MULTIPLICITY OF SOLUTIONS AND SOURCE TERMS IN A NONLINEAR PARABOLIC EQUATION UNDER DIRICHLET BOUNDARY CONDITION

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ABSTRACT. We investigate the existence of solutions of the nonlinear heat equation under Dirichlet boundary condition on $\Omega$ and periodic condition on the variable $t$, $Lu - D_t u + g(u) = f(x, t)$. We also investigate a relation between multiplicity of solutions and the source terms of the equation.

0. Introduction

In this paper, we investigate multiplicity of solutions $u(x, t)$ for a nonlinear perturbation $g(u)$ of the parabolic operator $(L - D_t)$ under Dirichlet boundary condition on $\Omega$ and periodic condition on the variable $t$,

\begin{equation}
Lu - D_t u + g(u) = f(x, t) \quad \text{in } \Omega \times R, \\
\quad u = 0 \quad \text{on } \partial \Omega, \\
\quad u(x, t) = u(x, t + T),
\end{equation}

where $\Omega$ is a bounded domain in $R^n$ with smooth boundary $\partial \Omega$ and the nonlinear perturbation $g(u)$ is piecewise linear one $bu^+ - au^-$ with $a < \lambda_0 < b < \lambda_1$. Here $L$ is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact linear inverse, with
eigenvalues $-\lambda_i$, each repeated as often as multiplicity

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \rightarrow +\infty.$$  

Let $H$ be the Hilbert space defined by

$$H = \{ u \in L^2(\Omega \times [0,T]) | \quad u \text{ is } T \text{-periodic in } t \}.$$  

Then equation (0.1) is represented by

$$(0.2) \quad Lu - D_t u + bu^+ - au^- = f(x,t) \text{ in } H.$$  

In [6], the author showed by degree theory that equation (0.2), with the forcing term $f$ is supposed to be a multiple of the first positive eigenfunction, has at least two solutions if $n$ is even, and at least three solutions if $n$ is odd.

We suppose that $a < \lambda_{01} < b < \lambda_{02}$ and the source term $f$ is generated by $\varphi_{01}$ and $\varphi_{02}$. Our goal is to investigate a relation between multiplicity of solution and source terms in equation (0.2) when $f$ belongs to the two-dimensional subspace of $H$ that spanned by $\varphi_{01}$ and $\varphi_{02}$.

Let $V$ be the two dimensional subspace of $H$ spanned by $\varphi_{01}$ and $\varphi_{02}$. Let $P$ be the orthogonal projection $H$ onto $V$. Let $\Phi : V \rightarrow V$ be a map (cf. (1.7)) defined by

$$\Phi(v) = Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V.$$  

In section 1, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue $\lambda_{01}$. And we use the variational reduction method to reduce the problem from an infinite dimensional one to a finite dimensional one. In section 2, we investigate the properties of the map $\Phi$ and we reveal a relation between multiplicity of solutions and source terms in equation (0.2) when $f(x,t)$ belongs to the two-dimensional space $V$.

1. A variational reduction

We consider the parabolic equation under Dirichlet boundary condition and periodic condition on the variable $t$,
Multiplicity of solutions and source terms in a nonlinear parabolic equation

\[ Lu - D_t u + g(u) = f(x, t) \quad \text{in } \Omega \times R, \]
\[ u = 0 \quad \text{on } \partial \Omega, \]
\[ u(x, t) = u(x, t + T). \]

Here the nonlinear term \( g(u) \) is piecewise linear \( bu^+ - au^- \) with \( a < \lambda_{01} < b < \lambda_{02} \). We consider the boundary problem

\[ Lu - D_t u + bu^+ - au^- = f(x, t) \quad \text{in } \Omega \times R, \]
\[ u = 0 \quad \text{on } \partial \Omega, \]
\[ u(x, t) = u(x, t + T). \]

We denote \( \varphi_n \) to be the eigenfunctions corresponding to eigenvalues \( \lambda_n \) and \( \varphi_1(x) > 0 \) in \( \Omega \). Let \( H \) be the Hilbert space defined by

\[ H = \{ u \in L^2(\Omega) \times [0, T] \mid u \text{ is } T\text{-periodic in } t \}. \]

Then the set \( \{ \varphi_{mn} = \frac{1}{\sqrt{2\pi}} \varphi_n(x) e^{int} \mid n \geq 1, m = 0, \pm 1, \pm 2, \cdots \} \) is orthogonal in \( H \) and \( \varphi_{01} > 0 \).

We are concerned with the multiplicity of solutions of (1.2) only when \( f \) is generated by the eigenfunctions \( \varphi_{01} \) and \( \varphi_{02} \). That is, we study the equation

\[ Lu - D_t u + bu^+ - au^- = f \quad \text{in } H, \]

where \( f = s_1 \varphi_{01} + s_2 \varphi_{02} (s_1, s_2 \in R) \).

**Theorem 1.1.** If \( s_1 < 0 \), then (1.3) has no solution.

**Proof.** We rewrite (1.3) as

\[ (L - D_t + \lambda_{01})u + (b - \lambda_{01})u^+ - (a - \lambda_{01})u^- = s_1 \varphi_{01} + s_2 \varphi_{02}. \]

Multiply across by \( \varphi_{01} \) and integrate over \( H \). Since \( (L - D_t + \lambda_{01}) \varphi_{01} = 0 \) and \( ((L - D_t + \lambda_{01})u, \varphi_{01}) = 0 \), we have

\[ \int_{\Omega} ((b - \lambda_{01})u^+ - (a - \lambda_{01})u^-) \varphi_{01} = (s_1 \varphi_{01} + s_2 \varphi_{02}, \varphi_{01}) = s_1 \int_{\Omega} \varphi_{01}^2 = s_1. \]

However, we know that \( (b - \lambda_{01})u^+ - (a - \lambda_{01})u^- \geq 0 \) for all real-valued function \( u \). Also \( \varphi_{01} > 0 \) in \( H \). Therefore

\[ \int_{\Omega} ((b - \lambda_{01})u^+ - (a - \lambda_{01})u^-) \varphi_{01} \geq 0. \]
Hence, there is no solution of (1.3) if \( s_1 < 0 \). \( \square \)

To study equation (1.3), we use the contraction mapping theorem to reduce the problem from an infinite-dimensional one to a finite-dimensional one.

Let \( V \) be two-dimensional subspace of \( H \) spanned by \( \{ \varphi_{01}, \varphi_{02} \} \) and \( W \) be the orthogonal complement of \( V \) in \( H \). Let \( P \) be the orthogonal projection of \( H \) onto \( V \). Then every \( u \in H \) can be written as \( u = v + w \), where \( v = Pu \) and \( w = (I - P)u \). Hence equation (1.3) is equivalent to a system

\[
(1.4) \quad Lw - D_t w + (I - P)(b(v + w)^+ - a(v + w)^-) = 0,
\]

\[
(1.5) \quad Lv - D_t v + P(b(v + w)^+ - a(v + w)^-) = s_1 \varphi_{01} + s_2 \varphi_{02}.
\]

**Lemma 1.2.** For a fixed \( v \in V \), equation (1.4) has a unique solution \( w = \theta(v) \). Furthermore, \( \theta(v) \) is Lipschitz continuous (with respect to the \( L^2 \)-norm) in \( v \).

The proof of the lemma is similar to that of [5].

By Lemma 1.2, the study of multiplicity of solutions of (1.3) is reduced to one of an equivalent problem

\[
(1.6) \quad Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = s_1 \varphi_{01} + s_2 \varphi_{02}
\]

defined on the two dimensional subspace \( V \) spanned by \( \{ \varphi_{01}, \varphi_{02} \} \).

While one feels intuitively that (1.6) ought to be easier to solve than (1.3), there is the disadvantage of an implicitly defined term \( \theta(v) \) in the equation. However, in our case, it turns out that we know \( \theta(v) \) for some special \( c \)'s.

**Corollary.** If \( v \geq 0 \) or \( v \leq 0 \), then \( \theta(v) \equiv 0 \).

**Proof.** Now, take \( v \geq 0 \) and \( \theta(v) = 0 \) since \( v \in V, (I - P)v = 0 \). Then equation (1.4) is reduced to

\[
(L - D_t) \cdot 0 + (I - P)(bv^+ - av^-) = 0
\]

because \( v^+ = v, v^- = 0 \) and \( (I - P)v = 0 \). By Lemma 1.2, \( \theta(v) \equiv 0 \). \( \square \)
Multiplicity of solutions and source terms in a nonlinear parabolic equation

Since $V = \text{span}\{\varphi_{01}, \varphi_{02}\}$ and $\varphi_{01}$ is a positive eigenfunction, there exists a cone $C_1$ defined by

$$C_1 = \{v = c_1 \varphi_{01} + c_2 \varphi_{02} : c_1 \geq 0, |c_2| \leq \varepsilon_0 |c_1|\}$$

for some $\varepsilon_0 > 0$, so that $v \geq 0$ for all $v \in C_1$, and a cone $C_3$ defined by

$$C_3 = \{v = c_1 \varphi_{01} + c_2 \varphi_{02} : c_1 \leq 0, |c_2| \leq \varepsilon_0 |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$. Thus, we do not know $\theta(v)$ for all $v \in PH$, but we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. And $C_2$ and $C_4$ are defined as follows

$$C_2 = \{v = c_1 \varphi_{01} + c_2 \varphi_{02} | c_2 \geq 0, c_2 \geq \varepsilon_0 |c_1|\},$$
$$C_4 = \{v = c_1 \varphi_{01} + c_2 \varphi_{02} | c_2 \leq 0, |c_2| \geq \varepsilon_0 |c_1|\}.$$

Then the union of $C_1, C_3$ and $C_2, C_4$ is the space $V$. Now we define a map $\Phi : V \longrightarrow V$ given by

$$(1.7) \quad \Phi(v) = Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-, v \in V.$$

Then $\Phi$ is continuous on $V$, since $\theta$ is continuous on $V$ and we have the following lemma.

**Lemma 1.3.** For $v \in V$ and $c \geq 0$, $\Phi(cv) = c\Phi(v)$.

**Proof.** Let $c \geq 0$. If $v$ satisfies

$$L\theta(v) - D_t \theta(v) + (I - P)(b(v + \theta(v))^+ - a(v + \theta(v))^-) = 0,$$

then

$$L(c\theta(v) - D_t (c\theta(v)) + (I - P)(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\Phi(cv) = L(cv) - D_t(WV) + P(b(cv + \theta(cv))^+ - a(cv + \theta(cv))^-)$$
$$= L(cv) - D_t(cv) + P(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-)$$
$$= cL(v) - cD_t v + cP(b(v + \theta(v))^+ - a(v + \theta(v))^-)$$
$$= c\Phi(v).$$

□
2. Multiplicity of solutions and source terms

Now we investigate the image of the cone \(C_1, C_3\) under \(\Phi\). First we consider the image of \(C_1\) under \(\Phi\). If \(v = c_1 \varphi_{01} + c_2 \varphi_{02}\), then we have

\[
\Phi(v) = \begin{cases} 
L v - D_t v + P(b(v + \theta(v)) + a(v + \theta(v))) \leq c_1 \lambda_{01} \varphi_{01} - c_2 \lambda_{02} \varphi_{02} + b(c_1 \varphi_{01} + c_2 \varphi_{02}) \\
= c_1(b - \lambda_{01}) \varphi_{01} + c_2(b - \lambda_{02}) \varphi_{02}.
\end{cases}
\]

Thus the image of the rays \(c_1 \varphi_{01} \pm \varepsilon_0 c_2 \varphi_{02}(c_1 \geq 0)\) can be explicitly calculated and they are

\[
c_1(b - \lambda_{01}) \varphi_{01} \pm \varepsilon_0 c_1(b - \lambda_{02}) \varphi_{02} \quad (c_1 \geq 0).
\]

Therefore if \(a < \lambda_{01} < b < \lambda_{02}\), then \(\Phi\) maps \(C_1\) onto the cone

\[
R_1 = \left\{ d_1 \varphi_{01} + d_2 \varphi_{02} \mid d_1 \geq 0, |d_2| \leq \varepsilon_0 \left( \frac{\lambda_{02} - b}{b - \lambda_{01}} \right) d_1 \right\}.
\]

Second, we consider the image of \(C_3\). If \(v = -c_1 \varphi_{01} + c_2 \varphi_{02} \leq 0 \quad (c_1 \geq 0, |c_2| \leq \varepsilon_0 c_1\), then we have

\[
\Phi(v) = \begin{cases} 
L v - D_t v + P(b(v + \theta(v)) + a(v + \theta(v))) \leq c_1 \lambda_{01} \varphi_{01} - c_2 \lambda_{02} \varphi_{02} - ac_1 \varphi_{01} + ac_2 \varphi_{02} \\
= c_1(\lambda_{01} - a) \varphi_{01} + c_2(a - \lambda_{02}) \varphi_{02}.
\end{cases}
\]

Thus the image of the rays \(-c_1 \varphi_{01} \pm \varepsilon_0 c_1 \varphi_{02}\) can be explicitly calculated and they are

\[
c_1(\lambda_{01} - a) \varphi_{01} \pm \varepsilon_0 c_1(a - \lambda_{02}) \varphi_{02} \quad (c_1 \geq 0).
\]

Therefore \(\Phi\) maps \(C_3\) onto the cone

\[
R_3 = \left\{ d_1 \varphi_{01} + d_2 \varphi_{02} \mid d_1 \geq 0, |d_2| \leq \varepsilon_0 \left( \frac{\lambda_{02} - a}{\lambda_{01} - a} \right) d_1 \right\}.
\]

Here we have three cases, which are \(R_1 \subset R_3, R_3 \subset R_1, \) and \(R_1 = R_3\). The first relation \(R_1 \subset R_3\) holds if and only if the nonlinearity \(-(bu^+ - au^-)\) satisfies \(b > \frac{2\lambda_{01} \lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a}\). The second relation \(R_3 \subset R_1\) holds if and only if the nonlinearity \(-(bu^+ - au^-)\) satisfies \(b < \frac{2\lambda_{01} \lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a}\).
Multiplicity of solutions and source terms in a nonlinear parabolic equation

The last case $R_1 = R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{2\lambda_0\lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a}$.

**Lemma 2.1.** For every $v = c_1 \varphi_{01} + c_2 \varphi_{02} \in V$, there exists a constant $d > 0$ such that $(\Phi(v), \varphi_{01}) \geq d|c_2|$.

Lemma 2.1 tells us that the image of $\Phi$ is contained in the right half-plane of $V$. That is, $\Phi(C_2)$ and $\Phi(C_4)$ are the cone in the right half-plane of $V$.

We consider the restriction $\Phi|_{C_i}(1 \leq i \leq 4)$ of $\Phi$ to the cone $C_i$. Let $\Phi_i = \Phi|_{C_i}(0 \leq i \leq 4)$, i.e.,

$$\Phi_i : C_i \longrightarrow V.$$

First, we consider $\Phi_1$. It maps $C_1$ onto $R_1$. Let $l_1$ be the segment defined by

$$l_1 = \left\{ \varphi_{01} + d_2 \varphi_{02} \mid |d_2| \leq \varepsilon_0 \left( \frac{\lambda_{02} - b}{b - \lambda_{01}} \right) \right\}.$$

Then the inverse image $\Phi_1^{-1}(l_1)$ is the segment

$$L_1 = \Phi_1^{-1}(l_1) = \left\{ \frac{1}{b - \lambda_{01}}(\varphi_{01} + c_2 \varphi_{02}) \mid |c_2| \leq \varepsilon_0 \right\}.$$

By Lemma 1.3, $\Phi_1 : C_1 \longrightarrow R_1$ is bijective.

Next we consider $\Phi_3$. It maps $C_3$ onto $R_3$. Let $l_3$ be the segment defined by

$$l_3 = \left\{ \varphi_{01} + d_2 \varphi_{02} \mid |d_2| \leq \varepsilon_0 \left( \frac{\lambda_{02} - a}{a - \lambda_{01}} \right) \right\}.$$

Then the inverse image $\Phi_3^{-1}(l_3)$ is the segment

$$L_3 = \Phi_3^{-1}(l_3) = \left\{ \frac{1}{a - \lambda_{01}}(\varphi_{01} + c_2 \varphi_{02}) \mid |c_2| \leq \varepsilon_0 \right\}.$$

By Lemma 1.3, $\Phi_3 : C_3 \longrightarrow R_3$ is bijective.
2.1. The nonlinearity \(-(bu^+-au^-)\) satisfies \(b > \frac{2\lambda_0^2 - a(\lambda_0 + 1)}{\lambda_0 + \lambda_0 - 2a}\)

The relation \(R_1 \subset R_3\) holds if and only if the nonlinearity \(-(bu^+-au^-)\) satisfies \(b > \frac{2\lambda_0^2 - a(\lambda_0 + 1)}{\lambda_0 + \lambda_0 - 2a}\). Now we find the images of the cones \(C_2\) and \(C_4\) under \(\Phi\), where

\[
C_2 = \{ v = c_1\varphi_0 + c_2\varphi_2 \mid c_2 \geq 0, \varepsilon_0 |c_1| \leq c_2 \},
\]

\[
C_4 = \{ v = c_1\varphi_0 + c_2\varphi_2 \mid c_2 \leq 0, \varepsilon_0 |c_1| \leq |c_2| \}.
\]

By Theorem 1.1 and Lemma 1.2, the image of \(C_2\) under \(\Phi\) is a cone containing

\[
R_2 = \left\{ d_1\varphi_0 + d_2\varphi_2 \mid d_1 \geq 0, \varepsilon_0 \left( \frac{\lambda_0^2 - b}{b - \lambda_0} \right) d_1 \leq d_2 \leq \varepsilon_0 \left( \frac{\lambda_0^2 - a}{\lambda_0 - a} \right) d_1 \right\}
\]

and the image of \(C_4\) under \(\Phi\) is a cone containing

\[
R_4 = \left\{ d_1\varphi_0 + d_2\varphi_2 \mid d_1 \geq 0, -\varepsilon_0 \left( \frac{\lambda_0^2 - a}{\lambda_0 - a} \right) d_1 \leq d_2 \leq -\varepsilon_0 \left( \frac{\lambda_0^2 - b}{b - \lambda_0} \right) d_1 \right\}.
\]

We consider the restrictions \(\Phi_2\) and \(\Phi_4\), and define the segments \(l_2, l_4\) as follows:

\[
l_2 = \left\{ \varphi_0 + d_2\varphi_2 \mid \varepsilon_0 \left( \frac{\lambda_0^2 - b}{b - \lambda_0} \right) \leq d_2 \leq \varepsilon_0 \left( \frac{\lambda_0^2 - a}{\lambda_0 - a} \right) \right\},
\]

\[
l_4 = \left\{ \varphi_0 + d_2\varphi_2 \mid \varepsilon_0 \left( \frac{a - \lambda_0^2}{\lambda_0 - a} \right) \leq d_2 \leq \varepsilon_0 \left( \frac{b - \lambda_0^2}{b - \lambda_0} \right) \right\}.
\]

We investigate the inverse image \(\Phi_2^{-1}(l_2)\) and \(\Phi_4^{-1}(l_4)\). Hence, we want to prove that \(\Phi_2\) and \(\Phi_4\) are surjective.

**Lemma 2.2.** Let \(\gamma_i (i = 2, 4)\) be any simple path in \(R_i\) with end points on \(\partial R_i\), where each ray (starting from the origin) in \(R_i\) intersects only one point of \(\gamma_i\). Then the inverse image \(\Phi_i^{-1}(\gamma_i)\) of \(\gamma_i\) is also a simple path in \(C_i\) with end points on \(\partial C_i\), where any ray (starting from the origin) in \(C_i\) intersects only one point of this path.

**Proof.** We note that \(\Phi_i^{-1}(\gamma_i)\) is closed since \(\Phi\) is continuous and \(\gamma_i\) is closed in \(V\). Suppose that there is a ray (starting from the origin) in \(C_i\) which intersects two points of \(\Phi_i^{-1}(\gamma_i)\), say \(p\) and \(\alpha p (\alpha > 1)\). Then, by lemma 3.1.3 \(\Phi_i(\alpha p) = \alpha \Phi_i(p)\) which implies that \(\Phi_i(p) \in \gamma_i\) and
Multiplicity of solutions and source terms in a nonlinear parabolic equation

\[ \Phi_i(\alpha p) \in \gamma_i. \] This contradicts the assumption that each ray (starting from the origin) in \( C_i \) intersects only one point of \( \gamma_i \).

We regard a point \( p \) as a radius vector in the plane \( V \). Then for a point \( p \) in \( V \), we define the argument \( \arg p \) of \( p \) by the angle from the positive \( \varphi_{01} \)-axis to \( p \).

We claim that \( \Phi_i^{-1}(\gamma_i) \) meets all the rays (starting from the origin) in \( C_i \). If not, \( \Phi_i^{-1}(\gamma_i) \) is disconnected in \( C_i \). Since \( \Phi_i^{-1}(\gamma_i) \) is closed and meet at most one point of any ray in \( C_i \), there are two points \( p_1 \) and \( p_2 \) in \( C_2 \) such that \( \Phi_i^{-1}(\gamma_i) \) does not contain any point \( p \in C_i \) with

\[ \arg p_1 < \arg p < \arg p_2. \]

On the other hand, if we set \( l \) be the segment with end points \( p_1 \) and \( p_2 \), then \( \Phi_i(l) \) is a path in \( R_i \), where \( \Phi_i(p_1) \) and \( \Phi_i(p_2) \) belong to \( \gamma_i \). Choose a point \( q \) in \( \Phi_i(l) \) such that \( \arg q \) is between \( \arg \Phi_i(p_1) \) and \( \arg \Phi_i(p_2) \). Then there exist a point \( q' \) of \( \gamma_i \) such that \( q' = \beta q \) for some \( \beta > 0 \). Hence \( \Phi_i^{-1}(q) \) and \( \Phi_i^{-1}(q') \) are on the same ray (starting from the origin) in \( C_i \) and

\[ \arg p_1 < \arg \Phi_i^{-1}(q') < \arg p_2, \]

which is a contraction. This completes the proof.  \( \square \)

Lemma 2.2 implies that \( \Phi_i(i = 2, 4) \) is surjective. Hence we have the following theorem.

**Theorem 2.3.** For \( 1 \leq i \leq 4 \), the restriction \( \Phi_i \) maps \( C_i \) onto \( R_i \). Therefore, \( \Phi \) maps \( V \) onto \( R_3 \). In particular, \( \Phi_1 \) and \( \Phi_3 \) are bijective.

The above theorem also implies the following result.

**Theorem 2.4.** Suppose \( a < \lambda_{01} < b < \lambda_{02} \) and \( b > \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} \).

Let \( f = s_1 \varphi_{01} + s_2 \varphi_{02} \). Then we have:

1. If \( f \in \overline{R}_1 \), then (1.3) has exactly two solutions, one of which is positive and the other is negative.
2. If \( f \) belongs to interior of \( R_2 \) or interior of \( R_4 \), then (1.3) has a negative solution and at least one sign changing solution.
3. If \( f \) belongs to boundary of \( R_3 \), then (1.3) has a negative solution.
4. If \( f \) does not belong to \( R_3 \), then (1.3) has no solution.
2.2. The nonlinearity \(-(bu^+ - au^-)\) satisfies \(b < \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a}\)

The relation \(R_3 \subset R_1\) holds if and only if the nonlinearity \(-(bu^+ - au^-)\) satisfies \(b < \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a}\). We investigate the image of the cones \(C_2\) and \(C_4\) under \(\Phi\), where

\[
\begin{align*}
C_2 &= \{ v = c_1 \varphi_{01} + c_2 \varphi_{02} \mid c_2 \geq 0, \varepsilon_0 |c_1| \leq c_2 \}, \\
C_4 &= \{ v = c_1 \varphi_{01} + c_2 \varphi_{02} \mid c_2 \leq 0, \varepsilon_0 |c_1| \leq |c_2| \}.
\end{align*}
\]

By Theorem 1.1 and Lemma 1.2, the image of \(C_2\) under \(\Phi\) is a cone containing

\[
R'_2 = \left\{ d_1 \varphi_{01} + d_2 \varphi_{02} \mid d_1 \geq 0, \varepsilon_0 \left( \frac{\lambda_{02} - a}{\lambda_{01} - a} \right) d_1 \leq d_2 \leq \varepsilon_0 \left( \frac{\lambda_{02} - b}{b - \lambda_{01}} \right) d_1 \right\}
\]
and the image of \(C_4\) under \(\Phi\) is a cone containing

\[
R'_4 = \left\{ d_1 \varphi_{01} + d_2 \varphi_{02} \mid d_1 \geq 0, \varepsilon_0 \left( \frac{b - \lambda_{02}}{b - \lambda_{01}} \right) d_1 \leq d_2 \leq \varepsilon_0 \left( \frac{a - \lambda_{02}}{\lambda_{01} - a} \right) d_1 \right\}.
\]

We consider the restrictions \(\Phi_2\) and \(\Phi_4\), and define the segments \(l'_2\) and \(l'_4\) as follows:

\[
\begin{align*}
l'_2 &= \{ \varphi_{01} + d_2 \varphi_{02} \mid \varepsilon_0 \left( \frac{\lambda_{02} - a}{\lambda_{01} - a} \right) \leq d_2 \leq \varepsilon_0 \left( \frac{\lambda_{02} - b}{b - \lambda_{01}} \right) \}, \\
l'_4 &= \{ \varphi_{01} + d_2 \varphi_{02} \mid \varepsilon_0 \left( \frac{b - \lambda_{02}}{b - \lambda_{01}} \right) \leq d_2 \leq \varepsilon_0 \left( \frac{a - \lambda_{02}}{\lambda_{01} - a} \right) \}.
\end{align*}
\]

We investigate the inverse images \(\Phi_2^{-1}(l'_2)\) and \(\Phi_4^{-1}(l'_4)\). We note that \(\Phi_2(C_2)\) and \(\Phi_4(C_4)\) contains \(R'_2\) and \(R'_4\).

**Lemma 2.5.** For \(i = 2, 4\), let \(\gamma'\) be a simple path in \(R'_i\) with end points on \(\partial R'_i\), where each ray in \(R'_i\) (starting from the origin) intersects only one point of \(\gamma'\). Then the inverse image \(\Phi_i^{-1}(\gamma')\) of \(\gamma'\) is also simple path in \(C_i\) with end point on \(\partial C_i\), where any ray in \(C_i\) (starting from the origin) intersects only one point of this path.

**Proof.** The proof is similar to that of Lemma 2.2. \(\square\)

Lemma 2.5 implies that \(\Phi_2\) and \(\Phi_4\) are surjective. Hence we have the following theorem.

706
Multiplicity of solutions and source terms in a nonlinear parabolic equation

**Theorem 2.6.** For \( i = 2, 4 \), the restriction \( \Phi_i \) maps \( C_i \) onto \( R'_i \). And \( \Phi_1 \) and \( \Phi_3 \) are bijective. Therefore, \( \Phi \) maps \( V \) onto \( R_1 \).

With the above theorem, we have the following results.

**Theorem 2.7.** Suppose \( a < \lambda_{01} < b < \lambda_{02} \) and \( b < \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} \).
Let \( f = s_1 \varphi_{01} + s_2 \varphi_{02} \in V \). Then we have

1. If \( f \in \bar{R}_3 \), then \( (1.3) \) has exactly two solutions one of which is positive and the other is negative.
2. If \( f \) belongs to interior of \( R'_2 \) or interior \( R'_4 \), then \( (1.3) \) has a negative solution and at least one sign changing solution.
3. If \( f \) belongs to boundary of \( R_1 \), then \( (1.3) \) has a negative solution.
4. If \( f \) does not belong to \( R_1 \), then \( (1.3) \) has no solution.

### 2.3. The nonlinearity \(-(bu^+ - au^-)\) satisfies \( b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} \)

The relation \( R_1 = R_3 \) holds if and only if the nonlinearity \(-(bu^+ - au^-)\) satisfies \( b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} \). Consider the map \( \Phi : V \to V \) defined by

\[
\Phi(v) = Lv - D_tv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V,
\]

where \( a < \lambda_{01} < b < \lambda_{02} \) and \( b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} \). Now we want to investigate the images of the cone \( C_2 \) and \( C_4 \) under \( \Phi \). For fixed \( v \), we define a map

\[
\Phi_v : (\lambda_{01}, \lambda_{02}) \to V
\]

as follows

\[
\Phi_v(b) = Lv - D_tv + P(b(v + w)^+ - a(v + w)^-), \quad b \in (\lambda_{01}, \lambda_{02}),
\]

where \( v \in V \) and \( a \) is fixed.

**Lemma 2.8.** If \( a \) is fixed and \( \lambda_{01} < b < \lambda_{02} \), then \( \Phi_v \) is continuous at

\[
b_0 = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a}.
\]

**Proof.** Let \( \delta = \frac{a + b_0}{2} \) and \( \lambda_{01} < b < \lambda_{02} \). Rewrite \( (1.4) \) as

\[
(2.1) \quad (-L + D_t - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w)),
\]

or equivalently

\[
(2.2) \quad w = (-L + D_t - \delta)^{-1}(I - P)g(b, w),
\]

707
where
\[ g(b, w) = b(v + w)^+ - a(v + w)^- - \delta(v + w). \]

By Lemma 1.2, (2.2) has a unique solution \( w = \theta_b(v) \), for a fixed \( b \). Let \( w_0 = \theta_{b_0}(v) \). Then we have
\[ w - w_0 = S[g(b, w) - g(b_0, w_0)] \]
\[ = S[g(b, w) - g(b, w_0) + g(b, w_0) - g(b_0, w_0)] \]
\[ = S[g(b, w) - g(b, w_0)] + S[g(b, w_0) - g(b_0, w_0)], \]
where \( S = (-L + D_t - \delta)^{-1}(I - P) \). Since
\[ \|g(b, w) - g(b, w_0)\| \leq \max\{|b - \delta|, |\delta - a|\}\|w - w_0\| \]
and
\[ \gamma = \frac{1}{|\lambda_{q_2} - a|} \max\{|b - \delta|, |\delta - a|\} < 1, \]
we have
\[ \|w - w_0\| \leq \gamma\|w - w_0\| + \frac{1}{|\lambda_{q_2} - a|}\|w - w_0\| \cdot |b - b_0|. \]
Hence
\[ \|w - w_0\| \leq \frac{1}{|\lambda_{q_2} - a||1 - \gamma|}\|v + w_0\| \cdot |b - b_0|, \]
which shows that \( \theta_b(v) \) is continuous at \( b_0 = \frac{2\lambda_{q_1}\lambda_{q_2} - a(\lambda_{q_1} + \lambda_{q_2})}{\lambda_{q_1} + \lambda_{q_2} - 2a} \). Thus \( \Phi_b(b) \) is continuous at \( b_0 \).

First, we investigate the image of the cone \( C_2 \) under \( \Phi \). Let \( q_1 = \varphi_{01} + \epsilon_0\frac{\lambda_{q_2} - b}{\lambda_{q_1} - a}\varphi_{02} \) and \( q_2 = \varphi_{01} + \epsilon_0\frac{\lambda_{q_2} - a}{\lambda_{q_1} - a}\varphi_{02} \). We fix \( a \) and define
\[ \theta = \begin{cases} \arg q_1 - \arg q_2, & \text{if } b > \frac{2\lambda_{q_1}\lambda_{q_2} - a(\lambda_{q_1} + \lambda_{q_2})}{\lambda_{q_1} + \lambda_{q_2} - 2a} \\ \arg q_2 - \arg q_1, & \text{if } b \leq \frac{2\lambda_{q_1}\lambda_{q_2} - a(\lambda_{q_1} + \lambda_{q_2})}{\lambda_{q_1} + \lambda_{q_2} - 2a} \end{cases}. \]
Then \( 0 \leq \theta \leq \frac{\pi}{2} \) and
\[ \tan \theta = \left| \frac{\epsilon_0(\lambda_{q_2} - b)(\lambda_{q_1} - a) - \epsilon_0(\lambda_{q_2} - a)(b - \lambda_{q_1})}{(b - \lambda_{q_1})(\lambda_{q_1} - a) + \epsilon_0^2(\lambda_{q_2} - b)(\lambda_{q_2} - a)} \right|. \]
When \( b \) converges to \( \frac{2\lambda_{q_1}\lambda_{q_2} - a(\lambda_{q_1} + \lambda_{q_2})}{\lambda_{q_1} + \lambda_{q_2} - 2a} \), \( \tan \theta \) converges to 0. Hence \( \theta \) converges to 0 since \( 0 \leq \theta \leq \frac{\pi}{2} \). We note that \( \Phi_2 \) maps \( C_2 \) onto \( R_2 \) when \( b > \frac{2\lambda_{q_1}\lambda_{q_2} - a(\lambda_{q_1} + \lambda_{q_2})}{\lambda_{q_1} + \lambda_{q_2} - 2a} \).
Multiplicity of solutions and source terms in a nonlinear parabolic equation

\[ \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \] and that \( \Phi_2 \) maps \( C_2 \) onto \( R'_2 \) when \( b < \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \).

So if \( b \) converges to \( \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \), the angle of two lines consisting \( \partial R_2 \) and \( \partial R'_2 \) converges to 0. Since \( \Phi_2 \) is continuous at \( b = \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \), \( \Phi_2 \) maps \( C_2 \) onto the ray

\[ S_2 = \left\{ d_1 \varphi_{01} + d_2 \varphi_{02} \mid d_1 \geq 0, d_2 = \varepsilon_0 \left( \frac{\lambda_{02} - b}{b - \lambda_{01}} \right) d_1 \right\}. \]

Second we investigate the image of the cone \( C_4 \) under \( \Phi \). Let \( \gamma_1 = \varphi_{01} - \varepsilon_0 \frac{\lambda_{02} - b}{b - \lambda_{01}} \varphi_{02} \) and \( \gamma_2 = \varphi_{01} - \varepsilon_0 \frac{\lambda_{02} - a}{\lambda_{01} - a} \varphi_{02} \). We fix \( a \). Define

\[ \theta' = \left\{ \begin{array}{ll}
\arg \gamma_1 - \arg \gamma_2, & \text{if } b < \frac{\lambda_{01} \lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} \\
\arg \gamma_2 - \arg \gamma_1, & \text{if } b > \frac{\lambda_{01} \lambda_{02} - a(\lambda_{01} + \lambda_{02})}{\lambda_{01} + \lambda_{02} - 2a} 
\end{array} \right. \]

Then \( 0 \leq \theta' \leq \frac{\pi}{2} \) and

\[ \tan \theta' = \frac{\varepsilon_0(\lambda_{02} - b)(\lambda_{01} - a) - \varepsilon_0(\lambda_{02} - a)(b - \lambda_{01})}{(b - \lambda_{01})(\lambda_{01} - a) + \varepsilon_0^2(\lambda_{02} - b)(\lambda_{02} - a)} \].

When \( b \) converges to \( \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \), \( \tan \theta' \) converges to 0. Hence \( \theta' \) converges to 0 since \( 0 \leq \theta' \leq \frac{\pi}{2} \). We note that \( \Phi_4 \) maps \( C_4 \) onto \( R_4 \) when \( b > \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \) and that \( \Phi_4 \) maps \( C_4 \) onto \( R'_4 \) when \( b < \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \).

So if \( b \) converges to \( \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \), the angle of two lines consisting \( \partial R_4 \) and \( \partial R'_4 \) converges to 0. Since \( \Phi_4 \) is continuous at \( b = \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \), \( \Phi_4 \) maps \( C_4 \) onto the ray

\[ S_4 = \left\{ d_1 \varphi_{01} + d_2 \varphi_{02} \mid d_1 \geq 0, d_2 = \varepsilon_0 \left( \frac{\lambda_{02} - a}{\lambda_{01} - a} \right) d_1 \right\}. \]

Hence we have the following results.

**Theorem 2.9.** For \( i = 2, 4 \), the restriction \( \Phi_i \) maps \( C_i \) onto \( S_i \). And \( \Phi_1 \) and \( \Phi_3 \) are bijective. Therefore, \( \Phi \) maps \( V \) onto \( R \), where \( R = R_1 = R_3 \).

**Theorem 2.10.** Suppose \( a < \lambda_{01} < b < \lambda_{02} \) and \( b = \frac{2\lambda_0 \lambda_2 - a(\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2a} \).

Let \( f = s_1 \varphi_{01} + s_2 \varphi_{02} \in V \). Then we have

1. If \( f \) belongs to interior of \( R \), then (1.3) has exactly two solutions, one of which is positive and the other is negative.
Q-Heung Choi and Zheng-Guo Jin

(2) If \( f \) belongs to boundary of \( R \), then (1.3) has a positive solution and a negative solution, and infinitely may sign changing solutions.

(3) If \( f \) does not belong to \( R \), then (1.3) has no solution.

References


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