FIXED POINTS OF A CERTAIN CLASS OF ASYMPTOTICALLY REGULAR MAPPINGS

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ABSTRACT. In this paper, we study in Banach spaces the existence of fixed points of asymptotically regular mapping \( T \) satisfying: for each \( x, y \) in the domain and for \( n = 1, 2, \cdots \),

\[
\|T^nx - T^ny\| \leq a_n\|x - y\| + b_n(\|x - T^nx\| + \|y - T^ny\|) + c_n(\|x - T^nx\| + \|y - T^nx\|),
\]

where \( a_n, b_n, c_n \) are nonnegative constants satisfying certain conditions. We also establish some fixed point theorems for these mappings in a Hilbert space, in \( L^p \) spaces, in Hardy spaces \( H^p \), and in Sobolev spaces \( H^{k,p} \) for \( 1 < p < \infty \) and \( k \geq 0 \). We extend results from papers [10], [11], and others.

1. Introduction and preliminaries

Let \( E \) be a real Banach space with norm \( \| \cdot \| \) and let \( K \) be a nonempty subset of \( E \). A mapping \( T : E \to E \) is said to be asymptotically regular \( [2] \) if \( \lim_{n \to \infty} \|T^{n+1}x - T^nx\| = 0 \) for all \( x \in E \). It is well known that if \( T \) is nonexpansive, then \( T_t = t \cdot I + (1 - t) \cdot T \) is asymptotically regular for all \( 0 < t < 1 \) (cf. [9]).

Lin [14] constructed an asymptotically regular Lipschitzian mapping acting on a weakly compact subset of \( l^2 \) which has no fixed point. Górnicki gave the sufficient condition for the existence of fixed points

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In this paper, we extend all the above results for the class of mappings whose $n$th iterate $T^n$ satisfy

$$\|T^n x - T^n y\| \leq a_n \|x - y\| + b_n \left(\|x - T^n x\| + \|y - T^n y\|\right)$$
$$+ c_n \left(\|x - T^n y\| + \|y - T^n x\|\right)$$

(1)

for each $x, y \in K$ and $n = 1, 2, \cdots$, where $a_n, b_n, c_n$ are the nonnegative constants such that there exists an integer $n_0$ satisfying $b_n + c_n < 1$ for all $n \geq n_0$.

This class of mappings are more general than nonexpansive mappings. Also by taking $b_n = c_n = 0$, it will be seen that this class of mappings are more general than asymptotically nonexpansive mappings defined by Goebel and Kirk [8].

The normal structure coefficient $N(E)$ (cf. Bynum [3]) of $E$ is the number:

$$N(E) = \inf \left\{ \frac{\text{diam} K}{r_K(K)} : K \text{ is a bounded convex subset of } E \right\},$$

consisting of more than one point,

where $\text{diam} K = \sup \{\|x - y\| : x, y \in K\}$ is the diameter of $K$ and $r_K(K) = \inf_{x \in K} \{\sup_{y \in K} \|x - y\|\}$ is the Chebyshev radius of $K$ relative to itself. $E$ is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure (cf. Danes [5]) and for a Hilbert space $H$, $N(H) = \sqrt{2}$. Recently, Pichugov [16] (cf. Prus [17]) calculated that

$$N(L^p) = \min \{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\}, \ 1 < p < \infty.$$ 

Some estimates for normal structure coefficients in other Banach spaces may be found in [18].

Let $p > 1$ and denote by $\lambda$ the number in $[0, 1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda)$. 

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The functional \( \| \cdot \|^p \) is said to be uniformly convex (cf. Zalinescu [23]) on the Banach space \( E \) if there exists a positive constant \( c_p \) such that for all \( \lambda \in [0, 1] \) and \( x, y \in E \) the following inequality holds:

\[
(2) \quad \| \lambda x + (1 - \lambda) y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - W_p(\lambda) \cdot c_p \cdot \| x - y \|^p.
\]

Xu [22] proved that the functional \( \| \cdot \|^p \) is uniformly convex on the whole Banach space \( E \) if and only if \( E \) is \( p \)-uniformly convex, i.e., there exists a constant \( c > 0 \) such that the modulus of convexity (see [9])

\[
\delta_E(\varepsilon) \geq c \cdot \varepsilon^p
\]

for all \( 0 \leq \varepsilon \leq 2 \).

2. Main results

Before presenting our main result, we need following lemmas.

**Lemma 1** ([22]). Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( E \) and \( \{ x_n \} \subset E \) a bounded sequence. Then there exists a unique point \( z \) in \( K \) such that

\[
(3) \quad \limsup_{n \to \infty} \| x_n - z \|^p \leq \limsup_{n \to \infty} \| x_n - x \|^p - c_p \cdot \| x - z \|^p
\]

for every \( x \) in \( K \), where \( c_p \) is the constant given in (2).

**Lemma 2** ([11]). Let \( K \) be a nonempty closed convex subset of a Banach space \( E \) and \( \{ n_i \} \) an increasing sequence of natural numbers. Assume that \( T : K \to K \) is an asymptotically regular mapping such that for some \( m \in \mathbb{N} \), \( T^m \) is continuous. If

\[
\limsup_{i \to \infty} \| x - T^{m_i} u \| = 0
\]

for some \( u \in K \) and \( x \in K \), then \( Tx = x \).

Now we are in position to give our result.

**Theorem 1.** Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( E \), and \( T : K \to K \)
an asymptotically regular mapping which holds the inequality (1) such that

\[(C) \quad \left[ \left( \alpha + \beta \right)^p \frac{(2^{p-1} \alpha^p - 1)}{(c_p - 2^{p-1} \beta^p) \cdot N^p} \right]^\frac{1}{p} < 1, \]

where

\[\alpha = \liminf_{n \to \infty} \frac{a_n + c_n}{1 - c_n}, \quad \beta = \liminf_{n \to \infty} \frac{b_n}{1 - c_n},\]

and $N$ is the normal structure coefficient of $E$. Suppose that there is a $z_0$ in $K$ for which $\{T^n z_0\}$ is bounded. Then $T$ has a fixed point in $K$.

**Proof.** Let $\{n_i\}$ be a sequence of natural numbers such that

\[\alpha = \liminf_{n \to \infty} \frac{a_n + c_n}{1 - c_n} = \lim_{i \to \infty} \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}},\]

and

\[\beta = \liminf_{n \to \infty} \frac{b_n}{1 - c_n} = \lim_{i \to \infty} \frac{b_{n_i}}{1 - c_{n_i}}.\]

Since $\{T^n z_0\}$ is bounded (and hence $\{T^n z\}$ is bounded for any $z$ in $K$), by Lemma 1, we can inductively construct a sequence $\{z_m\}$ such that $z_m$ is the unique asymptotic center of the sequence $\{T^{m_i} z_{m-1}\}_{i \geq 1}$ with respect to the functional

\[\limsup_{i \to \infty} \|x - T^{m_i} z_{m-1}\|^p\]

over $x$ in $K$.

Now for each $m \geq 1$, we set

\[D_m = \limsup_{i \to \infty} \|z_m - T^{m_i} z_m\|\]

and

\[r_m = \limsup_{i \to \infty} \|z_{m+1} - T^{m_i} z_m\|.\]
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Now, using (1), we have for each $x, y \in K$ and $k, l \geq 1$,

$$\|T^k x - T^l y\| \leq \|T^k x - T^{k+l} y\| + \|T^{k+l} y - T^l y\|$$

$$\leq a_k \|x - T^l y\| + b_k (\|x - T^k x\| + \|T^l y - T^{k+l} y\|)$$

$$+ c_k (\|T^l y - T^k x\| + \|x - T^{k+l} y\|) + \|T^{k+l} y - T^l y\|$$

which by simplification, gives

$$\|T^k x - T^l y\| \leq \frac{a_k + c_k}{1 - c_k} \cdot \|x - T^l y\| + \frac{b_k}{1 - c_k} \cdot \|x - T^k y\|$$

$$+ \frac{1 + b_k + c_k}{1 - c_k} \cdot \|T^{k+l} y - T^l y\|. \quad (4)$$

By inequality (4), the result of Casini and Maluta [4], and the asymptotic regularity of $T$, we have

$$r_m \leq \frac{1}{N} \cdot \lim_{n \to \infty} (\sup_{n_i, n_j} \|T^{n_i} z_m - T^{n_j} z_m\| : n_i, n_j \geq n)$$

$$\leq \frac{1}{N} \cdot \limsup_{i \to \infty} (\limsup_{j \to \infty} \|T^{n_i} z_m - T^{n_j} z_m\|)$$

$$\leq \frac{1}{N} \cdot \limsup_{i \to \infty} \left[ \limsup_{j \to \infty} \left( \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_j} z_m\| \right.ight.$$  

$$\left. + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i} z_m\| \left. + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|T^{n_i+n_j} z_m - T^{n_j} z_m\| \right) \right]$$

$$\leq \frac{1}{N} \cdot \limsup_{i \to \infty} \left[ \limsup_{j \to \infty} \left( \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_j} z_m\| \right. \right.$$  

$$\left. + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i} z_m\| \left. + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \sum_{l=0}^{n_i-1} \|T^{n_j+l+1} z_m - T^{n_j+l} z_m\| \right) \right]$$

which implies

$$r_m \leq \frac{1}{N} \cdot (\alpha + \beta) \cdot D_m, \quad m = 0, 1, 2, \cdots , \quad (5)$$
where $N$ is the normal structure coefficient of $E$. For each $m \geq 1$ and all $n_i$, $n_j$, we have from (2) and (4)

$$
\| \lambda z_{m+1} + (1 - \lambda)T^{m_j}z_{m+1} - T^{m_i}z_m \| + c_p \cdot W_p(\lambda) \cdot \| z_{m+1} - T^{n_j}z_{m+1} \|_p \\
\leq \lambda \cdot \| z_{m+1} - T^{m_i}z_m \| + (1 - \lambda) \cdot \left[ \frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \| z_{m+1} - T^{n_i}z_m \| \\
+ \frac{b_{n_j}}{1 - c_{n_j}} \cdot \| z_{m+1} - T^{m_j}z_{m+1} \| + \frac{1 + b_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \| T^{n_i+n_j}z_m - T^{n_i}z_m \| \right]_p \\
\leq \lambda \cdot \| z_{m+1} - T^{n_j}z_m \| + (1 - \lambda) \cdot \left[ \frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \| z_{m+1} - T^{n_i}z_m \| \\
+ \frac{b_{n_j}}{1 - c_{n_j}} \cdot \| z_{m+1} - T^{m_j}z_{m+1} \| \\
+ \frac{1 + b_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \sum_{i=0}^{n_j-1} \| T^{n_i+n_j}z_m - T^{n_i}z_m \| \right]_p .
$$

Taking the limit superior as $i \to \infty$ on each side, by definition of $z_m$ and the asymptotic regularity of $T$, we get

$$
r_m^p + c_p \cdot W_p(\lambda) \cdot \| z_{m+1} - T^{n_j}z_{m+1} \|_p \\
\leq \lambda r_m^p + (1 - \lambda) \left[ \frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot r_m + \frac{b_{n_j}}{1 - c_{n_j}} \cdot \| z_{m+1} - T^{m_j}z_{m+1} \| \right]_p \\
\leq \lambda r_m^p + (1 - \lambda) \left[ 2^{p-1} \left\{ \left( \frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \right)^p \cdot r_m^p \\
+ \left( \frac{b_{n_j}}{1 - c_{n_j}} \right)^p \cdot \| z_{m+1} - T^{n_j}z_{m+1} \|^p \right\} \right] .
$$

It then follows from (5) that

$$
r_m^p + c_p \cdot W_p(\lambda) \cdot D_{m+1}^p \leq \lambda r_m^p + (1 - \lambda) \left[ 2^{p-1} \left\{ \alpha p r_m^p + \beta p \cdot D_{m+1}^p \right\} \right] \\
or
$$

$$
D_{m+1}^p \leq \frac{(1 - \lambda) \cdot (2^{p-1} \cdot \alpha^p - 1)}{c_p \cdot W_p(\lambda) - (1 - \lambda) \cdot 2^{p-1} \cdot \beta p} \cdot r_m^p \\
\leq \frac{(1 - \lambda) \cdot (2^{p-1} \cdot \alpha^p - 1)}{\left\{ c_p \cdot W_p(\lambda) - (1 - \lambda) \cdot 2^{p-1} \cdot \beta p \right\} \cdot (\alpha + \beta)^p} \cdot D_m^p .
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Letting $\lambda \uparrow 1$, we conclude that

$$D_{m+1} \leq \left[\frac{(\alpha + \beta)^p 2^{p-1} \cdot \alpha^p - 1}{(c_p - 2^{p-1} \cdot \beta^p) \cdot N_p^p}\right]^\frac{1}{p} \cdot D_m$$

$$= A \cdot D_m, \quad m = 1, 2, \ldots,$$

where

$$A = \left[\frac{(\alpha + \beta)^p 2^{p-1} \cdot \alpha^p - 1}{(c_p - 2^{p-1} \cdot \beta^p) \cdot N_p^p}\right]^\frac{1}{p} < 1$$

by the assumption of the theorem. Since

$$\|z_{m+1} - z_m\| \leq r_m + D_m \leq 2D_m \leq \cdots \leq 2 \cdot A^m D_1 \to 0$$

as $m \to \infty$,

it follows that $\{z_m\}$ is a Cauchy sequence. Let $z = \lim_{m \to \infty} z_m$. Then we have

$$\|z - T^{n_i}z\|$$

$$\leq \|z - z_m\| + \|z_m - T^{n_i}z_m\| + \|T^{n_i}z_m - T^{n_i}z\|$$

$$\leq \|z - z_m\| + \|z_m - T^{n_i}z_m\| + a_{n_i} \cdot \|z_m - z\|$$

$$+ b_{n_i} (\|z_m - T^{n_i}z_m\| + \|z - T^{n_i}z\|) + c_{n_i} (\|z_m - T^{n_i}z_m\| + \|z - T^{n_i}z_m\|)$$

and so

$$\|z - T^{n_i}z\| \leq \frac{1 + a_{n_i} + 2c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot \|z - z_m\| + \frac{1 + b_{n_i} + c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot \|z_m - T^{n_i}z_m\|.$$

Taking the limit superior as $i \to \infty$ on each side, we get

$$\limsup_{i \to \infty} \|z - T^{n_i}z\| \leq \limsup_{i \to \infty} \frac{1 + a_{n_i} + 2c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot \|z - z_m\|$$

$$+ \limsup_{i \to \infty} \frac{1 + b_{n_i} + c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot D_m \to 0$$

as $m \to \infty$.

Therefore we obtain $Tz = z$ by Lemma 2. This completes the proof. □

Górnicki [11] proved the following theorem:
THEOREM ([Górnicki]). Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty bounded closed convex subset of \( E \), and \( T : K \to K \) an asymptotically regular mapping. If

\[
\liminf_{n \to \infty} |||T^n||| = k < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot c_p \cdot N_p} \right) \right]^\frac{1}{p},
\]

(where \( |||T^n||| \) is the Lipschitz constant of \( T^n \), i.e.,

\[
|||T^n||| = \sup \left\{ \frac{||T^n x - T^n y||}{\|x - y\|} : x \neq y, \ x, \ y \in K \right\},
\]

\( N \) is the normal structure coefficient of \( E \), and \( c_p \) is the constant in (2)), then \( T \) has a fixed point in \( K \).

If we put \( b_n = c_n = 0 \) in (1), then \( a_n \) is equal to \( |||T^n||| \) and the condition of Górnicki [11] that

\[
k < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot c_p \cdot N_p} \right) \right]^\frac{1}{p} \quad \text{or equivalently} \quad \left[ \frac{k_p (k_p - 1)}{c_p \cdot N_p} \right]^\frac{1}{p} < 1
\]

follows from condition (C) of Theorem 1, and hence the result of Górnicki [11] follows as special case of Theorem 1.

REMARK 1. In place of bounded subset \( K \) of [11], we have taken weaker assumption that there is an \( z_0 \) in \( K \) for which \( \{T^n z_0\} \) is bounded.

3. Some Applications

In a Hilbert space \( H \), the following equality holds:

\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2
\]

for all \( x, \ y \) in \( H \) and \( \lambda \in [0, 1] \).

By Theorem 1 and (6), we immediately obtain the following.
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Theorem 2. Let $K$ be a nonempty closed convex subset of a Hilbert space $H$ and $T : K \to K$ an asymptotically regular mapping which holds the inequality (1) such that

$$\left[ \frac{(\alpha + \beta)^2(2\alpha^2 - 1)} {2(1 - 2\beta^2)} \right]^\frac{1}{2} < 1,$$

where $\alpha$, $\beta$ as in Theorem 1. Suppose that there is a $z_0$ in $K$ for which \{${T^n z_0}$\} is bounded. Then $T$ has a fixed point in $K$.

If we put $b_n = c_n = 0$ in (1), then from Theorem 2, we have the following result.

Corollary 1 ([11, Corollary 2]). Let $K$ be a nonempty bounded closed convex subset of a Hilbert space $H$. If $T : K \to K$ is an asymptotically regular mapping such that

$$\lim_{n \to \infty} |||T^n||| < \sqrt{2},$$

then $T$ has a fixed point in $K$.

If $1 < p \leq 2$, then we have for all $x$, $y$ in $L^p$ and $\lambda \in [0, 1]$,

(7) $\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)(p - 1)\|x - y\|^2.$

(The inequality (7) is contained in Lim, Xu and Xu [13] and Smarzewski [21].)

Assume that $2 < p < \infty$ and $t_p$ is the unique zero of the function $g(x) = -x^{p-1} + (p - 1)x + p - 2$ in the interval $(1, \infty)$. Let

$$c_p = (p - 1)(1 + t_p)^{2-p} = \frac{1 + t_p^{p-1}} {(1 + t_p)^{p-1}}.$$

Then we have the following inequality

(8) $\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda) \cdot c_p \cdot \|x - y\|^p$

for all $x$, $y$ in $L^p$ and $\lambda \in [0, 1]$. (The inequality (8) is essentially due to Lim [12].)
THEOREM 3. Let $K$ be a nonempty closed convex subset of $L^p$, $1 < p < \infty$, and $T : K \to K$ an asymptotically regular mapping which holds (1) such that

$$\left[ \frac{(\alpha + \beta)^2 \cdot (2\alpha^2 - 1)}{((p-1) - 2\beta^2) \cdot 2^{\frac{p-1}{p}}} \right]^{\frac{1}{2}} < 1 \quad \text{for } 1 < p \leq 2$$

and

$$\left[ \frac{(\alpha + \beta)^p \cdot (2^{p-1}\alpha^p - 1)}{(c_p - 2^{p-1}\beta^p) \cdot 2} \right]^{\frac{1}{p}} < 1 \quad \text{for } 2 < p < \infty,$$

where $\alpha, \beta$ as in Theorem 1. Suppose that there is a $z_0$ in $K$ for which $\{T^n z_0\}$ is bounded. Then $T$ has a fixed point in $K$.

If we put $b_n = c_n = 0$ in (1), then from Theorem 3, we have the following result.

COROLLARY 2 ([11, Corollary 3, 4]). Let $K$ be a nonempty bounded closed convex subset of $L^p$ ($1 < p < \infty$). If $T : K \to K$ is an asymptotically regular mapping such that

$$\liminf_{n \to \infty} \|T^n\| = k < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot (p - 1) \cdot 2^{\frac{p-1}{p}}} \right) \right]^{\frac{1}{2}} \quad \text{for } 1 < p \leq 2$$

and

$$\liminf_{n \to \infty} \|T^n\| = k < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 8 \cdot c_p} \right) \right]^{\frac{1}{p}} \quad \text{for } 2 < p < \infty,$$

then $T$ has a fixed point in $K$.

Let $H^p$, $1 < p < \infty$, denote the Hardy space [7] of all functions $x$ analytic in unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty.$$
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Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega)$, $k \geq 0$, $1 < p < \infty$, the Sobolev space [1, p.149] of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm

$$
\| x \| = \left( \sum_{|\alpha| \leq k} \int\int_{\Omega} |D^\alpha x(\omega)|^p d\omega \right)^{\frac{1}{p}}.
$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where index set $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$ [15], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$
\| x \| = \left( \sum_{\alpha \in \Lambda} \left( \| x_\alpha \|_{p,\alpha} \right)^q \right)^{\frac{1}{q}},
$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$, where $1 < p < \infty$, $q = \max\{2, p\}$, and $(S_\iota, \sum_\iota, \mu_\iota)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [6, III. 2.10] of all measurable $L_p$-value function $x$ on $S_2$ such that

$$
\| x \| = \left( \int_{S_2} \left( \| x(s) \|_p \right)^q \mu_2(ds) \right)^{\frac{1}{q}}.
$$

These spaces are $q$-uniformly convex with $q = \max\{2, p\}$ [19, 20] and the norm in these spaces satisfies

$$
\| \lambda x + (1 - \lambda)y \| \leq \lambda \| x \| + (1 - \lambda)\| y \| - d \cdot W_q(\lambda) \cdot \| x - y \|^{q}
$$

with a constant

$$
d = \begin{cases} 
\frac{p - 1}{8} & \text{for } 1 < p \leq 2 \\
\frac{1}{p \cdot 2^p} & \text{for } 2 < p < \infty.
\end{cases}
$$

Now, from Theorem 1, we have the following result.
Theorem 4. Let $K$ be a nonempty bounded closed convex subset of the space $E$, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2, p\}$, $k \geq 0$. Let $T : K \to K$ be an asymptotically regular mapping which holds the inequality (1) such that

$$
\left[ \frac{(\alpha + \beta)^q \cdot (2^{q-1} \cdot \alpha^q - 1)}{(d - 2^{q-1} \cdot \beta^q) \cdot N^q} \right]^\frac{1}{q} < 1,
$$

where $\alpha$, $\beta$ as in Theorem 1. Suppose that there is a $z_0$ in $K$ for which $\{T^n z_0\}$ is bounded. Then $T$ has a fixed point in $K$.

If we put $b_n = c_n = 0$ in (1), then from Theorem 4, we have the following result:

Corollary 3 ([11, Corollary 5]). Let $K$ be a nonempty bounded closed convex subset of the space $E$, where $E$ is as in Theorem 4. If $T : K \to K$ is an asymptotically regular mapping such that

$$
\liminf_{n \to \infty} ||T^n|| = k < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot d \cdot N^q} \right) \right]^\frac{1}{q},
$$

then $T$ has a fixed point in $K$.

References


Asymptotically regular mappings


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