ACYCLIC DIGRAPHS WHOSE 2-STEP COMPETITION GRAPHS ARE $P_n \cup I_2$

HAN HYUK CHO, SUH-RYUNG KIM, AND YUNSUN NAM

ABSTRACT. The 2-step competition graph of $D$ has the same vertex set as $D$ and an edge between vertices $x$ and $y$ if and only if there exist $(x, z)$-walk of length 2 and $(y, z)$-walk of length 2 for some vertex $z$ in $D$. The 2-step competition number of a graph $G$ is the smallest number $k$ such that $G$ together with $k$ isolated vertices is the 2-step competition graph of an acyclic digraph. Cho, et al. showed that the 2-step competition number of a path of length at least two is two. In this paper, we characterize all the minimal acyclic digraphs whose 2-step competition graphs are paths of length $n$ with two isolated vertices and construct all such digraphs.

1. Introduction

Cohen [7] introduced the notion of competition graph in connection with a problem in ecology in 1968. The competition graph of a digraph $D$, denoted by $C(D)$, has the same set of vertices as $D$ and an edge between vertices $x$ and $y$ if and only if there is a vertex $z$ in $D$ such that $(x, z)$ and $(y, z)$ are arcs of $D$ (for all undefined graph theory terminology, see [2]). Since the notion of competition graph was introduced, there has been a very large literature on competition graphs. For surveys of the literature of competition graphs, see [8, 9, 12, 18]. In addition to ecology, their various applications include applications to channel assignments, coding, and modeling of complex economic and energy systems (see [14]). A

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variety of generalizations of the notion of competition graph have been introduced, including the common enemy graph (sometimes called the resource graph) in [13, 17], the competition-common enemy graph (sometimes called the competition-resource graph) in [16], the niche graph in [3], the p-competition graph in [11], and the competition multigraph in [1].

Recently, Cho, et al. [4, 5] introduced yet another such generalization, the m-step competition graph, and obtained results about m-step competition graphs analogous to the well-known results about ordinary competition graphs. Given a digraph $D$ and a positive integer $m$, the \textit{m-step digraph} $D^m$ of $D$ is defined as follows: $V(D^m) = V(D)$ and there exists an arc $(u,v)$ in $D^m$ if and only if there exists a directed walk of length $m$ from $u$ to $v$. If there is a directed walk of length $m$ from a vertex $x$ to a vertex $y$ in $D$, we call $y$ an \textit{m-step prey} of $x$, and if a vertex $w$ is an m-step prey of both vertices $u$ and $v$, then we say that $w$ is an \textit{m-step common prey} of $u$ and $v$. The m-step competition graph of $D$, denoted by $C^m(D)$, has the same vertex set as $D$ and an edge between vertices $x$ and $y$ if and only if $x$ and $y$ have an m-step common prey in $D$. Note that $C^1(D)$ is the (ordinary) competition graph of $D$.

In studying the (ordinary) competition graphs of acyclic digraphs, Roberts [15] observed that adding sufficiently many isolated vertices to an arbitrary graph $G$ makes it into the competition graph of some acyclic digraph. The smallest such number of isolated vertices was called the \textit{competition number} of $G$ and denoted by $\kappa(G)$. Much of the study of competition graphs of acyclic digraph has been focused on competition numbers, since the characterization of competition graphs of acyclic digraphs reduces to the question of computing the competition number of an arbitrary graph. Analogous to the competition number, Cho, et al. [4] defined the \textit{m-step competition number} $\kappa^{(m)}(G)$ of $G$, which is the smallest number $k$ such that $G$ together with $k$ isolated vertices is the m-step competition graph of an acyclic digraph. Also, the double competition number of Scott [16], the p-competition number of Kim, et al. [10], the niche number of Cable, et al. [3], and the multicompetition number of Anderson, et al. [1] have been introduced. Cho, et al. [4, 5, 6] computed the 2-step competition numbers of paths and cycles and characterized trees with 2-step competition number two. The following is one of their result about the 2-step competition number of a path of length at least two:
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**Proposition 1** (Cho, et al. [5]). For any integer \( n \geq 2 \), \( k^{(2)}(P_n) = 2 \).

It seems to be interesting to characterize the set \( M_n \) of all the minimal acyclic digraphs whose competition graphs are \( P_n \cup I_2 \), where \( P_n \) and \( I_k \), respectively, denote the path of order \( n \) and the graph consisting of \( k \) vertices with no edges. In this paper, we completely characterize \( M_n \) and, surprisingly, it is shown that for \( n \geq 6 \), \( M_n \) consists of only one digraph.

2. Main Result

Let \( D \) be an acyclic digraph with \( n \) vertices. An acyclic labeling of the vertex set \( V(D) \) of \( D \) is a labeling of \( V(D) \) using the set \( \{v_1, v_2, \ldots, v_n\} \) so that \( i < j \) holds whenever there is an arc \((v_i, v_j)\) in \( D \). An acyclic digraph is said to be acyclically labeled if its vertices are acyclically labeled. Given an acyclic digraph \( D \) with \( n \) vertices and an acyclic labeling \( v_1, v_2, \ldots, v_n \) of \( V(D) \), we call an arc \((v_i, v_j)\) in \( D \) a jump-arc when \( i + 1 < j \). To show our main result, we need the following lemmas.

**Lemma 2.** Let \( n \) be any integer greater than one and \( D \) be an acyclic digraph such that \( C^2(D) \) is \( P_n \) with two isolated vertices. Let \( v_1, v_2, \ldots, v_{n+2} \) be an acyclic labeling of \( V(D) \). Then the following are true:

(i) \( C^2(D - v_{n+2}) \) is \( P_{n-1} \) with two isolated vertices.

(ii) For any \( i \) with \( 2 \leq i \leq n + 1 \), there exists an arc \((v_i, v_{i+1})\) in \( D \).

(iii) If there is an incoming jump-arc toward \( v_j \), then there is no outgoing jump-arc from \( v_j \).

(iv) If \( v_i v_j \) (\( i < j \)) is an edge in \( C^2(D) \), then either \((v_i, v_{j+1})\) or \((v_{i+1}, v_{j+2})\) is a jump-arc of \( D \).

**Proof.** Proofs of (i) and (ii) were given in Cho, et al. [5].

We prove (iii) by contradiction. Suppose that \((v_i, v_j)\) and \((v_j, v_l)\) are jump-arcs in \( D \). Then \( v_l \) is a 2-step common prey of \( v_i, v_{j-1}, \) and \( v_{l-2} \). Since \( i < j - 1 \) and \( j < l - 1 \), the three vertices \( v_i, v_{j-1}, \) and \( v_{l-2} \) are distinct and form a cycle \( C_3 \) in \( C^2(D) \), which is a contradiction. Hence (iii) follows.

Finally we prove (iv). Suppose that \( v_i v_j \) (\( i < j \)) is an edge in \( C^2(D) \). Then there exists \( l \), \( l \geq j + 2 \), such that \( v_l \) is a 2-step common prey of \( v_i \) and \( v_j \). In fact \( l = j + 2 \), for otherwise \( v_l \) is a 2-step common prey of
three distinct vertices \( v_{i-2}, v_j \) and \( v_i \). Since \( v_{j+2} \) is a 2-step prey of \( v_i \), arcs \((v_1, v_p)\) and \((v_p, v_{j+2})\) are in \( D \) for some \( p \). By (iii), either \( p = i + 1 \) or \( p = j + 1 \). Hence (iv) follows. \( \square \)

Given a graph \( G \), we say that an acyclic digraph \( D \) is \emph{minimally associated with} \( G \) if \( C^2(D) \) is \( G \) with \( k^{(2)}(G) \) isolated vertices and the 2-step competition graph of any proper subdigraph of \( D \) is not \( G \) with \( k^{(2)}(G) \) isolated vertices. Let

\[
D(G) = \{ D \mid D \text{ is an acyclically labeled digraph which is minimally associated with } G \}.
\]

Now the following lemma holds:

**Lemma 3.** Let \( n \) be an integer with \( 2 \leq n \leq 5 \). Then \( D(P_n) \) consists of the digraphs \( D_{n^*} \) given in Figure 1.

**Proof.** It can easily be checked that each of the digraphs given in Figure 1 is minimally associated with \( P_n \). Let \( D(P_n) = D_n \). It is easy to see that \( D_2 = \{ D_{2a}, D_{2b} \} \). Take \( D \) from \( D_n \) (\( 3 \leq n \leq 5 \)). Let \( v_1, v_2, \ldots, v_{n+2} \) be an acyclic labeling of \( V(D) \). By Lemma 2 (i) and (ii), \( D \) contains arc \((v_{n+1}, v_{n+2})\) and one of the digraphs in \( D_{n-1} \) as a subdigraph.

For \( 3 \leq n \leq 5 \), denote by \( F_{n-1} \) the digraph in \( D_{n-1} \) that is contained in \( D \). Then \( C^2(F_{n-1}) \) is a path with two isolated vertices. Denote the labeled path by \( P^*_{n-1} \). Then \( v_n \) is adjacent to one of the end vertices of \( P^*_{n-1} \) in \( C^2(D) \). Denote by \( v_{F_{n-1}} \) the vertex adjacent to \( v_n \). By Lemma 2 (iv), there should be a jump-arc in \( D \) in order for \( v_n \) and \( v_{F_{n-1}} \) to be joined in \( C^2(D) \). Denote the jump-arc by \( a_{v_{F_{n-1}}} \). By Lemma 2 (iv),

\[
a_{v_{F_{n-1}}} = (v_i, v_{n+1}) \text{ or } (v_{i+1}, v_{n+2}) \text{ for } v_{F_{n-1}} = v_i.
\]

Then by containing digraph \( F_{n-1} \) and arcs \((v_{n+1}, v_{n+2})\) and \( a_{v_{F_{n-1}}} \), \( D \) becomes either the digraph in the corresponding row of the last column of Table 1 by the minimality of \( D \), or undefined for various reasons stated in Table 1. By checking Table 1, we can conclude that \( D(P_n) \) consists of the digraphs \( D_{n^*} \) given in Figure 1. \( \square \)

Denote by \( M_n \) the digraph with

\[ V(M_n) = \{ v_1, v_2, \ldots, v_{n+2} \} \]

and

\[
A(M_n) = \{ (v_1, v_3) \} \cup \{ (v_i, v_{i+1}) \mid 1 \leq i \leq n + 1 \}
\]

\[
\cup \{ (v_i, v_{i+3}) \mid i \text{ even and } 2 \leq i \leq n - 1 \}.
\]
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Let $l = \lfloor \frac{n}{2} \rfloor$. Then it is not difficult to check that $C^2(M_n)$ is the path $v_2v_2(l-1) \cdots v_4v_2v_1v_3 \cdots v_{2l-1}v_{2l+1}$ together with two isolated vertices $v_{n+1}$ and $v_{n+2}$ if $n$ is odd, and the path $v_2v_2(l-1) \cdots v_4v_2v_1v_3 \cdots v_{2l-3}v_{2l-1}$ with two isolated vertices $v_{n+1}$ and $v_{n+2}$ if $n$ is even. That is, the 2-step competition graph of $M_n$ is $P_n$ with two isolated vertices. The following theorem shows that for $n \geq 6$, every acyclic digraph whose competition graph is $P_n$ with two isolated vertices contains $M_n$ as a subdigraph.

**Theorem 4.** For $n \geq 6$, $M_n$ is the unique acyclic digraph minimally associated with $P_n$.

**Proof.** Induct on $n$. Let $D$ be an acyclic digraph minimally associated with $P_n$. Let $v_1, v_2, \ldots, v_8$ be the acyclic labeling of $V(D)$. By Lemma 2 (i) and (ii), $D$ contains arc $(v_7, v_6)$ and one of $D_{5a}, D_{5b}, D_{5c}$ in Figure 1. Suppose that $D$ contains $D_{5a}$. Then $v_6$ is adjacent to $v_4$ or $v_5$ in $C^2(D)$. If $v_6$ is adjacent to $v_4$, then $(v_4, v_7)$ or $(v_5, v_8)$ is in $D$. By Lemma 2 (iii), neither $(v_4, v_7)$ nor $(v_5, v_8)$ can be in $D$. Thus $v_6$ is adjacent to $v_5$ in $C^2(D)$. Then either $(v_5, v_7)$ or $(v_6, v_8)$ is in $D$. Again by Lemma 2 (iii), neither $(v_5, v_7)$ nor $(v_6, v_8)$ can be in $D$. Therefore $D$ cannot contain $D_{5a}$. Suppose that $D$ contains $D_{5b}$. Then arcs $(v_2, v_7)$, $(v_6, v_7)$, and $(v_7, v_8)$ are in $D$, and this implies that there is an edge $v_2v_6$ in $C^2(D)$, which is impossible. Thus $D$ does not contain $D_{5b}$. Hence $D$ contains $D_{5c}$. Now $D_{5c}$ together with arc $(v_7, v_8)$ is $M_6$ and the theorem follows for $n = 6$.

Assume that the theorem holds for $n - 1$ with $n > 6$. Let $D$ be an acyclic digraph minimally associated with $P_n$. Let $v_1, v_2, \ldots, v_{n+2}$ be an acyclic labeling of $V(D)$. By Lemma 2 (i), $C^2(D - v_{n+2})$ is $P_{n-1}$ with two isolated vertices $v_n, v_{n+1}$. By the induction hypothesis, $D - v_{n+2}$ contains $M_{n-1}$. Thus $D$ contains $M_{n-1}$. By Lemma 2 (ii), $D$ contains arc $(v_{n+1}, v_{n+2})$. If $n$ is even, $M_{n-1}$ with arc $(v_{n+1}, v_{n+2})$ is $M_n$ and the theorem follows. Now suppose that $n$ is odd. Since $v_{n-1}$ and $v_{n-2}$ are the end vertices of the path in $C^2(D - v_{n+2})$, $v_n$ is adjacent to $v_{n-1}$ or $v_{n-2}$ in $C^2(D)$. First, suppose that $v_n$ is adjacent to $v_{n-1}$. Then one of $(v_{n-1}, v_{n+1})$ or $(v_n, v_{n+2})$ is in $D$ by Lemma 2 (iv). By Lemma 2 (iii), $(v_n, v_{n+2})$ cannot be in $D$. But if $(v_{n-1}, v_{n+1})$ is in $D$, then $v_{n-1}$ is a 2-step common prey of $v_{n-1}$, $v_{n-2}$ and $v_{n-3}$, which is impossible. Thus $v_n$ is adjacent to $v_{n-2}$. Then either $(v_{n-2}, v_{n+1})$ or $(v_{n-1}, v_{n+2})$ is in $D$. By Lemma 2 (iii), $(v_{n-2}, v_{n+1})$ cannot be in $D$. Therefore $(v_{n-1}, v_{n+2})$ is in $D$. Since $M_{n-1}$ together with arcs $(v_{n-1}, v_{n+2})$ and $(v_{n+1}, v_{n+2})$ is $M_n$, $D$ is $M_n$. \[\Box\]
Figure 1. $D_{n^*}$ are the acyclic digraphs minimally associated with $P_n$: with two isolated vertices in each case, $C^2(D_2)$ are paths $v_1v_2$; $C^2(D_{3a})$ and $C^2(D_{3b})$ are paths $v_1v_2v_3$; $C^2(D_{3c})$ and $C^2(D_{3d})$ are paths $v_2v_1v_3$; $C^2(D_{4a})$, $C^2(D_{4b})$, and $C^2(D_{4d})$ are paths $v_3v_2v_1v_4$; $C^2(D_{4c})$ is path $v_1v_2v_3v_4$; $C^2(D_{4e})$ and $C^2(D_{4f})$ are paths $v_3v_1v_2v_4$; $C^2(D_{5a})$ and $C^2(D_{5b})$ are $v_4v_3v_2v_1v_5$; $C^2(D_{5c})$ is path $v_4v_2v_1v_3v_5$.
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TABLE 1. A part of the proof of Lemma 3. (1) $D_{2a}$ (resp. $D_{3d}$) together with arc $(v_4, v_6)$ (resp. $(v_5, v_6)$) is $D_{3a}$ (resp. $D_{4f}$). By the minimality of $D$, $D$ is $D_{3a}$ (resp. $D_{4f}$). (2) If $(v_2, v_5)$ is in $D$ in order for $v_5$ to be a 2-step prey of $v_1$, then $(v_1, v_2)$ must be in $D$. (3) By Lemma 2 (iii). (4) If $(v_2, v_6)$ is in $D$ in order for $v_6$ to be a 2-step prey of $v_1$, then $(v_1, v_2)$ must be in $D$, and so $D$ contains $D_{4b}$ together with arc $(v_1, v_5)$, contradicting the minimality of $D$. (5) If $(v_3, v_6)$ (resp. $(v_4, v_6)$) is in $D$, then $v_6$ is a 2-step common prey of $v_1$, $v_2$, $v_4$ (resp. $v_2$, $v_3$, $v_4$), which is impossible. (6) Since $D$ contains $D_{4b}$ and arc $(v_6, v_7)$, $v_7$ is a 2-step common prey of $v_5$ and $v_2$ in $D$. Hence $v_5$ is adjacent to $v_2$ in $C^2(D)$ and we reach a contradiction.

References


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