ON THE SPECTRUM OF THE $p$-LAPLACIAN
ON QUATERNIONIC KAHLER MANIFOLDS

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ABSTRACT. We study some spectral properties of the $p$-Laplacian on quaternionic Kaehler manifolds.

1. Introduction

Let $(M, g)$ be a compact manifold of dimension $n$ with metric tensor $g$. Let $\Delta^p = d\delta + \delta d$ be the Laplace-Beltrami operator acting on the space of smooth $p$-forms. Then we have the spectrum of $\Delta^p$ for each $0 \leq p \leq n$

$$\text{Spec}^p(M, g) := \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \cdots \uparrow +\infty\},$$

where each eigenvalue is repeated according to its multiplicity.

An interesting problem on the spectral geometry is as follows: Let $(M, g)$ and $(M', g')$ be compact Riemannian (resp. Kaehlerian) manifolds with $\text{Spec}^p(M, g) = \text{Spec}^p(M', g')$ for an arbitrary fixed $p \geq 0$. Then is it true that $(M, g)$ is constant sectional curvature (resp. constant holomorphic sectional curvature) $c$ if and only if $(M', g')$ is of constant sectional curvature (resp. constant holomorphic sectional curvature) $c'$ and $c = c'$?

The answer to the problem is affirmative for any $p \geq 0$ and the particular dimension of $M$ (cf. [4, 5, 6, 7, 8]).

The purpose of the present paper is to study quaternionic analogues for certain results (cf. [2, 5, 6, 7, 8]) of the problem.

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Main Theorem. Let \((M, g)\) and \((M', g')\) be compact quaternionic Kaehler manifolds with \(\text{Spec}^p(M, g) = \text{Spec}^p(M', g')\) for an arbitrary fixed \(p \geq 0\) (which implies \(\text{dim}M = \text{dim}M' = 4m\), \(m \geq 2\). If \((m, p) \notin \{(4, 2), (4, 14)\}\) and \(2m(4m - 1) - 3p(4m - p) \neq 0\), then \(M\) is of constant quaternionic sectional curvature \(c\) if and only if \(M'\) is of constant quaternionic sectional curvature \(c' = c\).

2. Preliminaries and Proof

Let \((M, g)\) be a real \(4m\)-dimensional compact quaternionic Kaehler manifold. Then there exists a 3-dimensional vector bundle \(V\) of tensors of type \((1, 1)\) with local basis of almost Hermitian structures \(F = (F_{ij})\), \(G = (G_{ij})\), \(H = (H_{ij})\) such that (i) \(FG = -GF = H\), and (ii) for any local cross section \(\phi\) of \(V\), \(\nabla_X \phi\) is also a cross section of \(V\), where \(X\) is an arbitrary vector field in \(M\) and \(\nabla\) the Levi-Civita connection on \(M\). It is known that any quaternionic Kaehler manifold \((M, g)\) is an Einstein manifold when \(\text{dim}M \geq 8\) (for details, see cf. [1]). By \(R = (R^i_{jkl}), \rho = (R_{ijkl}) = (R^i_{jkl})\) and \(\sigma = (g^{ij}R_{ijkl})\) we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively, and \(g = (g_{ij})\) is a Riemannian metric tensor on \(M\), \((g^{ij}) = (g_{ij})^{-1}\). For the tensor field \(T\) on \(M\) we denote \(|T|\) the norm of \(T\) with respect to \(g\). We define a quaternionically projective curvature tensor field \(Q = (Q_{kijh})[3]\) defined on \(M\) by

\[
Q_{kijh} = R_{kijh} + \frac{1}{4m + 8}(R_{kijh}g_{kk} - R_{kij}g_{kh} - 2R_{kli}F^i_{j}F_{ih} - R_{kl}F^i_{j}F_{j}h + R_{lij}F^i_{j}F_{k}h - 2R_{kli}G^i_{j}G_{ih} - R_{kli}G^i_{j}G_{j}h + R_{kl}H^i_{j}H_{ih} - R_{kl}H^i_{j}H_{j}h + R_{lij}H^i_{j}H_{k}h).
\]

Then we obtain

\[
(2.1) \quad |Q|^2 = |R|^2 - \frac{5m + 1}{(m + 2)^2} |E|^2 - \frac{5m + 1}{4m(m + 2)^2} \sigma^2,
\]

where we have put \(E := (E_{ij} = R_{ij} - \frac{\sigma}{4m} g_{ij})\).

Let \(S(X)\) be the so-called quaternionic section determined by \(X\), which is a 4-plane spanned by \(\{X, FX, GX, HX\}\), where \(X\) is a unit.
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vector on the quaternionic Kaehler manifold $M$. Any 2-plane in a quaternionic section is called a quaternionic plane. The sectional curvature of a quaternionic plane $\pi$ is called the quaternionic sectional curvature of $\pi$. It is known [3] that if a quaternionic Kaehler manifold $M$ quaternionically projective flat, i.e., $Q$ vanishes identically, then it is of constant quaternionic sectional curvature.

Now we introduce the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for $Spec^p(M,g)$, which is given by

$$
\sum_{\alpha=0}^{\infty} \exp(-\lambda_{\alpha,p} t) = (4\pi t)^{-\text{dim} M/2} [a_{0,p} + ta_{1,p} + \cdots + t^Na_{N,p}] \\
+ o(t^{N-m+\frac{1}{2}}) \quad \text{as} \quad t \downarrow 0,
$$

where $a_{0,p}, a_{1,p}, a_{2,p}, \cdots$ are numbers which is expressed by (see cf. [4])

$$
a_{0,p} = \binom{4m}{p} \int_M dM,
$$

$$
a_{1,p} = \frac{1}{6} \left[ \binom{4m}{p} - 6 \binom{4m-2}{p-1} \right] \int_M \sigma dM,
$$

$$
a_{2,p} = \frac{1}{360} \int_M \left\{ \left[ 5 \binom{4m}{p} - 60 \binom{4m-2}{p-1} + 180 \binom{4m-4}{p-2} \right] \sigma^2 \\
+ \left\{ -2 \binom{4m}{p} + 180 \binom{4m-2}{p-1} - 720 \binom{4m-4}{p-2} \right\} |\rho|^2 \\
+ \left\{ 2 \binom{4m}{p} - 30 \binom{4m-2}{p-1} + 180 \binom{4m-4}{p-2} \right\} |R|^2 \right\} dM,
$$

where $dM$ denotes the volume element of $M$.

On the other hand, we have for $p \notin \{0, 1, 2, 3, 4m-1, 4m-2\}$,

$$
\binom{4m}{p} = \frac{4m(4m-1)(4m-2)(4m-3)}{p(p-1)(4m-p)(4m-p-1)} \binom{4m-4}{p-2},
$$

$$
\binom{4m-2}{p-1} = \frac{(4m-2)(4m-3)}{(p+1)(4m-p-1)} \binom{4m-4}{p-2}.
$$
For $p \notin \{0, 1, 2, 3, 4m - 1, 4m\}$, substituting (2.1), (2.6), and (2.7) into (2.5) yields

\[(2.8) \quad a_{2,p} = \alpha \int_M \left[ 4P_1|Q|^2 + \frac{4}{(m + 2)^2} P_2|E|^2 + \frac{1}{m(m + 2)^2} P_3 \sigma^2 \right] dM, \]

where

\[P_1 := P_1(m, p) = 128m^4 - (480p + 192)m^3 + (840p^2 - 120p + 88)m^2 - (360p^3 - 30p^2 + 12)m + 45p^4,\]

\[P_2 := P_2(m, p) = -128m^6 + (2880p + 320)m^5 - (3600p^2 - 8400p + 664)m^4 + (1440p^3 - 10020p^2 + 7920p + 676)m^3 - (180p^4 - 3960p^3 + 12780p^2 + 1560p + 276)m^2 - (495p^4 - 5400p^3 - 390p^2 - 1440p - 36)m - 675p^4 - 360p^2,\]

\[P_3 := P_3(m, p) = 1270m^7 - (3840p - 3072)m^6 + (3840p^2 - 10560p - 1360)m^5 - (1440p^3 - 11280p^2 + 4944)m^4 + (180p^4 - 4320p^3 + 3600p^2 + 12720p + 3716)m^3 + (540p^4 - 1800p^3 - 13980p^2 - 4440p - 756)m^2 + (225p^4 + 5400p^3 + 1110p^2 + 1440p + 36)m - 675p^4 - 360p^2,\]

\[\alpha := \frac{(4m - 4)}{360p(p - 1)(4m - p)(4m - p - 1)}.\]
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For \( p \in \{0, 1, 2, 3, 4m - 1, 4m\} \), the formula (2.5) is of the form;

\[
(2.9) \quad a_{2,p} = \beta \int_M \left[ 4Q_1|Q|^2 + \frac{8}{m+2}Q_2|E|^2 + \frac{4}{m(m+1)}Q_3\sigma^2 \right] dM,
\]

where for \( i = 1, 2, 3 \) and \( m > 1 \)

(i) if \( p = 0 \), then

\[
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,0)}{4m(4m-1)(4m-2)(4m-3)},
\]

(ii) if \( p = 1 \), then

\[
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,1)}{(4m-1)(4m-2)(4m-3)},
\]

(iii) if \( p = 2 \), then

\[
\beta = \frac{1}{2 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2)}{(4m-2)(4m-3)},
\]

(iv) if \( p = 3 \), then

\[
\beta = \frac{1}{6 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,3)}{4m-3},
\]

(v) if \( p = 4m - 1 \), then

\[
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,4m-1)}{(4m-1)(4m-2)(4m-3)},
\]

(vi) if \( p = 4m \), then

\[
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,4m)}{4m(4m-1)(4m-2)(4m-3)}.
\]
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**Remark 1.** The sign of the coefficients of $|Q|^2$, $|E|^2$, and $\sigma^2$ in the formula (2.9) are respectively determined by the polynomials $P_1, P_2$, and $P_3$.

From now on we shall write (2.8) and (2.9) in the following form ; (2.10)

$$a_{2,p} = \gamma \int_M [4R_1|Q|^2 + \frac{4}{(m+2)^2} R_2|E|^2 + \frac{1}{m(m+2)^2} R_3 \sigma^2] \, dM,$$

where $\gamma$ is either $\alpha$ or $\beta$, and $R_i$ is either $P_i$ or $Q_i$ (i=1,2,3).

**Remark 2.** The equation $\binom{4m}{p} - 6 \binom{4m-2}{p-1} = 0$ if and only if $2m(4m-1)-3p(4m-p) = 0$ if and only if $u^2 - 12v^2 = 1$, where $m = \frac{u-1}{4}$, $p = \frac{u-1}{2} \pm v$. The least solutions are $(u,v) = (7,2), (97,28), (1351, 390), \cdots$, which give $(m,p) = (24,20), (24,76), \cdots$.

**Remark 3.** The polynomial $R_1$ has the only solutions $(m,p) = (4,2), (4,14)$ (for the proof, see Theorem 3.1 in [6]).

**Remark 4.** Assume that $\text{Spec}^p(M,g) = \text{Spec}^p(M',g')$. Then $\dim M = \dim M'$ is derived from (2.2).

**Proof of Main Theorem.** Since $M$ and $M'$ (dim $M$=dim $M'$=4$m$ $\geq 8$) are Einstein manifolds, $E = 0 = E'$ and $\sigma, \sigma'$ are constants. From (2.3) and (2.4), we have $\sigma = \sigma'$. And (2.10) with $\text{Spec}^p(M,g) = \text{Spec}^p(M',g')$ yields

$$\int_M 4R_1|Q|^2 \, dM = \int_{M'} 4R_1|Q'|^2 \, dM'.$$

But for $(m,p) \notin \{(4,2), (4,14)\}$, $R_1 \neq 0$ (Remark 3). Hence $Q = 0$ if and only if $Q' = 0$. \hfill $\square$

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References


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