A GENERALIZATION OF GIESEKER'S LEMMA

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ABSTRACT. We generalize Gieseker's lemma and use it to compute Picard number of a complete intersection surface.

1. Introduction

We work over the complex numbers $\mathbb{C}$. In [2], J. Harris gave a proof of the following Gieseker's lemma using monodromy:

GIESEKER'S LEMMA. Let $W \subseteq H^0(O_{\mathbb{P}^1}(d - 1))$ be a linear system and $V \subseteq H^0(O_{\mathbb{P}^1}(d))$ be a linear system containing the image of $W$ under the multiplication map $\mu$

$$\mu : W \otimes H^0(O_{\mathbb{P}^1}(1)) \to H^0(O_{\mathbb{P}^1}(d)).$$

Then either $\dim V \geq \dim W + 2$ or $|V|$ equals the complete series $|O_{\mathbb{P}^1}(l - 1)|$ plus $d - l + 1$ fixed points, where $l = \dim V$.

Though this looks simple, it has been used explicitly and implicitly in the proofs of important results. (See, for example, [6]). We generalize the lemma as follows:

THEOREM 1. Let $2 \leq d_1 \leq \cdots \leq d_{n-2}$, $n \geq 3$ and $E = \bigoplus_{j=1}^{n-2} O_{\mathbb{P}^1}(d_j)$. Let $W \subset H^0(\mathbb{P}^1, E)$ denote a subspace such that the evaluation map

$$f : W \otimes O_{\mathbb{P}^1},x \to E_x$$

is surjective for all $x \in \mathbb{P}^1$ and $\text{codim} W \geq 1$. Let $\mu$ denote the multiplication map

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\[ \mu : W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) \rightarrow H^0(\mathbb{P}^1, E \otimes O_{\mathbb{P}^1}(1)). \]

Then \( \dim(\text{im}\mu(W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)))) \geq m + n - 1 \), where \( m = \dim W \geq 1 \).

We will give an elementary proof of this theorem. As an application of this theorem, we will show that the Picard number of a general complete intersection surface in \( \mathbb{P}^n \) containing a line is 2.

2. Proof of the Theorem 1

**General Strategy.** We will show that a basis of \( W \) can be divided into at least \( n - 1 \) disjoint subsets with the property that the map \( \mu \) operates on each disjoint subset creating 1 extra dimension, resp. Here, we note that for the above map \( f \) to be surjective as in the hypothesis, \( \dim W \geq n - 1 \).

**Notation.** Let \( z_0, z_1 \) denote homogeneous coordinates for \( \mathbb{P}^1 \). Let \( B = \{v_1, \ldots, v_m\} \) be a basis of \( W \) in the Theorem 1. Each \( v_i \) can be written as

\[ v_i = z_0^{i_0} z_1^{i_1} (p_1^1, p_1^2, \ldots, p_1^{n-2}), \]

where neither \( z_0 \) nor \( z_1 \) is a common factor of \( p_1^1, p_1^2, \ldots, p_1^{n-2} \). We denote by

\[ p_i^* = (p_i^1, p_i^2, \ldots, p_i^{n-2}). \]

Here, \( p_i^k \) is a homogeneous polynomial of degree \( d_k = (i_0 + i_1) \) for \( i = 1, \ldots, m \), and \( k = 1, \ldots, n - 2 \).

**Definition 1.** Define for \( v_i, v_j \in B, i \neq j \),

\[ \text{Edge}(v_i, v_j) = 1 \quad \text{if} \quad i_0 - j_0 = j_1 - i_1 \quad \text{and} \quad p_i^* = p_j^* \]

\[ \text{Edge}(v_i, v_j) = 0 \quad \text{otherwise}. \]

Note that \( \text{Edge}(v_i, v_j) = \text{Edge}(v_j, v_i) \) and that for each \( v_j \in B \), there can be at most two \( v_i \)'s in \( B \) such that \( \text{Edge}(v_i, v_j) = 1 \).
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**Definition 2.** For a \( v_l \in B \), we define a subset \([v_l]\) of \( B \) recursively as follows:

1. \( v_l \in [v_l] \).
2. \( v_i \in [v_l] \) if \( \text{Edge}(v_l, v_i) = 1 \), and \( v_i \in B - [v_l] \).
3. \( v_j \in [v_l] \) if \( \text{Edge}(v_k, v_j) = 1 \) for some \( v_k \in [v_l] \), \( v_j \in B - [v_l] \).
4. Repeat (3) until there remains no such \( v_j \)'s.

**Facts.** One can easily observe the following facts:

1. Each element \( v_k \) of \([v_l]\) can be written as \( v_k = z_0^{k_0} z_1^{k_1} p_1^\ast \). That is, \( k_0 \) and \( k_1 \) depend on \( v_k \), but \( p_1^\ast = p_1^\ast \) for any \( v_k \in [v_l] \).
2. If there is a \( v_k \in B - [v_l] \), one can construct another subset \([v_k]\). By construction, \([v_l]\) is disjoint with \([v_k]\). Also, for any element \( v_i \) of \([v_l]\) and any element \( v_j \) of \([v_k]\), \( z_h v_i \neq z_\nu v_j \), where \( h, \nu \in \{0, 1\} \). Thus \( W \) can be divided into disjoint subset \([v_l]\)'s.
3. The map \( \mu \) operates on each disjoint \([v_i]\) creating 1 extra dimension respectively. That is, let \( V_i \subset H^0(\mathbb{P}^1, E) \) be the subspace generated by the elements of \([v_i]\). Then \( \dim V_i = ||v_i|| = \) the number of elements of \( v_i \) and
   \[
   \dim \left( \text{im}\mu \left( V_i \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) \right) \right) = \dim V_i + 1.
   \]

Moreover, if \( B \) is the union of disjoint subsets, say \([v_1], \ldots, [v_k]\), then

\[
\dim \left( \text{im}\mu \left( W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) \right) \right) = \sum_{i=1}^{k} \dim \left( \text{im}\mu \left( V_i \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) \right) \right) = \sum_{i=1}^{k} (||v_i|| + 1) = \dim W + k.
\]

From the above facts, we can see that to prove the theorem, all we need to show is the following:

**Lemma 1.** There are at least \( n - 1 \) disjoint subset \([v_i]\)'s in \( B \).
Proof. We will show this for \( n = 3 \) and then for any \( n \geq 4 \). Though the proof need not be separated for these 2 cases, we provide the proof for \( n = 3 \) as an illustration for the idea of the proof.

A. For \( n = 3 \), we claim that \( B \) is the union of at least 2 disjoint subsets, say, \([v_1]\) and \([v_k]\). If not, then \( B = [v_1] \) and \( v_i = z_0^{i_0}z_1^{i_1}p_1^{i_1} \), \( 1 \leq i \leq m \), and \( p_1^{i_1} \) is a homogeneous polynomial which is not divisible by either \( z_0 \) or \( z_1 \). Moreover, \( v_i \)'s can be rearranged so that

\[
\begin{align*}
v_1 &= z_0^{\alpha + m}z_1^{\beta}p_1^{i_1} \\
v_2 &= z_0^{\alpha + m - 1}z_1^{\beta + 1}p_1^{i_1} \\
&\vdots \\
v_m &= z_0^{\alpha + 1}z_1^{\beta + m - 1}p_1^{i_1}
\end{align*}
\]

for an integer \( \alpha \geq -1 \) and for some nonnegative integer \( \beta \).

If \( \text{deg } p_1^{i_1} \geq 1 \), then a zero of \( p_1^{i_1} \) is a base point of \( W \), at which the map \( f \) is not surjective.

If \( \text{deg } p_1 = 0 \), then \( m + \alpha + \beta = d_1 \). For the evaluation map \( f \) to be surjective at \( P = (1,0) \) and at \( Q = (0,1) \), we should have \( \beta = 0 \) and \( \alpha = -1 \). Hence we get \( m - 1 = d_1 \), i.e., \( \text{codim} W = 0 \), which is a contradiction.

B. If \( n \geq 4 \), we will show that \( B \) is a union of at least \( n - 1 \) disjoint \([v_i]\)'s.

If \( B \) is a union of \( k \) disjoint \([v_i]\)'s, then without loss of generality, we may assume that \( B = \bigcup_{i=1}^{k} [v_i] \) and

\[
\begin{align*}
[v_1] &= \{z_0^{\gamma_1}z_1^{\delta_1}p_1^{*}, z_0^{\gamma_1 - 1}z_1^{\delta_1 + 1}p_1^{*}, \ldots, z_0^{\gamma_1 - \alpha_1}z_1^{\gamma_1 + \alpha_1}p_1^{*}\} \\
&\vdots \\
[v_k] &= \{z_0^{\gamma_k}z_1^{\delta_k}p_k^{*}, z_0^{\gamma_k - 1}z_1^{\delta_k + 1}p_k^{*}, \ldots, z_0^{\gamma_k - \alpha_k}z_1^{\delta_k + \alpha_k}p_k^{*}\}
\end{align*}
\]

for nonnegative integers \( \gamma_i, \delta_i, \) and \( \alpha_i, 1 \leq i \leq k \) satisfying the following conditions:

(a) \( \sum_{i=1}^{k}(\alpha_i + 1) = m \)
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(b) \( \gamma_i - \alpha_i \geq 0 \)

(c) Fix a \( j \) with \( 1 \leq j \leq n - 2 \). Then, 
\[
\gamma_i + \delta_i + \deg p_i^j = d_j \text{ for any } i \text{ with } p_i^j \neq 0.
\]

We will show \( k \geq n - 1 \).

(1) If \( k \leq n - 3 \), then, at \( P = (z_0, z_1) \) with \( z_0 \neq 0 \) and \( z_1 \neq 0 \), the rank of the evaluation map \( f \) is at most \( k \leq n - 3 \), which contradicts the hypothesis.

(2) If \( k = n - 2 \), then we will find a point where the map \( f \) is not surjective. We consider the following matrix:

\[
\begin{pmatrix}
p_1^1 & p_1^2 & \cdots & p_1^{n-2} \\
p_2^1 & p_2^2 & \cdots & p_2^{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-2}^1 & p_{n-2}^2 & \cdots & p_{n-2}^{n-2}
\end{pmatrix}
\]

(i) If \( \deg p_i^j > 0 \) for some \( i \) and \( j \), then at the zeros of the determinant of the above matrix, the evaluation map \( f \) is not surjective.

(ii) If \( \deg p_i^j = 0 \) for every \( i \) and \( j \) in \( \{1, \ldots, n - 2\} \), then either \( p_i^j = 0 \) or \( p_i^j = 1 \). If the rank of the above matrix is \( < n - 2 \), then the map \( f \) is not surjective at any point \( P = (z_0, z_1) \) where \( z_0 \neq 0 \) and \( z_1 \neq 0 \). So it contradicts the hypothesis of the theorem and the proof is done.

But, for the above matrix to be of rank \( n - 2 \), the determinant of the matrix should not be equal to \( 0 \). This can happen when

\[
p_1^{j_1} p_2^{j_2} \cdots p_{n-2}^{j_{n-2}} \neq 0
\]

for at least one permutation \( (j_1 \ldots j_{n-2}) \) of \( \{1, \ldots, n - 2\} \). In this case, \( \deg p_i^j = 0 \) implies \( d_{j_i} = \gamma_i + \delta_i \) by the above condition (c). For the map \( f \) to be surjective at \((1, 0)\) and at \((0, 1)\), \( \delta_i = 0 \) for all \( i \) and \( \gamma_i = \alpha_i \). So

\[
m = \sum_{i=1}^{n-2} (\alpha_i + 1) = \sum_{i=1}^{n-2} (\gamma_i + 1)
\]

\[
= \sum_{i=1}^{n-2} (d_{j_i} + 1) = \sum_{i=1}^{n-2} (d_i + 1).
\]

This implies \( \text{codim} W = 0 \), which is a contradiction. \( \square \)
3. An Application

Using, the above theorem, we will show that the Picard number of a general complete intersection surface $S$ containing a line in $\mathbb{P}^n$ is 2, that is, $\text{Pic}(S)$ is generated by the hyperplane section curve and the line.

Let $2 \leq d_1 \leq d_2 \leq \cdots \leq d_{n-2}$ and $Y_{n,d_1,d_2,\ldots,d_{n-2}} = \{ \text{ smooth complete intersection surfaces of type } (d_1, \ldots, d_{n-2}) \text{ in } \mathbb{P}^n \}$. The Noether-Lefschetz locus is $\Sigma = \{ S \in Y_{n,d_1,d_2,\ldots,d_{n-2}} \mid \text{Pic}(S) \neq \mathbb{Z} \}$.

**Theorem 2.** Let $\sum_{i=1}^{n-2} d_i \geq n + 2$ and $n \geq 3$. Let $Z_1$ denote an irreducible component of $\Sigma$ whose generic member contains a line. Then, for a general $S$ in $Z_1$, the Picard number is 2.

The word "general" is used in the sense that a property is said to hold at a general point of a projective variety $V$ if the property holds at all the points of $V$ but the points in a countable union of subvarieties of $V$.

It is known that the codimension of $Z_1$ is $\geq \sum_{i=1}^{n-2} d_i - n$ (cf. [3]).

Using deformation theoretic technique, Lopez [5] figured the generators of the Picard group of a general complete intersection surface containing a fixed curve. In [5], he showed that for a general projectively Cohen-Macaulay surface $X$ in $\mathbb{P}^4$ defined by the maximal minors of a matrix with no zeros, $\text{Pic}(X) \cong \mathbb{Z}^2$ generated by $O_X(1)$ and $K_X$ unless $X$ is the Castelnuovo or Bordiga surface. He [6] also gave a new proof of the above Theorem 2 for a general surface in $\mathbb{P}^3$ containing a plane curve, which is infinitesimal Hodge theoretic and completely different from the one in [5].

Following Lopez's idea for the case $n = 3$ in [6], we can reduce Theorem 2 to Theorem 1.

**Proof of Theorem 2.** Let $z_0, \ldots, z_n$ denote homogeneous coordinates for $\mathbb{P}^n$. $C$ be the line with equations $z_0 = z_1 = \cdots = z_{n-2} = 0$ in $\mathbb{P}^n$. For a generic $S \in Z_1$, let $S = \cap_{i=1}^{n-2} \{ F_i = 0 \}$, where $F_i = \sum_{j=0}^{n-2} z_j G_j^i = 0$ and $F_i$ is an irreducible homogeneous polynomial of degree $d_i$, $i = 1, \ldots, n - 2$. Without loss of generality, we may assume
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that \( \bigcap_{j=1}^{k} \{ F_j = 0 \} \) are smooth for \( k = 1, \ldots, n - 2 \), and that, for \( i = 1, \ldots, n - 2 \), \( \bigcap_{j=0}^{n-2} \{ G_j^i = 0 \} \cap C = \emptyset \).

Let \( H_{prim}^{1,1}(S) \subset H^1(S, \Omega^1_S) \) denote the primitive (1,1)-cohomology of \( S \). Let \( L = O_S(C) \). \( \gamma = c_1(L) \in H_{prim}^{1,1}(S) \) defines an extension \( M \) of the tangent sheaf \( \Theta_S \) of \( S \) by the structure sheaf \( O_S \), i.e. \( M \) is defined by the exact sequence

\[
0 \to O_S \to M \to \Theta_S \to 0
\]

with the extension class \( \gamma \). The induced map \( H^1(S, \Theta_S) \to H^2(S, O_S) \) is given by the cup product with \( \gamma \). By dualizing the map,

\[
H^1(S, \Theta_S) \otimes H_{prim}^{1,1}(S) \to H^2(S, O_S),
\]

we get

\[
H^1(S, \Theta_S) \otimes H^{2,0}(S) \to H_{prim}^{1,1}(S)^*.
\]

Let \( E = \bigoplus_{i=1}^{n-2} O_{\mathbb{P}^n}(d_i), E(k) = E \otimes O_{\mathbb{P}^n}(k) \), and \( \nu \) denote the number \( \nu = \sum_{i=1}^{n-2} d_i - n - 1 \).

By algebraic identifications (cf. [1] or [5] for \( n = 3 \), [3] for \( n \geq 4 \)), the above map is the multiplication map

\[
\begin{array}{ccc}
\frac{H^0(\mathbb{P}^n, E)}{J} & \otimes & \frac{H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))}{I} \\
J & \to & J'
\end{array}
\]

where \( J, I, J' \) denote the appropriate subspaces; For \( n = 3 \), \( J \) is the Jacobian ideal of \( S \) in degree \( d_1 \) and \( I = 0 \). For \( n \geq 4 \),

\[
I = \text{im} H^0(\mathbb{P}^n, \bigoplus_{i=1}^{n-2} O_{\mathbb{P}^n}(\nu - d_i)) = \{ \sum_{i=1}^{n-2} a_i F_i | a_i \in H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu - d_i)) \}.
\]

Here \( a_i = (e_1, \ldots, e_i^{n-2}), e_i^k = 1 \) if \( i = k \), \( e_i^k = 0 \) otherwise (for precise definitions, see [3]).

Let \( W'_\gamma \subset H^1(S, \Theta_S) \) be the Zariski tangent space to \( Z_1 \) keeping \( \gamma \) of type \((1,1)\). Then the image of \( W'_\gamma \otimes H^{2,0}(S) \) is contained in \((\gamma)^{1}\). Let \( W_\gamma \) be the preimage of \( W'_\gamma \) under the projection \( H^0(\mathbb{P}^n, E) \to H^0(\mathbb{P}^n, E) \), and \( R_{J'} = \frac{H^0(\mathbb{P}^n, E(\nu))}{J'} \). Then we have a map

\[
\lambda : W_\gamma \otimes O_{\mathbb{P}^n,x} \to E_x \quad \text{is surjective for every } x \in \mathbb{P}^n \text{ (cf. [3])}.
\]
LEMMA 2. \( \operatorname{codim}_{R_j} \operatorname{im} \lambda(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \leq 1. \)

Proof. Let \( W = \operatorname{im}(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu)) \to H^0(\mathbb{P}^n, E(\nu))) \) and \( R = H^0(\mathbb{P}^n, E(\nu)) \). By definition, \( J' \subset W \) and hence it is enough to show \( \operatorname{codim}_R W \leq 1. \)

Let \( R|_C \) be the restriction of \( R \) to \( C \), and \( W|_C, W_\gamma|_C \), the restriction of \( W, W_\gamma \) to \( C \), resp. Recall that \( C = \mathbb{P}^1 \) with homogeneous coordinates \( z_{n-1}, z_n \).

Note \( R|_C = H^0(\mathbb{P}^1, E(\nu) \otimes O_{\mathbb{P}^1}), \) and \( W|_C = \operatorname{im}(W_\gamma|_C \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\nu)) \to R|_C) \). Let \( I_C \) be the ideal sheaf of \( C \), and \( I(C) = \operatorname{im}(H^0(\mathbb{P}^n, I_C \otimes E) \to H^0(E)) \). By construction, \( I(C) \subset W_\gamma \) and this implies \( \operatorname{codim}_R W = \operatorname{codim}_{R|_C} W|_C \). So it suffices to show \( \operatorname{codim}_{R|_C} W|_C \leq 1. \)

On the other hand, \( \{ z_k(G^1_j, \ldots, G^{n-2}_j) | 0 \leq j \leq n-2, k = n-1, n \} \subset W_\gamma|_C \) and so \( W_\gamma|_C = W' \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) \) for some \( W' \subset H^0(\mathbb{P}^1, E \otimes O_{\mathbb{P}^1}(-1)) \) containing \( \{ (G^1_j, \ldots, G^{n-2}_j) \mid 0 \leq j \leq n-2 \} \). So the evaluation map of \( W' \) is surjective and \( \dim W' \geq n-1. \) By applying Theorem 1 \((\nu + 1)\) times,

\[
\dim W|_C = \dim \left( \operatorname{im}(W' \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\nu+1))) \right) \geq n-1 + (n-1)(\nu+1).
\]

Hence \( \operatorname{codim}_{R|_C} W|_C \leq 1. \)

The rest of the proof of the theorem uses the idea of Lopez's proof for \( n = 3 \) which we restate: By the semicontinuity theorem, it is enough to prove that for each \( \gamma' \in H^{1,1}_{\text{prim}}(S) - \mathbb{C}\gamma \), there exists a deformation \( \eta \in \mathcal{W}_\gamma \) such that, when we deform \( S \) in the direction of \( \eta \) to a surface \( S' \), the class \( \gamma' \) is not of type \((1,1)\). That is, it is enough to show \( \mathcal{W}_\gamma \not\subset \mathcal{W}_{\gamma'} \). By Lemma 2, \( \gamma' \in \operatorname{im}(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \). Therefore, if \( \gamma' \neq 0 \) and \( W_\gamma \subset \mathcal{W}_{\gamma'} \), then

\[
\gamma' \in \operatorname{im}(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \subset \operatorname{im}(W_{\gamma'} \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \subset (\gamma')^\perp,
\]

which is a contradiction. \( \square \)
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 References


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