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# DEGENERATE VOLTERRA EQUATIONS IN BANACH SPACES

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Abstract. This paper is concerned with degenerate Volterra equations $Mu(t) + \int_0^t k(t-s)Lu(s)ds = f(t)$ in Banach spaces both in the hyperbolic case, and the parabolic one. The key assumption is played by the representation of the underlying space $X$ as a direct sum $X = N(T) \oplus \overline{R(T)}$, where $T$ is the bounded linear operator $T = ML^{-1}$. Hyperbolicity means that the part $\tilde{T}$ of $T$ in $\overline{R(T)}$ is an abstract potential operator, i.e., $-\tilde{T}^{-1}$ generates a $C_0$-semigroup, and parabolicity means that $-\tilde{T}^{-1}$ generates an analytic semigroup. A maximal regularity result is obtained for parabolic equations. We will also investigate the cases where the kernel $k(\cdot)$ is degenerate or singular at $t = 0$ using the results of Prüss [8] on analytic resolvents. Finally, we consider the case where $\lambda$ is a pole for $(\lambda L + M)^{-1}$.

This paper is concerned with the unique solvability of the following Volterra integral equations

$$(1) \qquad Mu(t) + \int_0^t k(t-s)Lu(s)ds = f(t), \quad t \in [0, \tau]$$

in a Banach space $X$. Only an outline is presented here, and the details will be published elsewhere.

Let $M$ and $L$ be closed linear operators such that $D(L) \subset D(M)$ and $0 \in \rho(L)$. The kernel $k(\cdot)$ is a numerical function defined in the closed interval $[0, \tau]$. In case $k(t) \equiv 1$ we get by formally differentiating (1)

$$\frac{d}{dt}Mu(t) + Lu(t) = \dot{f}(t), \quad t \in [0, \tau]$$
$$Mu(0) = f(0).$$

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This type of (possibly degenerate) equations is discussed in detail in the book of A. Favini and A. Yagi [3].

We consider the following two cases.

**Hyperbolic case** There exists a constant $K$ such that

$$\|(M(sM+L)^{-1})^k\| \le \frac{K}{s^k} \quad \forall s>0, \quad k=1,2,\dots. \tag{2}$$

**Parabolic case** There exists a constant $K$ such that

$$\|M(\lambda M+L)^{-1}\| \le \frac{K}{1+|\lambda|}, \quad \mathrm{Re}\lambda \ge 0. \tag{3}$$

In view of our assumptions, $T = ML^{-1}$ is a bounded linear operator in $X$. The following facts are established in A. Favini [2] and A. Favini and A. Yagi [3]. Under the assumption (2) or (3), $X$ has a direct decomposition representation $X = N(T) \oplus \overline{R(T)}$. Let $\tilde{T}$ denote the restriction of $T$ to $\overline{R(T)}$. If $X$ is reflexive and (2) is satisfied, then $-\tilde{T}^{-1}$ generates a $C_0$-semigroup. If $X$ is reflexive and (3) is satisfied, then $-\tilde{T}^{-1}$ generates an analytic semigroup.

We make the change of the unknown function $v(t) = Lu(t)$. Then, since $Mu(t) = ML^{-1}v(t) = Tv(t)$, the equation (1) is transformed into

$$Tv(t) + \int_0^t k(t-s)v(s)ds = f(t). \tag{4}$$

Let $P$ be the projection onto $N(T)$ along $\overline{R(T)}$. Then equation (4) splits into the two equations

$$\tilde{T}(1-P)v(t) + \int_0^t k(t-s)(1-P)v(s)ds = (1-P)f(t), \tag{5}$$

$$\int_0^t k(t-s)Pv(s)ds = Pf(t), \tag{6}$$

since $PT = TP = 0$. (5) is a Volterra equation of the second kind, and (6) is a Volterra equation of the first kind.

Another change of the unknown variable $w(t) = \tilde{T}(1-P)v(t)$ transforms the equation (5) into the equation

$$w(t) + \int_0^t k(t-s)\tilde{T}^{-1}w(s)ds = (1-P)f(t). \tag{7}$$

Formally differenting both sides of (7), we arrive at the following initial value problem

$$(8)\quad \dot{w}(t) + k(0)\tilde{T}^{-1}w(t) + \int_0^t \dot{k}(t-s)\tilde{T}^{-1}w(s)ds = (1-P)\dot{f}(t),$$

$$(9)\quad w(0) = (1-P)f(0).$$

Also by formal differentiation, the equation (6) is transformed into

$$(10)\quad k(0)Pv(t) + \int_0^t \dot{k}(t-s)Pv(s)ds = P\dot{f}(t).$$

We try to solve the original equation (1) by solving the problems (8), (9) and (10).

We denote by $AC([0,\tau])$ and $BV([0,\tau])$ the set of absolutely continuous functions in $[0,\tau]$ and the set of functions of bounded variation in $[0,\tau]$, respectively.

## Hyperbolic case

THEOREM 1. *Let $X$ be a reflexive* Banach *space and let* (2) *be satisfied. If $k \in AC([0,\tau]), \dot{k} \in BV([0,\tau]), k(0) > 0, f \in W^{2,1}(0,\tau;X)$ and $f(0) \in R(T)$, then the solution $u(\cdot) \in C([0,\tau];D(L))$ of problem* (1) *exists and is unique.*

*Proof.* The problem (8) and (9) can be solved by applying the following result of E. Sinestrari [9: Theorem 3.1]. □

LEMMA 1. *Let $A$ be a* Hille-Yosida *operator in $X$ and let $B$ be a linear closed operator with $D(A) \subset D(B)$. If $b \in BV([0,\tau])$, then for any $x_0 \in D(A)$ and $f \in W^{1,1}(0,\tau;X)$ such that $Ax_0 + f(0) \in \overline{D(A)}$ the integrodifferential problem*

$$\frac{d}{dt}x(t) = Ax(t) + \int_0^t b(t-s)Bx(s)ds + f(t), \quad 0 \le t \le \tau,$$
$$x(0) = x_0$$

*has a unique solution $x(\cdot) \in C\big([0,\tau];D(A)\big) \cap C^1([0,\tau];X)$.*

By the assumption of the theorem $(1-P)f(0) = f(0) \in R(T) = D(-\tilde{T}^{-1})$ and $D(-\tilde{T}^{-1})$ is dense in $\overline{R(T)}$. Hence the above result of Sinestrari can be applied to the problem (8) and (9), and there exists a unique solution $w$ such that $w \in C^1\left([0,\tau]\,;\overline{R(T)}\right)$ and $\tilde{T}^{-1}\,w \in C\left([0,\tau]\,;\overline{R(T)}\right)$. It is easy to show that the solution of (8) and (9) satisfies (7). The equation (10) can be solved by Neumann series expansion. Integrating (10) and using the hypothesis $Pf(0) = 0$, we can show that the solution of (10) satisfies (6).

EXAMPLE 1. Let $L = -\Delta, D(L) = H^2(\Omega) \cap H_0^1(\Omega), X = L^2(\Omega)$, where $\Omega$ is a bounded domain in $R^n$ with smooth boundary, and $M$ be the multiplication operator by a nonnegative superharmonic function $m \in C^2(\overline{\Omega})$. Then integration by parts yields for $u \in D(L)$

$$-\mathrm{Re}\int_\Omega mu\Delta\bar{u}dx = -\frac{1}{2}\int_\Omega \Delta m\cdot|u|^2dx + \int_\Omega m|\nabla u|^2dx \geq 0,$$

from which it follows that the condition (2) is satisfied with $K=1$.

## Pararbolic case

THEOREM 2. *Let $X$ be a reflexive* Banach *space and let $L, M$ be two closed linear operators in $X$ satisfying* (3). *Let $\tau > 0$ be fixed and assume that $k \in C^{1,\alpha}([0,\tau]), k(0) > 0$ for some $\alpha > 0$. If $f(\cdot) \in C^{1,\theta}([0,\tau];X), 0 < \theta < 1$, and $f(0) \in \overline{R(T)}$, $T = ML^{-1}$, then* (1) *has a unique solution $u(\cdot) \in C((0,\tau];D(L)) \cap C([0,\tau];X)$ which satisfies the equation in the following sense: $\int_\epsilon^t k(t-s)Lu(s)ds$ is uniformly bounded in $0 < \epsilon < t \leq \tau$, the improper integral*

$$\int_{+0}^t k(t-s)Lu(s)ds = \lim_{\epsilon\to 0}\int_\epsilon^t k(t-s)Lu(s)ds \tag{11}$$

*exists and* (1) *holds with the integral in the left hand side understood in the improper sense* (11). *Furthermore $u$ satisfies the additional regularity condition $Mu \in C^1(]0,\tau];X)$. If in addition $f(0) \in R(T)$, $T = ML^{-1}$, then $u$ is a strict solution belonging to $C\big([0,\tau];D(L)\big)$ with the regularity property $Mu \in C^1([0,\tau];X)$.*

*Proof.* The result of J. Prüss [6] is applied to the formally differentiated problem (8) and (9). In the present case, the initial value $f(0)$ is an arbitrary element of $\overline{R(T)}$. Therefore, $\tilde{T}^{-1}w(s)$ is not Bochner integrable near $s = 0$ in general; however, the improper integral

$$\int_{+0}^{t} \dot{k}(t-s)\tilde{T}^{-1}w(s)ds = \lim_{\epsilon\to 0}\int_{\epsilon}^{t} \dot{k}(t-s)\tilde{T}^{-1}w(s)ds \tag{12}$$

exists, and it can be shown that (8) holds with the integral in the left hand side replaced by (12). □

EXAMPLE 2. Let $X = H^{-1}(\Omega)$, $D(L) = H_0^1(\Omega)$, $Lu = -\Delta u$, $M$ is the multiplication operator by a nonnegative function in $C(\bar{\Omega})$, where $\Omega$ is a bounded domain in $R^n$ with smooth boundary. Then it has been proven by Favini and Yagi [3] that assumption (3) holds.

EXAMPLE 3. Let $X = H^{-1}(\Omega)$, $D(M) = H_0^1(\Omega)$, $Mu = -\Delta u$, and $D(L) = H^3(\Omega) \cap H_0^2(\Omega)$, $Lu = \Delta^2 u$, where $\Omega$ is a bounded domain in $R^n$ with smooth boundary. Let

$$(u,v)_1 = \int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_i}}dx, \quad (u,v)_2 = \int_\Omega \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i x_j}\overline{\frac{\partial^2 v}{\partial x_i x_j}}dx$$

be the inner product in $H_0^1(\Omega)$ and $H_0^2(\Omega)$, respectively. Following the idea of W. M. Greenlee[5], we consider the operator $\mathcal{A}$ defined by

$$(u,v)_2 = (\mathcal{A}u, v)_1 \ \text{ for every } \ v \in H_0^2(\Omega).$$

$\mathcal{A}$ is a positive definite selfadjoint operator in $H_0^1(\Omega)$ and $L = M\mathcal{A}$. Therefore we have

$$\|M(\lambda M + L)^{-1}\|_{\mathcal{L}(H^{-1}(\Omega))} = \|M(\lambda + \mathcal{A})^{-1}M^{-1}\|_{\mathcal{L}(H^{-1}(\Omega))} \le C_{\theta_0}(1+|\lambda|)^{-1}$$

for any sector $\{\lambda; |\arg\lambda| \le \theta_0\}$, $\frac{\pi}{2} < \theta_0 < \pi$.

REMARK 1. The operator $\mathcal{A}$ in Example 3 is characterized as

$$D(\mathcal{A}) = H^3(\Omega) \cap H_0^2(\Omega),\ \mathcal{A}u = -\Delta u + h,\ \Delta h = 0 \text{ in } \Omega,\ h = \Delta u \text{ on } \partial\Omega.$$

A related open problem is "Does the operator $-\mathcal{A}_p$, $1 < p < \infty$, defined by

$$D(\mathcal{A}_p) = W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega),\ \mathcal{A}_p u = -\Delta u + h,\ \Delta h = 0 \text{ in } \Omega,\ h = \Delta u \text{ on } \partial\Omega$$

generate an analytic semigroup in $W_0^{1,p}(\Omega)$?" It can be shown that the question is affirmative in case $n = 1$ with the aid of the explicite expression of the solution to

$$\mathcal{A}_p u + \lambda u = f$$

which takes the form if $\Omega = (0,1)$

$$\begin{aligned} &-u''(x) + h(x) + \lambda u(x) = f(x), \quad 0 < x < 1, \\ &u(0) = u'(0) = u(1) = u'(1) = 0, \end{aligned} \tag{13}$$

where $f$ is a given element of $W_0^{1,p}(0,1)$ and $h(x) = ax + b$ with $a$ and $b$ to be determined so that $h(0) = u''(0)$, $h(1) = u''(1)$. The solution of (13) is given by

$$\begin{aligned} u(x) = &-\frac{1}{2\sqrt{\lambda}} \int_0^x e^{-\sqrt{\lambda}(x-y)}(ay + b - f(y))dy \\ &- \frac{1}{2\sqrt{\lambda}} \int_x^1 e^{\sqrt{\lambda}(x-y)}(ay + b - f(y))dy, \end{aligned}$$

$$a = \frac{\lambda}{2 - \sqrt{\lambda} - 2e^{-\sqrt{\lambda}} - \sqrt{\lambda}e^{-\sqrt{\lambda}}} \left\{ \int_0^1 e^{-\sqrt{\lambda}y} f(y)dy - e^{-\sqrt{\lambda}} \int_0^1 e^{\sqrt{\lambda}y} f(y)dy \right\},$$

$$\begin{aligned} b = \frac{\sqrt{\lambda}}{1 - e^{-\sqrt{\lambda}}} \Bigg\{ &\frac{1 - \sqrt{\lambda} - e^{-\sqrt{\lambda}}}{2 - \sqrt{\lambda} - 2e^{-\sqrt{\lambda}} - \sqrt{\lambda}e^{-\sqrt{\lambda}}} \int_0^1 e^{-\sqrt{\lambda}y} f(y)dy \\ &+ \frac{e^{-\sqrt{\lambda}}\left(-\sqrt{\lambda}e^{-\sqrt{\lambda}} + 1 - e^{-\sqrt{\lambda}}\right)}{2 - \sqrt{\lambda} - 2e^{-\sqrt{\lambda}} - \sqrt{\lambda}e^{-\sqrt{\lambda}}} \int_0^1 e^{\sqrt{\lambda}y} f(y)dy \Bigg\}. \end{aligned}$$

Next result is concerned with the maximal regularity.

THEOREM 3. *Let $L, M$ be two closed linear operators acting in a reflexive* Banach *space $X$ and satisfying* (3). *Suppose $k(\cdot) \in C^{1,\alpha}([0,\tau])$ for some $\alpha > 0$ and $k(0) > 0$. If $f(\cdot) \in C^{1,\alpha}([0,\tau]; X)$ and $f(0) \in R(T^2)$, $\dot{f}(0) \in R(T)$, $T = ML^{-1}$, then problem* (1) *has a strict solution $u(\cdot)$ such that $Mu(\cdot) \in C^{1,\alpha}([0,\tau]; X)$ and $Lu(\cdot) \in C^{\alpha}([0,\tau]; X)$.*

*Proof.* Under the assumption of the theorem we have

$$(1 - P)\dot{f}(0) - k(0)\tilde{T}^{-1} f(0) \in R(T) \subset D_{-\tilde{T}^{-1}}(\alpha, \infty).$$

Hence the conclusion follows from Theorem 4.5 of E. Sinestrari [8].

Next, we consider the case where $k$ satisfies the following hypotheses:

$$k \in AC([0,\tau]),\ \dot{k} \in BV([0,\tau]),\ k(0) > 0. \tag{14}$$ □

THEOREM 4. *Let $X$ be a reflexive Banach space and let $M, L$ be two closed linear operators in $X$ satisfying (3). Suppose that (14) is satisfied. Then for each function $f$ satsifying $f \in AC([0,\tau];X)$, $\dot{f} \in BV([0,\tau];X)$, and $f(0) \in \overline{R(T)}$, there exists a function $u(\cdot)$ having the following properties:*

*$u$ is strongly measurable in $]0,\tau[$, $u(t) \in D(M)$ for every $t \in [0,\tau]$, $Mu \in C\big([0,\tau];X\big)$, $u(t) \in D(L)$ a.e. in $[0,\tau]$, $Mu$ is differentiable a.e. the functions $tdMu(t)/dt$, $tLu(t)$ belong to $L^\infty(0,\tau;X)$,*

*$\int_\epsilon^t k(t-s)Lu(s)ds$ is uniformly bounded in $0 < \epsilon < t \le \tau$,*

*and the improper integral*

$$\lim_{\epsilon\to 0}\int_\epsilon^t k(t-s)Lu(s)ds \equiv \int_{+0}^t k(t-s)Lu(s)ds \tag{15}$$

*exists, the equation (1) holds with the integral of the left hand side replaced by the the improper integral (15).*

*Furthermore the function $u$ which has the above properties is uniquely determined by the right hand side $f(\cdot)$ of (1).*

This theorem is proved by applying the following proposition to equation (7).

PROPOSITION 1. *Suppose that $A$ is a closed linear operator with dense domain $D(A)$ such that $-A$ generates the analytic semigroup $e^{-tA}$ in a* Banach *space $X$, and*

$$k \in AC([0,\tau]),\ \dot{k} \in BV([0,\tau]),\ k(0) > 0,$$
$$f \in AC([0,\tau];X),\ \dot{f} \in BV([0,\tau];X),\ 0 < \tau < \infty.$$

*Then a solution* $u(\cdot)$ *of the integral equation*

$$u(t) + \int_0^t k(t-s)Au(s)ds = f(t), \quad 0 < t \le \tau, \tag{16}$$

*with the following properties exists and is unique:*

$u(\cdot) \in C([0,\tau];X)$, $u(t) \in D(A)$ *a.e.*, $u(\cdot)$ *is differentiable a.e.*, *the functions* $tAu(t)$ *and* $tdu(t)/dt$ *belongs to* $L^\infty(]0,\tau];X)$, $\int_\epsilon^t k(t-s)Au(s)ds$ *is uniformly bounded for* $0 < \epsilon < t < \tau$, *the limit*

$$\int_{+0}^t k(t-s)Au(s)ds = \lim_{\epsilon \to 0} \int_\epsilon^t k(t-s)Au(s)ds \tag{17}$$

*exists, and equation* (16) *is satisfied with the integral of the left hand side replaced by the improper integral* (17).

*Proof.* Following the idea of Crandall and Nohel [1] the equation (16) is transformed into the initial value problem

$$\frac{d}{dt}u(t) + k(0)Au(t) = G(u)(t), \quad 0 < t \le \tau, \quad u(0) = f(0), \tag{18}$$

where

$$G(u)(t) = \dot{f}(t) + (r * \dot{f})(t) + r(t)f(0) - r(0)u(t) - (u * \dot{r})(t), \tag{19}$$

and $r$ is the solution of the integral equation

$$\dot{k} + k(0)r + \dot{k} * r = 0.$$

Problem (18) is further transformed into the integral equation

$$u(t) = e^{-tk(0)A}f(0) + \int_0^t e^{-(t-s)k(0)A}G(u)(s)ds. \tag{20}$$

Equation (20) can be solved by successive approximation, and there exists a unique solution $u \in C([0,\tau];X)$. Let

$$g(t) = \dot{f}(t) + (r * \dot{f})(t) + r(t)f(0).$$

Since $g(\cdot)$ is of bounded variation, by an elementary integral calculus it can be shown that $\int_0^t e^{-(t-s)k(0)A}g(s)ds$ is differentiable almost everywhere, and we have

$$\begin{aligned}\frac{d}{dt}\int_0^t e^{-(t-s)k(0)A}g(s)\,ds \\ &= \int_0^t e^{-(t-s)k(0)A}\,dg(s) + e^{-k(0)A}g(0) \\ &= g(t) - k(0)A\int_0^t e^{-(t-s)k(0)A}g(s)\,ds.\end{aligned}$$

Since $u(\cdot)$ is Hölder continuous, or rather by a direct use of the equation (20) itself, one can easily show that

$$\int_0^t e^{-(t-s)k(0)A}r(0)u(s)ds$$

is differentiable with bounded derivative. Noting that

$$\int_0^t e^{-(t-s)k(0)A}(u*\dot{r})(s)ds = \int_0^t\int_\xi^t e^{-(t-s)k(0)A}u(s-\sigma)dsdr(\sigma),$$

we see that $\int_0^t e^{-(t-s)k(0)A}(u*\dot{r})(s)ds$ is differentiable. Combining these and the equation (20), we conclude that $u(\cdot)$ is differentiable and satisfies (18) almost everywhere. Integrating (18) from $\epsilon$ to $t$ yields

$$u(t) - u(\epsilon) + k(0)\int_\epsilon^t Au(s)ds = \int_\epsilon^t G(u)(s)ds.$$

This shows that $\int_\epsilon^t Au(s)ds$ is uniformly bounded and the limit

$$\lim_{\epsilon\to 0} k(0)\int_\epsilon^t Au(s)ds = k(0)\int_{+0}^t Au(s)ds = \int_0^t G(u)(s)ds - u(t) + f(0)$$

exists. Integrating by parts yields

$$\begin{aligned}\int_\epsilon^t k(t-s)Au(s)ds &= \int_\epsilon^t k(t-s)\frac{d}{ds}\int_\epsilon^s Au(\sigma)d\sigma ds \\ &= k(0)\int_\epsilon^t Au(\sigma)d\sigma + \int_\epsilon^t \dot{k}(t-s)\int_\epsilon^s Au(\sigma)d\sigma ds.\end{aligned}$$

Hence (17) of the statement of the proposition follows. The details of the proof are carried out by the method of H. Tanabe [10], [11] □

REMARK 2. The equation (1) is the integrated version of the formally differentiated problem

$$\frac{d}{dt}Mu(t) + k(0)Lu(t) + \int_0^t \dot{k}(t-s)Lu(s)ds = \dot{f}(t), \tag{21}$$

$$\lim_{t\to+0}(Mu)(t) = f(0), \tag{22}$$

which is the special case of the integrodifferential equation with $L_1$ such that $D(L_1) \supset D(L)$ in place of $L$ in the integral of the left hand side of (21). Under the assumptions of Theorem 4, (21) holds with the integral of the left hand side understood in the improper sense:

$$\int_{+0}^t \dot{k}(t-s)Lu(s)ds = \lim_{\epsilon\to 0}\int_\epsilon^t \dot{k}(t-s)Lu(s)ds$$

as is shown in what follows. It suffices to show that $\int_{+0}^t k(t-s)Lu(s)ds$ is differentiable and

$$\frac{d}{dt}\int_{+0}^t k(t-s)Lu(s)ds = k(0)Lu(t) + \int_{+0}^t \dot{k}(t-s)Lu(s)ds. \tag{23}$$

By Fubini's theorem and integration by parts,

$$\begin{aligned}
&\int_\epsilon^t \int_\epsilon^\sigma k(\sigma-s)Lu(s)\,ds d_\sigma r(t-\sigma)\\
&= \int_\epsilon^t \int_s^t k(\sigma-s)d_\sigma r(t-\sigma)Lu(s)\,ds\\
&= \int_\epsilon^t \left\{\Big[k(\sigma-s)r(t-\sigma)\Big]_s^t - \int_s^t \dot{k}(\sigma-s)r(t-\sigma)d\sigma\right\} Lu(s)\,ds\\
&= \int_\epsilon^t \left\{k(t-s)r(0) - k(0)r(t-s) - (\dot{k}*r)(t-s)\right\} Lu(s)\,ds\\
&= \int_\epsilon^t \left\{k(t-s)r(0) + \dot{k}(t-s)\right\} Lu(s)\,ds\\
&= r(0)\int_\epsilon^t k(t-s)Lu(s)\,ds + \int_\epsilon^t \dot{k}(t-s)Lu(s)\,ds.
\end{aligned}$$

Hence $\int_\epsilon^t \dot{k}(t-s)Lu(s)ds$ is uniformly bounded and the limit $\int_{+0}^t \dot{k}(t-s)Lu(s)ds$ exists with

$$\int_0^t \int_{+0}^\sigma k(\sigma - s)Lu(s)\, ds d_\sigma r(t-\sigma)$$
$$= r(0)\int_{+0}^t k(t-s)Lu(s)\, ds \;+\; \int_{+0}^t \dot{k}(t-s)Lu(s)\, ds.$$

Letting $\epsilon \to 0$ in

$$\int_\epsilon^{t'} k(t'-s)Lu(s)ds - \int_\epsilon^t k(t-s)Lu(s)\, ds$$
$$= \int_t^{t'} \frac{d}{d\sigma}\int_\epsilon^\sigma k(\sigma-s)Lu(s)\, dsd\sigma$$
$$= \int_t^{t'} \left\{ k(0)Lu(\sigma) + \int_\epsilon^\sigma \dot{k}(\sigma-s)Lu(s)\, ds \right\} d\sigma,$$

we get

$$\int_{+0}^{t'} k(t'-s)Lu(s)\, ds - \int_{+0}^t k(t-s)Lu(s)\, ds$$
$$= \int_t^{t'} \left\{ k(0)Lu(\sigma) + \int_{+0}^\sigma \dot{k}(\sigma-s)Lu(s)\, ds \right\} d\sigma$$

for $0 < t < t'$. This shows that (23) is true.

**Case where $k(0)=0$ or $k$ is singular at $t=0$**

Suppose that $k(\cdot)$ is Laplace transformable, its Laplace transform $\hat{k}(\lambda)$ has a meromorphic extenstion to the sector

$$\Sigma\left(0, \theta_0 + \frac{\pi}{2}\right) = \left\{\lambda \in \mathbf{C}; |\arg\lambda| < \theta_0 + \frac{\pi}{2}\right\}$$

and $\hat{k}(\lambda) \neq 0$ for $\lambda \in \Sigma\left(0, \theta_0 + \frac{\pi}{2}\right)$ for some $\theta_0 \in \left]0, \frac{\pi}{2}\right]$. Suppose also that there exists an angle $\theta_1 \in \left]\frac{\pi}{2}, \pi\right[$ such that

$$\|L(zM+L)^{-1}\| \le \text{const for any } z \in \Sigma = \{0\} \cup \Sigma(0,\theta_1),$$
$$\frac{1}{\hat{k}(\lambda)} \in \Sigma \text{ for all } \lambda \in \Sigma\left(0, \theta_0 + \frac{\pi}{2}\right).$$

Then we can apply the result on analytic resolvents of J. Prüss [7: Theorem 2.1] to solve the integral equation (5). Equation (6) has a unique solution given by

$$Pv(t) = \int_0^t \omega(t-s)(Pf)'(s)ds$$

for any $f$ such that $Pf \in C^1\Big([0,\tau]; N(T)\Big)$, $Pf(0) = 0$, where $\omega$ is the inverse Laplace transform of $1/\lambda\hat{k}(\lambda)$.

EXAMPLE 4. Let $k(t) = t^\alpha$, $-1 < \alpha < 0$. Then $\omega(t) = \dfrac{t^{-\alpha-1}}{\Gamma(-\alpha)\Gamma(\alpha+1)}$.

**Case of $\lambda = 0$ is a pole for $(\lambda L + M)^{-1}$**

Here $X$ is a complex Banach space, not necessarily reflexive. Suppose that $\lambda = 0$ is a pole for the resolvent of $T = ML^{-1}$. Then $R(T)$ is closed and $X$ has the direct decomposition representation $X = N(T) \oplus R(T)$. Since $\tilde{T}^{-1}$ is bounded in this case, we can easily establish the following results.

THEOREM 5. *Let $z = 0$ be a simple pole for the resolvent of $ML^{-1}$. If $k \in C^1([0,\tau])$, $k(0) > 0$, and $f \in C^1([0,\tau]; X)$, $f(0) \in R(T)$, then* (1) *has a unique strict solution.*

THEOREM 6. *Let $\lambda = 0$ be a simple pole for $(\lambda + T)^{-1}, T = ML^{-1}$. Let $k \in L^1_{loc}\big([0,\infty[\big)$, with $1/(\lambda\hat{k}(\lambda)) = \hat{\omega}, \omega \in L^1_{loc}\big([0,\infty[\big)$. Then, for all $f \in C^1([0,\tau]; X), f(0) \in R(T)$, problem* (1) *has a strict solution.*

THEOREM 7. *Let $\lambda = 0$ be a simple pole for $(\lambda + T)^{-1}, T = ML^{-1}$. Let $k \in L^1_{loc}([0,\infty[) \cap C^m([0,\tau]) \cap C^{m+1}((0,\tau])$, $k^{(j)}(0) = 0$, $j = 0, 1, \dots, m$, $m \in \mathbf{N} \cup \{0\}$. If $f \in C^{(m+2)}([0,\tau]; X), Pf^{(j)}(0) = 0, j = 0, 1, \dots, m+1$, then problem* (P) *has a strict solution provided that $1/(\lambda^{m+2}\hat{k}(\lambda))$ is the* Laplace *transform of a function in $L^1_{loc}(0,\infty)$.*

Examples of the case where $\lambda = 0$ is a simple pole for $(\lambda L + M)^{-1}$ are found in section 5 of A. Favini, L. Pandolfi and H. Tanabe[4]. Analogous results also hold in the case of multiple pole.

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