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THE INDEX FOR A TOPOLOGICAL DEGREE THEORY FOR DENSELY DEFINED OPERATORS OF TYPE $(S_+)_{0,L}$ IN BANACH SPACES

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Abstract. This is a summary of results involving the development of a theory of an index of an isolated critical point for densely defined nonlinear operators of type $(S_+)_{0,L}$. This index theory is associated with a degree theory, for such operators, which has been recently developed by the authors.

1. Introduction and preliminaries

Let X be a real separable reflexive Banach space with dual space X^*. The norm of the space X (X^*) will be denoted by $\|\cdot\|$ $(\|\cdot\|_*)$. We let $\mathcal{R}^n$ denote the Euclidean space of dimension n and set $\mathcal{R} = \mathcal{R}^1$. For $x_0 \in X$ and $r > 0$, we let $B_r(x_0)$ denote the open ball $\{x \in X \ : \ \|x - x_0\| < r\}$. Unless otherwise stated, N is the set of natural numbers. An operator $A : X \supset D(A) \to X^*$ is "bounded" if it maps bounded subsets of its domain onto bounded sets in X^*. It is "compact" if it is strongly continuous and maps bounded subsets of $D(A)$ onto relatively compact sets in X^*. In what follows, the single term "continuous" means "strongly continuous". We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. We consider an operator $A : X \supset D(A) \to X^*$ with domain $D(A)$ dense in some open set $D_0 \subset X$. We assume that there exists a subspace L of the space X such that

$$D_0 \cap L \subset D(A), \quad \overline{L} = X. \tag{1.1}$$

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Let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of L.

DEFINITION 1.1. We say that the operator A satisfies Condition $(S_+)_{0,L}$ if for every sequence $\{u_j\} \subset D(A)$ with

$$(1.2) \qquad u_j \rightharpoonup u_0, \quad \limsup_{j\to\infty} \langle Au_j, u_j\rangle \le 0, \quad \lim_{j\to\infty} \langle Au_j, v\rangle = 0,$$

for some $u_0 \in X$ and any $v \in L$, we have

$$(1.3) \qquad u_j \to u_0, \quad u_0 \in D(A), \quad Au_0 = 0.$$

In (1.2), and the sequel, $\langle h, u\rangle$ denotes the value of the functional $h \in X^*$ at the element $u \in X$.

DEFINITION 1.2. We say that the operator A satisfies Condition $(S_+)_L$ if the operator $A_h : D(A) \to X^*$, defined by $A_h u = Au - h$ satisfies Condition $(S_+)_{0,L}$ for any $h \in X^*$.

A degree theory was developed in [3] for an operator $A : D(A) \subset X \to X^*$, with respect to a bounded open set $D \subset X$, under the assumption that

$$(1.4) \qquad Au \neq 0, \quad \text{for } u \in D(A) \cap \partial D, \ \overline{D} \subset D_0,$$

while A satisfies the following additional conditions:

A_1) there exists a subspace L of X satisfying (1.1) and such that the operator A satisfies Condition $(S_+)_{0,L}$;

A_2) for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $a(F, v) : F \to \mathcal{R}$, defined by $(a(F, v))(u) = \langle Au, v\rangle$, is continuous.

We now define the index of a critical point for an operator A satisfying A_1), A_2).

Definition 1.3. A point $u_0 \in D(A) \cap D_0$ is called a "critical point" of the operator A if $Au_0 = 0$. A critical point $u_0 \in D(A) \cap D_0$ is an "isolated critical point" of the operator A if there exists a ball $B_r(u_0) \subset D_0$ which contains no other critical point of the operator A. From the proof of Theorem 2.1 in [3], we can show that

$$\mathrm{Deg}(A, B_{r'}(u_0), 0) = \mathrm{Deg}(A, B_r(u_0), 0),$$

for every $r' \in (0, r]$.

From the above definition we have

Definition 1.4. The number

$$\lim_{\rho \to 0} \mathrm{Deg}(A, B_\rho(u_0), 0) \tag{1.5}$$

is called the "index" of the isolated critical point u_0 of the operator A and is denoted by $\mathrm{Ind}(A, u_0)$.

Our purpose here is to calculate the index $\mathrm{Ind}(A, u_0)$ by using a certain linearization of the nonlinear operator A at the critical point. In the known results, for Leray-Schauder operators [5, Theorem 4.7] and bounded demicontinuous operators of type (S_+) [8, Theorem 4.2], this linearization is given by means of Fréchet or Gateaux derivatives at the critical points of the nonlinear operators under consideration. We may further assume that $u_0 = 0$.

We now recall the assumptions for the calculation of the index in [9]. These assumptions are given in a form that can be used later for the relevant unbounded linear operators.

Let $A : X \supset B_r(0) \to X^*$ be a nonlinear operator which satisfies Condition (S_+) and $A(0) = 0$. Assume that A has the Fréchet derivative $A' : X \to X^*$ at zero. Let

$$Z_\varepsilon = \cup_{t \in [0,1]} \{u \ : \ tAu + (1-t)A'u = 0, \ 0 < \|u\| \le \varepsilon\}. \tag{1.6}$$

A') The equation $A'u = 0$ has only the zero solution. There exists a compact linear operator $\Gamma : X \to X^*$ such that

$$\begin{aligned} &\langle (A' + \Gamma)u, u \rangle > 0, \quad \text{for } u \in D(A'), \ u \ne 0, \\ &\langle (A' + \Gamma)^* v, v \rangle > 0, \quad \text{for } v \in D((A')^*), \ v \ne 0, \end{aligned} \tag{1.7}$$

and the operator $T = (A' + \Gamma)^{-1}\Gamma : X \to X$ is well defined and compact;

C) the weak closure of the set

$$\sigma_\varepsilon = \left\{ v = \frac{u}{\|u\|} \ : \ u \in Z_\varepsilon \right\} \tag{1.8}$$

does not contain zero for some sufficiently small $\varepsilon > 0$.

In (1.7) $(A')^*$ is the adjoint of the operator A'. We note that in [9] $D(A') = X$ and the second inequality in (1.7) follows from the first. By the assumptions A'), C) in [9], the value of $\mathrm{Ind}(A, 0)$ is calculated in terms of the multiplicities of the characteristic values of the operator T. We also note that in [8] there is an example demonstrating the fact that it is generally impossible to calculate $\mathrm{Ind}(A, 0)$ without Condition C).

A natural question arises now: how do we introduce a workable concept of linearization for a densely defined operator? Before we formulate our new linearization concept, we introduce the auxiliary operator $A_0 : X \supset D(A_0) \to X^*$ which satisfies the following condition:

A_0) A_0 is a bounded nonlinear operator which satisfies Conditions $(S_+)_L$, A_2) and is such that $D_0 \cap L \subset D(A_0)$, $A_0(0) = 0$ and

$$\lim_{\substack{u \to 0 \\ u \in D(A_0)}} \frac{\|A_0 u\|_*}{\|u\|} = 0.$$

We solve the problem of linearizing for the nonlinear operator A, satisfying Conditions A_1), A_2), in the following way. We assume that there exist a nonlinear operator A_0 satisfying A_0) and a linear operator $A' : X \supset D(A') \to X^*$ such that $D(A) \subset D(A')$ and the next condition holds:

ω) for the operator $\omega : D(A) \to X^*$, defined by $\omega(u) = Au - A'u$, we have

$$\frac{\omega(u)}{\|u\|} \to 0 \quad \text{as } u \to 0, \ u \in Z'_\varepsilon,$$

for some $\varepsilon > 0$, where

$$\begin{aligned} Z'_\varepsilon &= \cup_{t \in [0,1]} \left\{ u \in D(A_t^{(1)}) \ : \ A_t^{(1)} u = 0, \ 0 < \|u\| \le \varepsilon \right\}, \\ A_t^{(1)} u &= tAu + (1-t)\left[A_0 u + A' u\right]. \end{aligned} \tag{1.9}$$

We remark that Condition ω) is weaker that the conditions in terms of derivatives in [9]. Using Condition ω), it is possible to evaluate the indices of the critical points even for operators which are defined everywhere, but not differentiable in the usual sense. We shall formulate the relevant assertions in Section 2.

We shall assume that the operator A' satisfies Condition $(S')_L$ which is given in the following definition.

DEFINITION 1.5. We say that the operator A' satisfies Condition $(S')_L$ if for every sequence $\{u_j\} \subset D(A')$ such that

$$(1.10)\quad u_j \rightharpoonup u_0, \quad \limsup_{j\to\infty} \langle A'u_j - h, u_j\rangle \le 0, \quad \lim_{j\to\infty} \langle A'u_j - h, v\rangle = 0,$$

for some $u_0 \in X$, $h \in X^*$ and any $v \in L$, it follows that

$$(1.11)\qquad u_0 \in D(A'), \quad A'u_0 = h, \quad \lim_{j\to\infty} \langle A'u_j, u_0\rangle = \langle h, u_0\rangle.$$

The main result of this paper is the evaluation of the index $\mathrm{Ind}(A,0)$ under the conditions A'), $(S')_L$, ω) and C) (the last condition is satisfied with a special choice of the set Z_ε). We are going to show, under some additional conditions, that zero is an isolated critical point of the operator A and

$$(1.12)\qquad \mathrm{Ind}(A,0) = (-1)^\nu,$$

where ν is the sum of the multiplicities of the characteristic values of the operator T lying in the interval $(0,1)$.

The exact formulation of the results concerning the value of the index of the critical point is given in Section 3. In Theorem 2.1 we give a result of the general situation of an unbounded operator A and an unbounded linearization operator A'. More specific cases are given in the subsequent theorems of Section 2. In Theorem 2.2 we consider the case of a bounded operator A of type $(S_+)_{0,L}$ with bounded linearization operator. The evaluation of the index for a bounded operator A

satisfying Condition $(S_+)_L$ is given in Theorem 2.3. Finally, the case of operators in Hilbert spaces is considered in Theorem 2.4.

In Section 3 we discuss a problem of bifurcation points for densely defined operators. We consider only the case of unbounded operators A, A'.

The results of this work open the possibility of studying problems of branching of solutions and the evaluation of the number of solutions for nonlinear elliptic problems in Sobolev spaces with strong coefficient growth. These problems can be reduced to operator equations with unbounded densely defined operators, and cannot be studied by the methods contained in the monograph [9]. Such a problem has been studied by the authors in [4]. Namely, we consider there the Dirichlet problem

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ [e^u + a(x)] \frac{\partial u}{\partial x_i} \right\} - \lambda u = 0, \quad x \in \Omega, \tag{1.13}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.14}$$

where $a(x)$ is a positive, bounded and measurable function, and Ω is a bounded open set in $\mathcal{R}^n$ with boundary $\partial\Omega \in C^2$. We showed in [4] that every eigenvalue of odd multiplicity of the linear equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ [1 + a(x)] \frac{\partial u}{\partial x_i} \right\} - \lambda u = 0, \quad x \in \Omega,$$

with the boundary condition (1.14) is a point of bifurcation for the problem ((1.13),(1.14)).

2. Formulation of the main results

Let X be a real separable reflexive Banach space satisfying the following conditions:

X_1) there exists a bounded demicontinuous operator $J : \overline{B_r(0)} \to X^*$, with $J(0) = 0$, satisfying Condition (S_+) for some $r > 0$;

X_2) there exists a bounded linear operator $K : X \to X^*$ such that $\langle Kx, x \rangle > 0$, for $x \neq 0$.

Condition (S_+) in $X_1)$ coincides with Condition $(S_+)_L$ when $L = X$. We also note that the condition $X_1)$ is satisfied, e.g., if $X,\ X^*$ are uniformly convex. In this case we can choose the operator J as in [9]. Condition $X_2)$ is satisfied if the space X is included in some real Hilbert space H and the embedding operator $X \to H$ is continuous.

Let $A : X \supset D(A) \to X^*$ be an operator which satisfies Conditions $A_1),\ A_2)$ and is such that

$$\langle Au, u - v\rangle \geq -C(v) \tag{2.1}$$

holds for $u,\ v \in L,\ \|u\| \leq r_0$, where $r_0 > 0$ is a constant and $C(v)$ depends only on v.

In order to formulate the main results of the paper we introduce certain subspaces of the spaces $X,\ X^*$ connected with the operators $A' + \Gamma,\ T$ which are defined in Condition $A_1)$. We first define two invariant subspaces of the compact operator $T : X \to X$. Denote by F the direct sum of all invariant subspaces of the operator T corresponding to the characteristic values of this operator lying in the interval $(0, 1)$. Let R be the closure of the direct sum of all those invariant subspaces of the operator T not included in F. Then F and R are invariant subspaces of the operator T and the splitting

$$X = F + R \tag{2.2}$$

holds in the sense of a direct sum. F is a finite-dimensional subspace of X and

$$\dim F = \nu, \tag{2.3}$$

where ν is the same number as in (1.12).

We introduce a projection $\Pi : X \to F$ corresponding to the splitting (2.2):

$$\Pi(f + r) = f, \quad \text{for } f \in F,\ r \in R. \tag{2.4}$$

We define, for small enough $\varepsilon > 0$, the sets

$$\begin{aligned} &Z_\varepsilon = Z'_\varepsilon \cup Z''_\varepsilon, \\ &Z''_\varepsilon = \cup_{t\in[0,1]} \left\{u \in D(\overline{A}_t)\ :\ \overline{A}_t u = 0,\ 0 < \|u\| \leq \varepsilon\right\}, \end{aligned} \tag{2.5}$$

where $\overline{A}_t u = tA_0 u + A'u$, the operators $A_0,\ A'$ are defined according to the condition $\omega)$ and the set Z'_ε is introduced in (1.6).

THEOREM 2.1. *Let* $A : X \supset D(A) \to X^*$ *satisfy* Conditions $A_1)$, $A_2)$, *the inequality* (2.1) *and be such that* $0 \in D(A) \cap D_0$ *and* $A(0) = 0$. *Assume that there exist operators* $A_0 : X \supset D(A_0) \to X^*$, $A' : X \supset D(A') \to X^*$ *satisfying* Conditions $A_0)$, $A')$, $(S')_L$ *and* $\omega)$, *and such that the operator* $A + qA' : X \supset D(A) \to X^*$ *satisfies* $(S_+)_L$, *for any number* $q > 0$. *Suppose that the following conditions are satisfied:*

1) *the operator* $\Pi(A'+\Gamma)^{-1} : X^* \supset (A'+\Gamma)D(A') \to X$ *is bounded, where the operators* Π, Γ *are defined by* (2.4) *and* $A')$, *respectively;*

2) Condition $C)$ *is satisfied with the set* Z_ε *defined by* (2.5).

Then zero is an isolated critical point of the operator A *and its index is equal to* $(-1)^\nu$, *where* ν *is the sum of the multiplicities of the characteristic values of the operator* T *lying in the interval* $(0, 1)$.

We formulate below some particular cases of Theorem 2.1. In Theorem 2.2 we assume that the operators A, A' are bounded. Thus, Conditions $(S')_L$, (2.1) are automatically satisfied. Furthermore, by changing some arguments in the proof of Theorem 2.1 we can establish an analogous result without assuming Condition $(S_+)_L$ for the operator $A + qA'$. We also note that in this case it suffices to assume only the first of (1.7) in Condition $A')$.

THEOREM 2.2. *Let* $A : X \supset D(A) \to X^*$ *be bounded and satisfy* Conditions $A_1)$, $A_2)$. *Assume that* $0 \in D(A) \cap D_0$ *and* $A(0) = 0$. *Let there exist bounded operators* $A_0 : X \supset D(A_0) \to X^*$, $A' : X \supset D(A') \to X^*$ *satisfying* Conditions $A_0)$, $A')$ *and* $\omega)$. *Suppose that* Conditions 1) *and* 2) *of* Theorem 2.1 *are satisfied. Then zero is an isolated critical point of the operator* A *and its index equals* $(-1)^\nu$, *with the same number* ν *as in* Theorem 2.1.

In the next theorem we assume that the operator A satisfies Condition $(S_+)_L$ and both operators A, A' are bounded. Then we can pick the operator A_0 as the operator $\|u\|^2 Au$. In this case we can assume Condition $\omega)$ with the set

$$\widetilde{Z}_\varepsilon = \cup_{t\in[0,1]} \left\{ u \in D(\widetilde{A}_t) \ : \ \widetilde{A}_t u = 0, \ 0 < \|u\| \le \varepsilon \right\}, \tag{2.6}$$

where $\widetilde{A}_t u = tAu + (1-t)A'u$ and in Condition $C)$ we can take $Z_\varepsilon = \widetilde{Z}_\varepsilon$.

THEOREM 2.3. *Let L be a subspace of the space X satisfying* (1.1), *and let $A : X \supset D(A) \to X^*$ be a bounded operator satisfying* Conditions $(S_+)_L$, A_2) *and such that* $0 \in D(A) \cap A_0$, $A(0) = 0$. *Assume that there exists a bounded operator $A' : X \supset D(A') \to X^*$ satisfying* Condition A') *and such that* Conditions ω) *and* C) *are also satisfied with the set $\widetilde{Z}_\varepsilon$ defined by* (2.6). *Suppose that* Condition 1) *of* Theorem 2.1 *is satisfied. Then zero is an isolated critical point of the operator A and its index equals $(-1)^\nu$, where ν is as in* Theorem 2.1.

Finally, we state one result for the special case of a Hilbert space H in place of X. For simplicity, we consider only bounded operators A, A'. We use the following assumption: there exists a positive constant c such that

$$\langle (A' + \Gamma)u, u\rangle \geq c\|u\|^2 \tag{2.7}$$

holds for all $u \in H$, where the brackets denote now the scalar product of the space H.

THEOREM 2.4. *Let H be a real separable* Hilbert *space and $A : H \supset D(A) \to H$ a bounded operator satisfying* Conditions $(S_+)_L$, A_2), $0 \in D(A) \cap D_0$ *and* $A(0) = 0$. *Assume that there exist a bounded linear operator $A' : H \to H$ and a compact linear operator $\Gamma : H \to H$ such that the inequality* (2.7) *holds. Assume, further, that* Condition ω) *is satisfied with $Z'_\varepsilon = \widetilde{Z}_\varepsilon$, where $\widetilde{Z}_\varepsilon$ is defined by* (2.6). *Suppose that the equation $A'u = 0$ has only the zero solution. Then zero is an isolated critical point of the operator A and its index equals $(-1)^\nu$, where ν is as in* Theorem 2.1.

REMARK 2.1. It is easy to verify that in the case of a bounded operator A', as in Theorems 2.2-2.4, we can assume instead on Condition ω) a weaker condition: in Condition ω) we replace $\|u\|^{-1}\omega(u) \to 0$ by $\|u\|^{-1}\omega(u) \rightharpoonup 0$.

REMARK 2.2. Theorems 2.3, 2.4 are also new even for operators which satisfy Condition (S_+), are defined everywhere in a neighborhood of the critical point and have no derivatives in the usual sense.

3. Branching of solutions

In this section we present an application of the previous results to the bifurcation problem. In what follows, D_0 is an open set containing the origin in a separable reflexive Banach space X. We consider a nonlinear operator $A : X \supset D(A) \to X^*$ satisfying Conditions $(S_+)_L$, $A_2)$, for some subspace L of X such that $D_0 \cap L \subset D(A)$, $\overline{L} = X$. Let $C : D_0 \to X^*$ be a nonlinear compact operator. Assume further that $A(0) = C(0) = 0$. We can easily verify that the operator $A + \lambda C$ satisfies Condition $(S_+)_L$ for any real $\lambda > 0$.

We consider the bifurcation problem for the pair of operators A, C.

Definition 3.1. A real number λ_0 is called a "bifurcation point" of the operators A, C if for every $\varepsilon > 0$ there exist $u_\varepsilon \in D(A)$ and $\lambda_\varepsilon \in \mathcal{R}$ such that

$$(3.1) \qquad Au_\varepsilon + \lambda_\varepsilon C u_\varepsilon = 0, \quad |\lambda_\varepsilon - \lambda_0| < \varepsilon, \quad 0 < \|u_\varepsilon\| < \varepsilon.$$

We study necessary and sufficient conditions that λ_0 be a bifurcation point. For this, we may assume that there is some $\delta > 0$ such that zero is an isolated critical point of the operator $A + \lambda C$, for each λ from the interval $|\lambda - \lambda_0| < \delta$, since otherwise λ_0 itself would be a bifurcation point. Thus, the index $\operatorname{Ind}(A + \lambda C, 0)$ of the operator $A + \lambda C$ at 0 is defined for $|\lambda - \lambda_0| < \delta$ according to Definition 1.4.

Let

$$(3.2) \quad \bar{i}_\pm(\lambda_0) = \limsup_{\lambda \to \lambda_0 \pm} \operatorname{Ind}(A + \lambda C, 0), \quad \underline{i}^\pm(\lambda_0) = \liminf_{\lambda \to \lambda_0 \pm} \operatorname{Ind}(A + \lambda C, 0).$$

Theorem 3.1. *Let $A : X \supset D(A) \to X^*$ be a nonlinear operator satisfying* Conditions (S_+), $A_2)$ *and let $C : D_0 \to X^*$ be a nonlinear compact operator. Assume that $A(0) = C(0) = 0$ and that at least two of the numbers*

$$(3.3) \qquad \underline{i}^-(\lambda_0),\ \underline{i}^+(\lambda_0), \bar{i}_-(\lambda_0),\ \bar{i}_+(\lambda_0),\ \operatorname{Ind}(A + \lambda_0 C, 0)$$

are distinct. Then λ_0 is a bifurcation point of the pair A, C.

Now, we are going to establish necessary conditions for a number λ_0 to be a bifurcation point. We need to state new forms of Conditions ω), C) from section 1 so that they can be used in both necessary and sufficient conditions.

Assume that the operator C has Fréchet derivative at zero denoted by C'. The operator C' is compact [5]. We assume that there exist a nonlinear operator A_0 satisfying Condition A_0) and a linear operator $A' : X \supset D(A') \to X^*$ such that $D(A) \subset D(A')$ and the condition

$\overline{\omega}$) for the operator $\omega : D(A) \to X^*$, defined by $\omega(u) = Au - A'u$, we have

$$\frac{\omega(u)}{\|u\|} \to 0, \quad \text{as } u \to 0, \; u \in Z'_{\varepsilon,\Lambda},$$

holds for every $\Lambda > 0$ and some $\varepsilon > 0$ depending on Λ, where

$$Z'_{\varepsilon,\Lambda} = \cup_{\substack{t\in[0,1] \\ |\lambda|\leq\Lambda}} \left\{ u \in D(A^{(1)}_{t,\lambda}) \; : \; A^{(1)}_{t,\lambda}(u) = 0, \; 0 < \|u\| \leq \varepsilon \right\}. \tag{3.4}$$

Here,

$$A^{(1)}_{t,\lambda}(u) = t(Au + \lambda Cu) + (1-t)(A_0 u + A'u + \lambda C'u).$$

Define the sets $Z_{\varepsilon,\Lambda} = Z'_{\varepsilon,\Lambda} \cup Z''_{\varepsilon,\Lambda}$ with

$$Z''_{\varepsilon,\Lambda} = \cup_{\substack{t\in[0,1] \\ |\lambda|\leq\Lambda}} \left\{ u \in D(A^{(2)}_{t,\lambda}) \; : \; A^{(2)}_{t,\lambda}(u) = 0, \; 0 < \|u\| \leq \varepsilon \right\}, \tag{3.5}$$

where

$$A^{(2)}_{t,\lambda}(u) = tA_0 u + A'u + \lambda C'u.$$

We also introduce the condition

$\overline{C}$) the weak closure of the set

$$\sigma_{\varepsilon,\Lambda} = \left\{ v = \frac{u}{\|u\|} \; : \; u \in Z_{\varepsilon,\lambda} \right\} \tag{3.6}$$

does not contain zero for any $\Lambda > 0$ and all sufficiently small positive ε depending on Λ.

Theorem 3.2. *Let A, C satisfy the conditions of* Theorem 3.1 *and let C' be the Fréchet derivative of the operator C at zero. Assume that there exist an operator $A_0 : X \supset D(A_0) \to X^*$ and a linear operator $A' : X \supset D(A') \to X^*$ satisfying* Conditions $A_0)$ *and* $(S')_L$, *respectively, as well as conditions $\overline{\omega})$ and $\overline{C})$. Then a necessary condition that λ_0 be a bifurcation point of the pair A, C is that the equation*

$$A'u + \lambda_0 C'u = 0 \tag{3.7}$$

has a nonzero solution.

A sufficient condition that λ_0 be a bifurcation point is given in the following theorem.

Theorem 3.3. *Assume that X is a real reflexive separable Banach space satisfying* Conditions $X_1)$ *and* $X_2)$ *of* Section 2. *Let A, C satisfy all the assumptions of* Theorem 3.2, *respectively, and be such that* (2.1) *is satisfied and $A + qA'$ satisfies* Condition $(S_+)_L$ *for every number $q > 0$. Suppose that $\langle A'u, u\rangle > 0$ for all $u \in D(A')$ with $u \neq 0$ and that the operator $T = -(A')^{-1}C' : X \to X$ is well defined, compact and* Condition 1) *of* Theorem 2.1 *is satisfied with $\Gamma = 0$. Then each characteristic value of odd multiplicity of the operator T is a bifurcation point of the pair A, C.*

Remark 3.1. For a bounded operator A, or in the case of a Hilbert space X, the restrictions on the operator A in Theorems 3.2 and 3.3 can be weakened according to Theorem 2.4.

Remark 3.2. The conditions of Theorem 3.2 guarantee that the set of bifurcation points of the pair A, C is discrete. In general, this set can contain entire intervals of the real line. For an operator A defined everywhere, such an example can be found in [9, p. 63].

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