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ACYCLIC DIGRAPHS WHOSE 2-STEP COMPETITION GRAPHS ARE $P_n \cup I_2$

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ABSTRACT. The 2-step competition graph of D has the same vertex set as D and an edge between vertices x and y if and only if there exist (x, z)-walk of length 2 and (y, z)-walk of length 2 for some vertex z in D. The 2-step competition number of a graph G is the smallest number k such that G together with k isolated vertices is the 2-step competition graph of an acyclic digraph. Cho, *et al.* showed that the 2-step competition number of a path of length at least two is two. In this paper, we characterize all the minimal acyclic digraphs whose 2-step competiton graphs are paths of length n with two isolated vertices and construct all such digraphs.

1. Introduction

Cohen [7] introduced the notion of competition graph in connection with a problem in ecology in 1968. The *competition graph* of a digraph D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices x and y if and only if there is a vertex z in D such that (x, z) and (y, z) are arcs of D (for all undefined graph theory terminology, see [2]). Since the notion of competition graph was introduced, there has been a very large literature on competition graphs. For surveys of the literature of competition graphs, see [8, 9, 12, 18]. In addition to ecology, their various applications include applications to channel assignments, coding, and modeling of complex economic and energy systems (see [14]). A

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variety of generalizations of the notion of competition graph have been introduced, including the common enemy graph (sometimes called the resource graph) in [13, 17], the competition-common enemy graph (sometimes called the competition-resource graph) in [16], the niche graph in [3], the p-competition graph in [11], and the competition multigraph in [1].

Recently, Cho, *et al.* [4, 5] introduced yet another such generalization, the m-step competition graph, and obtained results about m-step competition graphs analogous to the well-known results about ordinary competition graphs. Given a digraph D and a positive integer m, the *m-step digraph* D^m of D is defined as follows: $V(D^m) = V(D)$ and there exists an arc (u, v) in D^m if and only if there exists a directed walk of length m from u to v. If there is a directed walk of length m from a vertex x to a vertex y in D, we call y an *m-step prey* of x, and if a vertex w is an m-step prey of both vertices u and v, then we say that w is an *m-step common prey* of u and v. The *m-step competition graph* of D, denoted by $C^m(D)$, has the same vertex set as D and an edge between vertices x and y if and only if x and y have an m-step common prey in D. Note that $C^1(D)$ is the (ordinary) competition graph of D.

In studying the (ordinary) competition graphs of *acyclic* digraphs, Roberts [15] observed that adding sufficiently many isolated vertices to an arbitrary graph G makes it into the competition graph of some acyclic digraph. The smallest such number of isolated vertices was called the *competition number* of G and denoted by $k(G)$. Much of the study of competition graphs of acyclic digraph has been focused on competition numbers, since the characterization of competition graphs of acyclic digraphs reduces to the question of computing the competition number of an arbitrary graph. Analogous to the competition number, Cho, *et al.* [4] defined the *m-step competition number* $k^{(m)}(G)$ of G, which is the smallest number k such that G together with k isolated vertices is the m-step competition graph of an acyclic digraph. Also, the double competition number of Scott [16], the p-competition number of Kim, *et al.* [10], the niche number of Cable, *et al.* [3], and the multicompetition number of Anderson, *et al.* [1] have been introduced. Cho, *et al.* [4, 5, 6] computed the 2-step competition numbers of paths and cycles and characterized trees with 2-step competition number two. The following is one of their result about the 2-step competition number of a path of length at least two:

PROPOSITION 1 (Cho, *et al.* [5]). *For any integer* $n \geq 2$, $k^{(2)}(P_n) = 2$.

It seems to be interesting to characterize the set M_n of all the minimal acyclic digraphs whose competition graphs are $P_n \cup I_2$, where P_n and I_k, respectively, denote the path of order n and the graph consisting of k vertices with no edges. In this paper, we completely characterize M_n and, surprisingly, it is shown that for $n \geq 6$, M_n consists of only one digraph.

2. Main Result

Let D be an acyclic digraph with n vertices. An *acyclic labeling* of the vertex set $V(D)$ of D is a labeling of $V(D)$ using the set $\{v_1, v_2, \dots, v_n\}$ so that $i < j$ holds whenever there is an arc (v_i, v_j) in D. An acyclic digraph is said to be *acyclically labeled* if its vertices are acyclically labeled. Given an acyclic digraph D with n vertices and an acyclic labeling $v_1, v_2, \dots, v_n$ of $V(D)$, we call an arc (v_i, v_j) in D a *jump-arc* when $i + 1 < j$. To show our main result, we need the following lemmas.

LEMMA 2. *Let n be any integer greater than one and D be an acyclic digraph such that $C^2(D)$ is P_n with two isolated vertices. Let $v_1, v_2, \dots, v_{n+2}$ be an acyclic labeling of $V(D)$. Then the following are true:*

(i) *$C^2(D - v_{n+2})$ is P_{n-1} with two isolated vertices.*
(ii) *For any i with $2 \leq i \leq n+1$, there exists an arc (v_i, v_{i+1}) in D.*
(iii) *If there is an incoming jump-arc toward v_j, then there is no outgoing jump-arc from v_j.*
(iv) *If v_iv_j $(i < j)$ is an edge in $C^2(D)$, then either (v_i, v_{j+1}) or (v_{i+1}, v_{j+2}) is a jump-arc of D.*

Proof. Proofs of (i) and (ii) were given in Cho, *et al.* [5].

We prove (iii) by contradiction. Suppose that (v_i, v_j) and (v_j, v_l) are jump-arcs in D. Then v_l is a 2-step common prey of v_i, v_{j-1}, and v_{l-2}. Since $i < j - 1$ and $j < l - 1$, the three vertices v_i, v_{j-1} and v_{l-2} are distinct and form a cycle C_3 in $C^2(D)$, which is a contradiction. Hence (iii) follows.

Finally we prove (iv). Suppose that v_iv_j $(i < j)$ is an edge in $C^2(D)$. Then there exists l, $l \geq j + 2$, such that v_l is a 2-step common prey of v_i and v_j. In fact $l = j + 2$, for otherwise v_l is a 2-step common prey of three distinct vertices v_{l-2}, v_j and v_i. Since v_{j+2} is a 2-step prey of v_i, arcs (v_i, v_p) and (v_p, v_{j+2}) are in D for some p. By (iii), either $p = i+1$ or $p = j+1$. Hence (iv) follows. □

Given a graph G, we say that an acyclic digraph D is *minimally associated with* G if $C^2(D)$ is G with $k^{(2)}(G)$ isolated vertices and the 2-step competition graph of any proper subdigraph of D is not G with $k^{(2)}(G)$ isolated vertices. Let

$$\mathcal{D}(G) = \{D \mid D \text{ is an acyclically labeled digraph which is minimally associated with } G.\}$$

Now the following lemma holds:

LEMMA 3. *Let n be an integer with $2 \leq n \leq 5$. Then $\mathcal{D}(P_n)$ consists of the digraphs D_{n*} given in* Figure 1.

Proof. It can easily be checked that each of the digraphs given in Figure 1 is minimally associated with P_n. Let $\mathcal{D}(P_n) = \mathcal{D}_n$. It is easy to see that $\mathcal{D}_2 = \{D_{2a}, D_{2b}\}$. Take D from $\mathcal{D}_n$ $(3 \leq n \leq 5)$. Let $v_1, v_2, \ldots, v_{n+2}$ be an acyclic labeling of $V(D)$. By Lemma 2 (i) and (ii), D contains arc (v_{n+1}, v_{n+2}) and one of the digraphs in $\mathcal{D}_{n-1}$ as a subdigraph.

For $3 \leq n \leq 5$, denote by F_{n-1} the digraph in $\mathcal{D}_{n-1}$ that is contained in D. Then $C^2(F_{n-1})$ is a path with two isolated vertices. Denote the labeled path by P^*_{n-1}. Then v_n is adjacent to one of the end vertices of P^*_{n-1} in $C^2(D)$. Denote by $v_{F_{n-1}}$ the vertex adjacent to v_n. By Lemma 2 (iv), there should be a jump-arc in D in order for v_n and $v_{F_{n-1}}$ to be joined in $C^2(D)$. Denote the jump-arc by $a_{v_{F_{n-1}}}$. By Lemma 2 (iv), $a_{v_{F_{n-1}}} = (v_i, v_{n+1})$ or (v_{i+1}, v_{n+2}) for $v_{F_{n-1}} = v_i$. Then by containing digraph F_{n-1} and arcs (v_{n+1}, v_{n+2}) and $a_{v_{F_{n-1}}}$, D becomes either the digraph in the corresponding row of the last column of Table 1 by the minimality of D, or undefined for various reasons stated in Table 1. By checking Table 1, we can conclude that $\mathcal{D}(P_n)$ consists of the digraphs D_{n*} given in Figure 1. □

Denote by M_n the digraph with

$$V(M_n) = \{v_1, v_2, \ldots, v_{n+2}\}$$

and

$$A(M_n) = \{(v_1, v_3)\} \cup \{(v_i, v_{i+1}) \mid 1 \leq i \leq n+1\} \cup \{(v_i, v_{i+3}) \mid i \text{ even and } 2 \leq i \leq n-1\}.$$

Let $l = \lfloor \frac{n}{2} \rfloor$. Then it is not difficult to check that $C^2(M_n)$ is the path $v_{2l}v_{2(l-1)} \cdots v_4 v_2 v_1 v_3 \cdots v_{2l-1} v_{2l+1}$ together with two isolated vertices v_{n+1} and v_{n+2} if n is odd, and the path $v_{2l}v_{2(l-1)} \cdots v_4 v_2 v_1 v_3 \cdots v_{2l-3} v_{2l-1}$ with two isolated vertices v_{n+1} and v_{n+2} if n is even. That is, the 2-step competition graph of M_n is P_n with two isolated vertices. The following theorem shows that for $n \geq 6$, every acyclic digraph whose competition graph is P_n with two isolated vertices contains M_n as a subdigraph.

THEOREM 4. *For $n \geq 6$, M_n is the unique acyclic digraph minimally associated with P_n.*

Proof. Induct on n. Let D be an acyclic digraph minimally associated with P_6. Let $v_1, v_2, \ldots, v_8$ be the acyclic labeling of $V(D)$. By Lemma 2 (i) and (ii), D contains arc (v_7, v_8) and one of D_{5a}, D_{5b}, D_{5c} in Figure 1. Suppose that D contains D_{5a}. Then v_6 is adjacent to v_4 or v_5 in $C^2(D)$. If v_6 is adjacent to v_4, then (v_4, v_7) or (v_5, v_8) is in D. By Lemma 2 (iii), neither (v_4, v_7) nor (v_5, v_8) can be in D. Thus v_6 is adjacent to v_5 in $C^2(D)$. Then either (v_5, v_7) or (v_6, v_8) is in D. Again by Lemma 2 (iii), neither (v_5, v_7) nor (v_6, v_8) can be in D. Therefore D cannot contain D_{5a}. Suppose that D contains D_{5b}. Then arcs (v_2, v_7), (v_6, v_7), and (v_7, v_8) are in D, and this implies that there is an edge v_2v_6 in $C^2(D)$, which is impossible. Thus D does not contain D_{5b}. Hence D contains D_{5c}. Now D_{5c} together with arc (v_7, v_8) is M_6 and the theorem follows for $n = 6$.

Assume that the theorem holds for $n - 1$ with $n > 6$. Let D be an acyclic digraph minimally associated with P_n. Let $v_1, v_2, \ldots, v_{n+2}$ be an acyclic labeling of $V(D)$. By Lemma 2 (i), $C^2(D - v_{n+2})$ is P_{n-1} with two isolated vertices v_n, v_{n+1}. By the induction hypothesis, $D - v_{n+2}$ contains M_{n-1}. Thus D contains M_{n-1}. By Lemma 2 (ii), D contains arc (v_{n+1}, v_{n+2}). If n is even, M_{n-1} with arc (v_{n+1}, v_{n+2}) is M_n and the theorem follows. Now suppose that n is odd. Since v_{n-1} and v_{n-2} are the end vertices of the path in $C^2(D - v_{n+2})$, v_n is adjacent to v_{n-1} or v_{n-2} in $C^2(D)$. First, suppose that v_n is adjacent to v_{n-1}. Then one of (v_{n-1}, v_{n+1}) or (v_n, v_{n+2}) is in D by Lemma 2 (iv). By Lemma 2 (iii), (v_n, v_{n+2}) cannot be in D. But if (v_{n-1}, v_{n+1}) is in D, then v_{n+1} is a 2-step common prey of v_{n-1}, v_{n-2} and v_{n-3}, which is impossible. Thus v_n is adjacent to v_{n-2}. Then either (v_{n-2}, v_{n+1}) or (v_{n-1}, v_{n+2}) is in D. By Lemma 2 (iii), (v_{n-2}, v_{n+1}) cannot be in D. Therefore (v_{n-1}, v_{n+2}) is in D. Since M_{n-1} together with arcs (v_{n-1}, v_{n+2}) and (v_{n+1}, v_{n+2}) is M_n, D is M_n. □

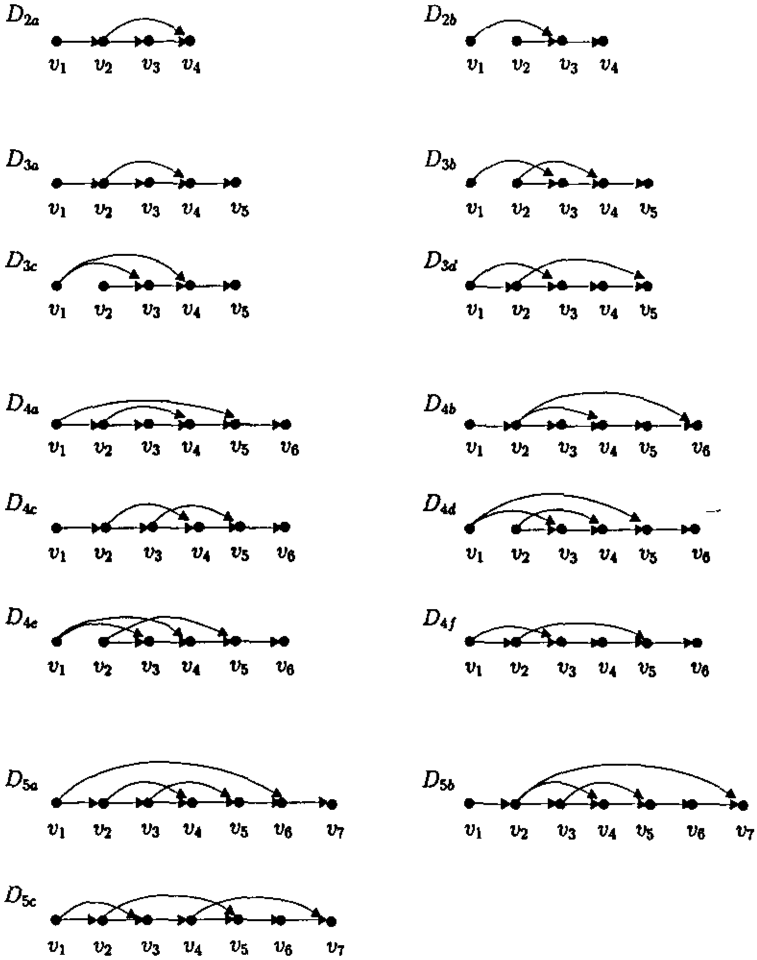

FIGURE 1. D_{n*} are the acyclic digraphs minimally associated with P_n: with two isolated vertices in each case, $C^2(D_{2*})$ are paths v_1v_2; $C^2(D_{3a})$ and $C^2(D_{3b})$ are paths $v_1v_2v_3$; $C^2(D_{3c})$ and $C^2(D_{3d})$ are paths $v_2v_1v_3$; $C^2(D_{4a})$, $C^2(D_{4b})$, and $C^2(D_{4d})$ are paths $v_3v_2v_1v_4$; $C^2(D_{4c})$ is path $v_1v_2v_3v_4$; $C^2(D_{4e})$ and $C^2(D_{4f})$ are $v_3v_1v_2v_4$; $C^2(D_{5a})$ and $C^2(D_{5b})$ are $v_4v_3v_2v_1v_5$; $C^2(D_{5c})$ is path $v_4v_2v_1v_3v_5$.

n	F_{n-1}	P^*_{n-1}	$v_{F_{n-1}}$	$a_{v_{F_{n-1}}}$	D
3	D_{2a}	v_1v_2	v_2	(v_2, v_4)	D_{3a} (see (1))
	D_{2b}	v_1v_2	v_1	(v_1, v_4)	D_{3c}
				(v_2, v_5)	D_{3d} (see (2))
			v_2	(v_2, v_4)	D_{3b}
				(v_3, v_5)	undefined (see (3))
4	D_{3a}	$v_1v_2v_3$	v_1	(v_1, v_5)	D_{4a}
				(v_2, v_6)	D_{4b}
			v_3	(v_3, v_5)	D_{4c}
				(v_4, v_6)	undefined (see (3))
	D_{3b}	$v_1v_2v_3$	v_1	(v_1, v_5)	D_{4d}
				(v_2, v_6)	undefined (see (4))
			v_3	(v_3, v_5) or (v_4, v_6)	undefined (see(3))
	D_{3c}	$v_2v_1v_3$	v_2	(v_2, v_5)	D_{4e}
				(v_3, v_6)	undefined (see (3))
			v_3	(v_3, v_5) or (v_4, v_6)	undefined (see (3))
	D_{3d}	$v_2v_1v_3$	v_2	(v_2, v_5)	D_{4f} (see (1))
5	D_{4a}	$v_3v_2v_1v_4$	v_3	(v_3, v_6)	undefined (see (5))
				(v_4, v_7)	undefined (see (3))
			v_4	(v_4, v_6) or (v_5, v_7)	undefined (see (3))
	D_{4b}	$v_3v_2v_1v_4$			undefined (see (6))
	D_{4c}	$v_1v_2v_3v_4$	v_1	(v_1, v_6)	D_{5a}
				(v_2, v_7)	D_{5b}
			v_4	(v_4, v_6) or (v_5, v_7)	undefined (see (3))
	D_{4d}	$v_3v_2v_1v_4$	v_3	(v_3, v_6) or (v_4, v_7)	undefined (see (3))
			v_4	(v_4, v_6) or (v_5, v_7)	undefined (see (3))
	D_{4e}	$v_3v_1v_2v_4$	v_3	(v_3, v_6) or (v_4, v_7)	undefined (see (3))
			v_4	(v_4, v_6) or (v_5, v_7)	undefined (see (3))
	D_{4f}	$v_3v_1v_2v_4$	v_3	(v_3, v_6)	undefined (see (3))
				(v_4, v_7)	D_{5c}
			v_4	(v_4, v_6)	undefined (see (5))
				(v_5, v_7)	undefined (see (3))

TABLE 1. A part of the proof of Lemma 3. (1) D_{2a} (resp. D_{3d}) together with arc (v_4, v_5) (resp. (v_5, v_6)) is D_{3a} (resp. D_{4f}). By the minimality of D, D is D_{3a} (resp. D_{4f}). (2) If (v_2, v_5) is in D in order for v_5 to be a 2-step prey of v_1, then (v_1, v_2) must be in D. (3) By Lemma 2 (iii). (4) If (v_2, v_6) is in D in order for v_6 to be a 2-step prey of v_1, then (v_1, v_2) must be in D, and so D contains D_{4b} together with arc (v_1, v_3), contradicting the minimality of D. (5) If (v_3, v_6) (resp. (v_4, v_6)) is in D, then v_6 is a 2-step common prey of v_1, v_2, v_4 (resp. v_2, v_3, v_4), which is impossible. (6) Since D contains D_{4b} and arc (v_6, v_7), v_7 is a 2-step common prey of v_5 and v_2 in D. Hence v_5 is adjacent to v_2 in $C^2(D)$ and we reach a contradiction.

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