.81]$$
$$P_{33}=\begin{bmatrix} 11.01 & 6.99 & 1.55 \\ 6.99 & 21.72 & 7.17 \\ 1.55 & 7.17 & 5.53 \end{bmatrix}$$
$$P_{34}=\begin{bmatrix} 5.95 & 4.66 & 6.66 \\ 10.87 & 4.80 & -14.28 \\ -4.20 & 9.31 & -4.46 \end{bmatrix}$$
$$P_{44}=\begin{bmatrix} 86.69 & 0.64 & -3.73 \\ 0.64 & 63.19 & 19.41 \\ -3.73 & 19.41 & 34.20 \end{bmatrix} \tag{25}$$

and set $\mu=0.13$. Note here that P_{11}, P_{12}, P_{13} and P_{14} are not needed, because the order $N(s)$ is zero.

Fig. 2~4 show the transient responses of the controlled output $y(t)$ and control input $u(t)$ corresponding to the cases of $W=1$, 10 and 100, respectively. Here, we assume that the unstructured uncertain element is

$$\tilde{\Delta}(s)=-\frac{0.9s+4}{s+5} \tag{26}$$

the reference signal is

$$r(t)=1,\ t\geq 0 \tag{27}$$

and all the initial states are 0. The solid lines show the perturbed behaviors, while the dashed line is the nominal behavior in case there is no uncertainty in the plant. We see that we can achieve a fast tracking response by increasing W.

5. Concluding Remarks

In this paper, we have presented a robust stability condition of the 2DOF servosystem for the case when the plant have structured and unstructured uncertainty. We have derived a robust stability condition in terms of a linear matrix inequality, which is independent of the gain of the integral compensator. This result implies that we can tune the gain to achieve a desirable transient response of the servosystem preserving robust stability. We have presented an example demonstrating that under the robust stability condition, the transient response can be improved by increasing the gain of the integral compensator.

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Appendix

Before proving the lemma, consider the state equation of unstructured uncertainty $\tilde{\Delta}(s)$ represented by

$$\begin{aligned} \dot{x}_d(t) &= A_d x_d(t) + B_d u_z(t) \\ u_w &= C_d x_d(t) \end{aligned} \tag{A1}$$

Here if the stability condition (14) is satisfied then there exists a positive definite symmetric matrix P_d such that a Riccati inequality

$$P_d A_d + A_d^T P_d < -\frac{1}{\mu} P_d B_d B_d^T P_d - \frac{\mu}{\gamma^2} C_d^T C_d \tag{A2}$$

holds. Using this fact, let us prove

$$\begin{bmatrix} P_d & 0 \\ 0 & \bar{P}(I) \end{bmatrix} \begin{bmatrix} A_d & B_d\bar{C}(I) \\ \bar{B}C_d & \bar{A}(I) \end{bmatrix} + \begin{bmatrix} A_d & B_d\bar{C}(I) \\ \bar{B}C_d & \bar{A}(I) \end{bmatrix}^T \begin{bmatrix} P_d & 0 \\ 0 & \bar{P}(I) \end{bmatrix} < 0 \tag{A3}$$

holds.

Proof : From the Eq. (14) and (A2), Eq. (A3) is represented by

$$\begin{bmatrix} P_d A_d + A_d^T P_d & 0 \\ 0 & \bar{P}(I)\bar{A}(I) + \bar{A}^T(I)P(I) \end{bmatrix}$$

$$
\begin{aligned}
&+\begin{bmatrix} \frac{1}{\sqrt{\mu}} P_d B_d \\ 0 \end{bmatrix}[0\ \sqrt{\mu}\,\bar{C}(I)]
+[0\ \sqrt{\mu}\,\bar{C}(I)]^T\begin{bmatrix} \frac{1}{\sqrt{\mu}} P_d B_d \\ 0 \end{bmatrix}^T \\
&+\begin{bmatrix} 0 \\ \frac{\gamma}{\sqrt{\mu}}\bar{P}(I)\bar{B} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{\mu}}{\gamma} C_d & 0 \end{bmatrix}
+\begin{bmatrix} \frac{\sqrt{\mu}}{\gamma} C_d & 0 \end{bmatrix}^T\begin{bmatrix} 0 \\ \frac{\gamma}{\sqrt{\mu}}\bar{P}(I)\bar{B} \end{bmatrix}^T \\
&< -\begin{bmatrix} \frac{1}{\mu}P_d B_d B_d^T P_d + \frac{\mu}{\gamma^2} C_d^T C_d & 0 \\ 0 & \frac{\gamma^2}{\mu}\bar{P}(I)\bar{B}\bar{B}^T P(I) + \mu \bar{C}^T(I)\bar{C}(I) \end{bmatrix} \\
&+\begin{bmatrix} \frac{1}{\sqrt{\mu}} P_d B_d \\ 0 \end{bmatrix}[0\ \sqrt{\mu}\,\bar{C}(I)]
+[0\ \sqrt{\mu}\,\bar{C}(I)]^T\begin{bmatrix} \frac{1}{\sqrt{\mu}} P_d B_d \\ 0 \end{bmatrix}^T \\
&+\begin{bmatrix} 0 \\ \frac{\gamma}{\sqrt{\mu}}\bar{P}(I)\bar{B} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{\mu}}{\gamma} C_d & 0 \end{bmatrix}
+\begin{bmatrix} \frac{\sqrt{\mu}}{\gamma} C_d & 0 \end{bmatrix}^T\begin{bmatrix} 0 \\ \frac{\gamma}{\sqrt{\mu}}\bar{P}(I)\bar{B} \end{bmatrix}^T \\
&= -\begin{bmatrix} \frac{1}{\sqrt{\mu}} P_d B_d \\ -\sqrt{\mu}\,\bar{C}^T(I) \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{\mu}} P_d B_d \\ -\sqrt{\mu}\,\bar{C}^T(I) \end{bmatrix}^T
-\begin{bmatrix} -\frac{\sqrt{\mu}}{\gamma} C_d^T \\ -\frac{\gamma}{\sqrt{\mu}}\bar{P}(I)\bar{B} \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{\mu}}{\gamma} C_d^T \\ -\frac{\gamma}{\sqrt{\mu}}\bar{P}(I)\bar{B} \end{bmatrix}^T
\end{aligned}
\tag{A4}
$$

This result implies that Eq. (A3) is negative definite.

[***Proof of lemma***] For the 2DOF servosystem of Fig. 1, consider the following representation to derive the transfer function (from the output to the input of W) :

$$
\hat{A}=\begin{bmatrix}
A_m & 0 & B_n F_0 & B_n F_0 & 0 \\
0 & A_m & 0 & 0 & 0 \\
0 & 0 & A_0+A_\delta+(B_0+B_\delta)F_0 & (B_0+B_\delta)F_0 & 0 \\
0 & -LC_m & -(A_\delta+B_\delta F_0+LC_\delta) & A_0-B_\delta F_0+LC_0 & 0 \\
0 & -(F_1L+I)C_m & -(F_1L+I)C_\delta & (F_1L+I)C_0 & 0
\end{bmatrix}
$$

$$
\hat{B}=\begin{bmatrix} B_n G \\ 0 \\ (B_0+B_\delta)G \\ -B_\delta G \\ F_1 B_0 G \end{bmatrix}
$$

$$
\hat{C}=[0\ 0\ 0\ 0\ I] \tag{A5}
$$

Then the transfer function is given by $\hat{C}(sI-\hat{A})^{-1}\hat{B}$. And the zeros of this transfer function are the values of s such that

$$
\Pi(s)=\begin{bmatrix} \hat{A}-sI & \hat{B} \\ \hat{C} & 0 \end{bmatrix} \tag{A6}
$$

is nonsingular. From simple calculation, the fact that $\Pi(s)$ is nonsingular is equivalent to

$$
\hat{\Pi}(s)=\begin{bmatrix}
A_n-sI & 0 & B_nF_0 & B_nF_0 & B_nG_0 \\
0 & A_m-sI & 0 & 0 & 0 \\
0 & 0 & A_0+A_\delta+(B_0+B_\delta)F_0-sI & (B_0+B_\delta)F_0 & (B_0+B_\delta)G_0 \\
0 & -LC_m & -(A_\delta+B_\delta F_0+LC_\delta) & A_0-B_\delta F_0+LC_0-sI & -B_\delta G_0 \\
0 & -(F_1L+I)C_m & -(F_1L+I)C_\delta & (F_1L+I)C_0 & F_1B_0G_0
\end{bmatrix}
$$

is nonsingular. Here, let us describe $\hat{\Pi}(s)$ as follows :

$$
\begin{aligned}
\hat{\Pi}(s)&=\hat{\Pi}_r+\hat{\Pi}_i(s) \\
&=\begin{bmatrix}
A_n & 0 & B_nF_0 & B_nF_0 & B_nG_0 \\
0 & A_m & 0 & 0 & 0 \\
0 & 0 & A_0+A_\delta+(B_0+B_\delta)F_0 & (B_0+B_\delta)F_0 & (B_0+B_\delta)G_0 \\
0 & -LC_m & -(A_\delta+B_\delta F_0+LC_\delta) & A_0-B_\delta F_0+LC_0 & -B_\delta G_0 \\
0 & -(F_1L+I)C_m & -(F_1L+I)C_\delta & (F_1L+I)C_0 & F_1B_0G_0
\end{bmatrix} \\
&+\begin{bmatrix}
-sI & 0 & 0 & 0 & 0 \\
0 & -sI & 0 & 0 & 0 \\
0 & 0 & -sI & 0 & 0 \\
0 & 0 & 0 & -sI & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{aligned}
\tag{A7}
$$

and define a positive definite matrix

$$\hat{P}(I)=\begin{bmatrix} P_d & 0 \\ 0 & \bar{P}(I) \end{bmatrix} \tag{A8}$$

From these facts, the following representation is obtained ;

$$\begin{aligned}
&\hat{P}(I)\hat{\Pi}(s)+\hat{\Pi}^*(s)\hat{P}(I) \\
&=\hat{P}(I)\hat{\Pi}_r+\hat{\Pi}_r^T\hat{P}(I) \\
&-2\hat{P}(I)\begin{bmatrix} \mathrm{Re}sI & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Re}sI & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Re}sI & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Re}sI & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{A9}$$

where * represents the conjugate transpose of a matrix.

In the Eq. (A9),

$$\hat{P}(I)\hat{\Pi}_r+\hat{\Pi}_r^T\hat{P}(I)$$

equals the Eq. (A3). If the robust stability condition (14) is satisfied then

$$\hat{P}(I)\hat{\Pi}_r+\hat{\Pi}_r^T\hat{P}(I)<0$$

holds.

Here, if we assume that there exists a negative s such that $\Pi(s)$ is nonsingular, then this assumption means that the left term of Eq. (A9) is positive. But, if we consider that the robust stability condition (14) holds, it is necessary that the right term of Eq. (A9) is negative. Therefore, under the robust stability condition, we can see that the real parts of all zeros of the transfer function (from the output to the input of tuning gain W) are negative values.