DEGENERATE VOLterra EQUATIONS
IN BANACH SPACES

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ABSTRACT. This paper is concerned with degenerate Volterra equations $Mu(t) + \int_0^t k(t - s)Lu(s)ds = f(t)$ in Banach spaces both in the hyperbolic case, and the parabolic one. The key assumption is played by the representation of the underlying space $X$ as a direct sum $X = N(T) \oplus \overline{R(T)}$, where $T$ is the bounded linear operator $T = ML^{-1}$. Hyperbolicity means that the part $\tilde{T}$ of $T$ in $\overline{R(T)}$ is an abstract potential operator, i.e., $-\tilde{T}^{-1}$ generates a $C_0$-semigroup, and parabolicity means that $-\tilde{T}^{-1}$ generates an analytic semigroup. A maximal regularity result is obtained for parabolic equations. We will also investigate the cases where the kernel $k(\cdot)$ is degenerate or singular at $t = 0$ using the results of Prüss [8] on analytic resolvents. Finally, we consider the case where $\lambda$ is a pole for $(\lambda L + M)^{-1}$.

This paper is concerned with the unique solvability of the following Volterra integral equations

(1) $Mu(t) + \int_0^t k(t - s)Lu(s)ds = f(t), \quad t \in [0, \tau]$

in a Banach space $X$. Only an outline is presented here, and the details will be published elsewhere.

Let $M$ and $L$ be closed linear operators such that $D(L) \subseteq D(M)$ and $0 \in \rho(L)$. The kernel $k(\cdot)$ is a numerical function defined in the closed interval $[0, \tau]$. In case $k(t) \equiv 1$ we get by formally differentiating (1)

$$\frac{d}{dt}Mu(t) + Lu(t) = \hat{f}(t), \quad t \in [0, \tau]$$

$$Mu(0) = f(0).$$

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This type of (possibly degenerate) equations is discussed in detail in the book of A. Favini and A. Yagi [3].

We consider the following two cases.

**Hyperbolic case** There exists a constant $K$ such that

$$\| (M(sM + L)^{-1})^k \| \leq \frac{K}{s^k} \quad \forall s > 0, \quad k = 1, 2, \ldots .$$

**Parabolic case** There exists a constant $K$ such that

$$\| M(\lambda M + L)^{-1} \| \leq \frac{K}{1 + |\lambda|}, \quad \text{Re}\lambda \geq 0.$$

In view of our assumptions, $T = ML^{-1}$ is a bounded linear operator in $X$. The following facts are established in A. Favini [2] and A. Favini and A. Yagi [3]. Under the assumption (2) or (3), $X$ has a direct decomposition representation $X = N(T) \oplus \overline{R(T)}$. Let $\tilde{T}$ denote the restriction of $T$ to $\overline{R(T)}$. If $X$ is reflexive and (2) is satisfied, then $-\tilde{T}^{-1}$ generates a $C_0$-semigroup. If $X$ is reflexive and (3) is satisfied, then $-\tilde{T}^{-1}$ generates an analytic semigroup.

We make the change of the unknown function $v(t) = Lu(t)$. Then, since $Mu(t) = ML^{-1}v(t) = T v(t)$, the equation (1) is transformed into

$$Tv(t) + \int_0^t k(t-s)v(s)ds = f(t).$$

Let $P$ be the projection onto $N(T)$ along $\overline{R(T)}$. Then equation (4) splits into the two equations

$$\tilde{T}(1-P)v(t) + \int_0^t k(t-s)(1-P)v(s)ds = (1-P)f(t),$$

$$\int_0^t k(t-s)Pv(s)ds = Pf(t),$$

since $PT = TP = 0$. (5) is a Volterra equation of the second kind, and (6) is a Volterra equation of the first kind.

Another change of the unknown variable $w(t) = \tilde{T}(1-P)v(t)$ transforms the equation (5) into the equation

$$w(t) + \int_0^t k(t-s)\tilde{T}^{-1}w(s)ds = (1-P)f(t).$$
Formally differentiating both sides of (7), we arrive at the following initial value problem

\begin{equation}
\dot{w}(t) + k(0)\bar{T}^{-1}w(t) + \int_{0}^{t} \dot{k}(t-s)\bar{T}^{-1}w(s)ds = (1 - P)\dot{f}(t),
\end{equation}

\begin{equation}
w(0) = (1 - P)f(0).
\end{equation}

Also by formal differentiation, the equation (6) is transformed into

\begin{equation}
k(0)Pv(t) + \int_{0}^{t} \dot{k}(t-s)Pv(s)ds = Pf(t).
\end{equation}

We try to solve the original equation (1) by solving the problems (8), (9) and (10).

We denote by $AC([0, \tau])$ and $BV([0, \tau])$ the set of absolutely continuous functions in $[0, \tau]$ and the set of functions of bounded variation in $[0, \tau]$, respectively.

**Hyperbolic case**

**Theorem 1.** Let $X$ be a reflexive Banach space and let (2) be satisfied. If $k \in AC([0, \tau]), k \in BV([0, \tau]), k(0) > 0, f \in W^{2,1}(0, \tau; X)$ and $f(0) \in R(T)$, then the solution $u(\cdot) \in C([0, \tau]; D(L))$ of problem (1) exists and is unique.

**Proof.** The problem (8) and (9) can be solved by applying the following result of E. Sinestrari [9: Theorem 3.1].

**Lemma 1.** Let $A$ be a Hille-Yosida operator in $X$ and let $B$ be a linear closed operator with $D(A) \subset D(B)$. If $b \in BV([0, \tau]),$ then for any $x_0 \in D(A)$ and $f \in W^{1,1}(0, \tau; X)$ such that $Ax_0 + f(0) \in D(A)$ the integro-differential problem

\begin{equation}
\frac{d}{dt}x(t) = Ax(t) + \int_{0}^{t} b(t-s)Bx(s)ds + f(t), \quad 0 \leq t \leq \tau,
\end{equation}

\begin{equation}
x(0) = x_0
\end{equation}

has a unique solution $x(\cdot) \in C([0, \tau]; D(A)) \cap C^1([0, \tau]; X)$. 

By the assumption of the theorem $(1 - P) f(0) = f(0) \in R(T) = D(-\tilde{T}^{-1})$ and $D(-\tilde{T}^{-1})$ is dense in $\overline{R(T)}$. Hence the above result of Sinestrari can be applied to the problem (8) and (9), and there exists a unique solution $w$ such that $w \in C^1([0, \tau] ; \overline{R(T)})$ and $\tilde{T}^{-1} w \in C([0, \tau] ; \overline{R(T)})$. It is easy to show that the solution of (8) and (9) satisfies (7). The equation (10) can be solved by Neumann series expansion. Integrating (10) and using the hypothesis $P f(0) = 0$, we can show that the solution of (10) satisfies (6).

**Example 1.** Let $L = -\Delta, D(L) = H^2(\Omega) \cap H_0^1(\Omega), X = L^2(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $M$ be the multiplication operator by a nonnegative superharmonic function $m \in C^2(\Omega)$. Then integration by parts yields for $u \in D(L)$

$$-\text{Re} \int_{\Omega} m \Delta \bar{u} d\Omega = -\frac{1}{2} \int_{\Omega} \Delta m \cdot |u|^2 d\Omega + \int_{\Omega} m |\nabla u|^2 d\Omega \geq 0,$$

from which it follows that the condition (2) is satisfied with $K = 1$.

**Parabolic case**

**Theorem 2.** Let $X$ be a reflexive Banach space and let $L, M$ be two closed linear operators in $X$ satisfying (3). Let $\tau > 0$ be fixed and assume that $k \in C^{1, \alpha}([0, \tau]), k(0) > 0$ for some $\alpha > 0$. If $f(\cdot) \in C^{1, \theta}([0, \tau]; X), 0 < \theta < 1$, and $f(0) \in \overline{R(T)}, T = ML^{-1}$, then (1) has a unique solution $u(\cdot) \in C^0([0, \tau]; D(L)) \cap C^1([0, \tau]; X)$ which satisfies the equation in the following sense: $\int_{\tau}^{t} k(t-s) Lu(s) ds$ is uniformly bounded in $0 < \epsilon < \tau$, the improper integral

$$(11) \quad \int_{\tau}^{t} k(t-s) Lu(s) ds = \lim_{\epsilon \to 0} \int_{\tau}^{t} k(t-s) Lu(s) ds$$

exists and (1) holds with the integral in the left hand side understood in the improper sense (11). Furthermore $u$ satisfies the additional regularity condition $Mu \in C^1([0, \tau]; X)$. If in addition $f(0) \in R(T), T = ML^{-1}$, then $u$ is a strict solution belonging to $C^0([0, \tau]; D(L))$ with the regularity property $Mu \in C^1([0, \tau]; X)$.

**Proof.** The result of J. Prüss [6] is applied to the formally differentiated problem (8) and (9). In the present case, the initial value $f(0)$ is an
arbitrary element of $R(T)$. Therefore, $\tilde{T}^{-1}w(s)$ is not Bochner integrable near $s = 0$ in general; however, the improper integral

$$\int_{0}^{t} k(t-s)\tilde{T}^{-1}w(s)ds = \lim_{\epsilon \to 0} \int_{\epsilon}^{t} k(t-s)\tilde{T}^{-1}w(s)ds$$

exists, and it can be shown that (8) holds with the integral in the left hand side replaced by (12).

Example 2. Let $X = H^{-1}(\Omega)$, $D(L) = H^1_0(\Omega)$, $Lu = -\Delta u$, $M$ is the multiplication operator by a nonnegative function in $C(\bar{\Omega})$, where $\Omega$ is a bounded domain in $R^n$ with smooth boundary. Then it has been proven by Favini and Yagi [3] that assumption (3) holds.

Example 3. Let $X = H^{-1}(\Omega)$, $D(M) = H^1_0(\Omega)$, $Mu = -\Delta u$, and $D(L) = H^3(\Omega) \cap H^2_0(\Omega)$, $Lu = \Delta^2 u$, where $\Omega$ is a bounded domain in $R^n$ with smooth boundary. Let

$$(u, v)_1 = \int_{\Omega} \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \quad (u, v)_2 = \int_{\Omega} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i x_j} \frac{\partial^2 v}{\partial x_i x_j} dx$$

be the inner product in $H^1_0(\Omega)$ and $H^2_0(\Omega)$, respectively. Following the idea of W. M. Greenlee[5], we consider the operator $A$ defined by

$$(u, v)_2 = (Au, v)_1 \quad \text{for every} \quad v \in H^2_0(\Omega).$$

$A$ is a positive definite selfadjoint operator in $H^1_0(\Omega)$ and $L = MA$. Therefore we have

$$||M(\lambda M + L)^{-1}||_{L(H^{-1}(\Omega))} = ||M(\lambda + A)^{-1}M^{-1}||_{L(H^{-1}(\Omega))} \leq C\theta_0 (1 + |\lambda|)^{-1}$$

for any sector $\{\lambda; |\arg \lambda| \leq \theta_0\}, \quad \frac{\pi}{2} < \theta_0 < \pi$.

Remark 1. The operator $A$ in Example 3 is characterized as

$$D(A) = H^3(\Omega) \cap H^2_0(\Omega), \quad Au = -\Delta u + h, \quad \Delta h = 0 \text{ in } \Omega, \quad h = \Delta u \text{ on } \partial \Omega.$$ 

A related open problem is "Does the operator $-A_p$, $1 < p < \infty$, defined by

$$D(A_p) = W^{3,p}(\Omega) \cap W^{2,p}_0(\Omega), \quad A_p u = -\Delta u + h, \quad \Delta h = 0 \text{ in } \Omega, \quad h = \Delta u \text{ on } \partial \Omega"
generate an analytic semigroup in $W^{1,p}_0(\Omega)$. It can be shown that the question is affirmative in case $n = 1$ with the aid of the explicit expression of the solution to

$$\mathcal{A}_p u + \lambda u = f$$

which takes the form if $\Omega = (0, 1)$

$$-u''(x) + h(x) + \lambda u(x) = f(x), \quad 0 < x < 1,$$

$$u(0) = u'(0) = u(1) = u'(1) = 0,$$

where $f$ is a given element of $W^{1,p}_0(0, 1)$ and $h(x) = ax + b$ with $a$ and $b$ to be determined so that $h(0) = u''(0)$, $h(1) = u''(1)$. The solution of (13) is given by

$$u(x) = -\frac{1}{2\sqrt{\lambda}} \int_0^x e^{-\sqrt{\lambda}(x-y)}(ay + b - f(y))dy$$

$$-\frac{1}{2\sqrt{\lambda}} \int_x^1 e^{\sqrt{\lambda}(x-y)}(ay + b - f(y))dy,$$

$$a = \frac{\lambda}{2 - \sqrt{\lambda} - 2e^{-\sqrt{\lambda}} - \sqrt{\lambda}e^{-\sqrt{\lambda}}} \left\{ \int_0^1 e^{-\sqrt{\lambda}y}f(y)dy - e^{-\sqrt{\lambda}} \int_0^1 e^{\sqrt{\lambda}y}f(y)dy \right\},$$

$$b = \frac{\sqrt{\lambda}}{1 - e^{-\sqrt{\lambda}}} \left\{ \frac{1 - \sqrt{\lambda} - e^{-\sqrt{\lambda}}}{2 - \sqrt{\lambda} - 2e^{-\sqrt{\lambda}} - \sqrt{\lambda}e^{-\sqrt{\lambda}}} \int_0^1 e^{-\sqrt{\lambda}y}f(y)dy + \frac{e^{-\sqrt{\lambda}y}(-\sqrt{\lambda}e^{-\sqrt{\lambda}} + 1 - e^{-\sqrt{\lambda}})}{2 - \sqrt{\lambda} - 2e^{-\sqrt{\lambda}} - \sqrt{\lambda}e^{-\sqrt{\lambda}}} \int_0^1 e^{\sqrt{\lambda}y}f(y)dy \right\}.$$

Next result is concerned with the maximal regularity.

**Theorem 3.** Let $L, M$ be two closed linear operators acting in a reflexive Banach space $X$ and satisfying (3). Suppose $k(\cdot) \in C^{1,\alpha}([0, \tau])$ for some $\alpha > 0$ and $k(0) > 0$. If $f(\cdot) \in C^{1,\alpha}([0, \tau]; X)$ and $f(0) \in R(T^2), \tilde{f}(\cdot) \in R(T), T = ML^{-1}$, then problem (1) has a strict solution $u(\cdot)$ such that $Mu(\cdot) \in C^{1,\alpha}([0, \tau]; X)$ and $Lu(\cdot) \in C^{\alpha}([0, \tau]; X)$.

**Proof.** Under the assumption of the theorem we have

$$(1 - P)f(0) - k(0)\tilde{T}^{-1}f(0) \in R(T) \subset D_{\mathcal{T}^{-1}}(\alpha, \infty).$$

Hence the conclusion follows from Theorem 4.5 of E. Sinestrari [8].
Next, we consider the case where \( k \) satisfies the following hypotheses:

\[
(14) \quad k \in AC([0, \tau]), \quad \dot{k} \in BV([0, \tau]), \quad k(0) > 0.
\]

\[\Box\]

**Theorem 4.** Let \( X \) be a reflexive Banach space and let \( M, L \) be two closed linear operators in \( X \) satisfying (3). Suppose that (14) is satisfied. Then for each function \( f \) satisfying \( f \in AC([0, \tau]; X), \quad \dot{f} \in BV([0, \tau]; X), \) and \( f(0) \in \overline{R(T)} \), there exists a function \( u(\cdot) \) having the following properties:

- \( u \) is strongly measurable in \( [0, \tau] \), \( u(t) \in D(M) \) for every \( t \in [0, \tau] \), \( Mu \in C([0, \tau]; X) \), \( u(t) \in D(L) \) a.e. in \( [0, \tau] \), \( Mu \) is differentiable a.e. the functions \( tdMu(t)/dt, tLu(t) \) belong to \( L^\infty(0, \tau; X) \), \( \int_{\epsilon}^{t} k(t-s)Lu(s)ds \) is uniformly bounded in \( 0 < \epsilon < t \leq \tau \), and the improper integral

\[
(15) \quad \lim_{\epsilon \to 0} \int_{\epsilon}^{t} k(t-s)Lu(s)ds \equiv \int_{-\epsilon}^{t} k(t-s)Lu(s)ds
\]

exists, the equation (1) holds with the integral of the left hand side replaced by the the improper integral (15).

Furthermore the function \( u \) which has the above properties is uniquely determined by the right hand side \( f(\cdot) \) of (1).

This theorem is proved by applying the following proposition to equation (7).

**Proposition 1.** Suppose that \( A \) is a closed linear operator with dense domain \( D(A) \) such that \(-A\) generates the analytic semigroup \( e^{-tA} \) in a Banach space \( X \), and

\[
(16) \quad k \in AC([0, \tau]), \quad \dot{k} \in BV([0, \tau]), \quad k(0) > 0,
\]

\[
f \in AC([0, \tau]; X), \quad \dot{f} \in BV([0, \tau]; X), \quad 0 < \tau < \infty.
\]
Then a solution $u(\cdot)$ of the integral equation
\begin{equation}
    u(t) + \int_0^t k(t - s)Au(s)ds = f(t), \quad 0 < t \leq \tau,
\end{equation}
with the following properties exists and is unique:
- $u(\cdot) \in C([0, \tau]; X)$, $u(t) \in D(A)$ a.e., $u(\cdot)$ is differentiable a.e.,
- the functions $tAu(t)$ and $tdu(t)/dt$ belongs to $L^\infty([0, \tau]; X)$,
- $\int_\epsilon^t k(t - s)Au(s)ds$ is uniformly bounded for $0 < \epsilon < t < \tau$,
- the limit
\begin{equation}
    \int_{\epsilon}^{t} k(t - s)Au(s)ds = \lim_{\epsilon \to 0} \int_{\epsilon}^{t} k(t - s)Au(s)ds
\end{equation}
exists, and equation (16) is satisfied with the integral of the left hand side replaced by the improper integral (17).

Proof. Following the idea of Crandall and Nohel [1] the equation (16) is transformed into the initial value problem
\begin{equation}
    \frac{d}{dt}u(t) + k(0)Au(t) = G(u)(t), \quad 0 < t \leq \tau, \quad u(0) = f(0),
\end{equation}
where
\begin{equation}
    G(u)(t) = \hat{f}(t) + (r \ast \hat{f})(t) + \tau(t)f(0) - r(0)u(t) - (u \ast r)(t),
\end{equation}
and $r$ is the solution of the integral equation
$$
\dot{k} + k(0)r + \dot{k} \ast r = 0.
$$
Problem (18) is further transformed into the integral equation
\begin{equation}
    u(t) = e^{-tk(0)A}f(0) + \int_0^t e^{-(t-s)k(0)A}G(u)(s)ds.
\end{equation}
Equation (20) can be solved by successive approximation, and there exists a unique solution $u \in C([0, \tau]; X)$. Let
$$
g(t) = \hat{f}(t) + (r \ast \hat{f})(t) + \tau(t)f(0).
$$
Since $g(\cdot)$ is of bounded variation, by an elementary integral calculus it can be shown that $\int_0^t e^{-(t-s)k(0)A}g(s)ds$ is differentiable almost everywhere, and we have
\[
\frac{d}{dt} \int_0^t e^{-\frac{(t-s)}{k(0)A}} g(s) \, ds \\
= \int_0^t e^{-\frac{(t-s)}{k(0)A}} g(s) \, ds + e^{-k(0)A} g(0) \\
= g(t) - k(0)A \int_0^t e^{-\frac{(t-s)}{k(0)A}} g(s) \, ds.
\]

Since \( u(\cdot) \) is Hölder continuous, or rather by a direct use of the equation (20) itself, one can easily show that

\[
\int_0^t e^{-\frac{(t-s)}{k(0)A}} r(0) u(s) \, ds
\]

is differentiable with bounded derivative. Noting that

\[
\int_0^t e^{-\frac{(t-s)}{k(0)A}} (u \ast \dot{r})(s) \, ds = \int_0^t \int_0^t e^{-\frac{(t-s)}{k(0)A}} u(s - \sigma) \, ds \, d\sigma 
\]

we see that \( \int_0^t e^{-\frac{(t-s)}{k(0)A}} (u \ast \dot{r})(s) \, ds \) is differentiable. Combining these and the equation (20), we conclude that \( u(\cdot) \) is differentiable and satisfies (18) almost everywhere. Integrating (18) from \( \epsilon \) to \( t \) yields

\[
u(t) - u(\epsilon) + k(0) \int_\epsilon^t A u(s) \, ds = \int_\epsilon^t G(u)(s) \, ds.
\]

This shows that \( \int_\epsilon^t A u(s) \, ds \) is uniformly bounded and the limit

\[
\lim_{\epsilon \to 0} k(0) \int_\epsilon^t A u(s) \, ds = k(0) \int_0^t A u(s) \, ds = \int_0^t G(u)(s) \, ds - u(t) + f(0)
\]

exists. Integrating by parts yields

\[
\int_\epsilon^t k(t-s) A u(s) \, ds = \int_\epsilon^t k(t-s) \frac{d}{ds} \int_s^t A u(\sigma) \, d\sigma \, ds \\
= k(0) \int_\epsilon^t A u(\sigma) \, d\sigma + \int_\epsilon^t k(t-s) \int_s^t A u(\sigma) \, d\sigma \, ds.
\]

Hence (17) of the statement of the proposition follows. The details of the proof are carried out by the method of H. Tanabe [10], [11] □
 Remark 2. The equation (1) is the integrated version of the formally
differentiated problem

\[
(21) \quad \frac{d}{dt}Mu(t) + k(0)Lu(t) + \int_0^t k(t-s)Lu(s)ds = \dot{f}(t),
\]

\[
(22) \quad \lim_{t \to 0^+} (Mu)(t) = f(0),
\]

which is the special case of the integrodifferential equation with \( L_1 \) such
that \( D(L_1) \supset D(L) \) in place of \( L \) in the integral of the left hand side of
(21). Under the assumptions of Theorem 4, (21) holds with the integral
of the left hand side understood in the improper sense:

\[
\int_{+0}^t \dot{k}(t-s)Lu(s)ds = \lim_{\epsilon \to 0^-} \int_{\epsilon}^t \dot{k}(t-s)Lu(s)ds
\]
as is shown in what follows. It suffices to show that \( \int_{+0}^t k(t-s)Lu(s)ds \)
is differentiable and

\[
(23) \quad \frac{d}{dt} \int_{+0}^t k(t-s)Lu(s)ds = k(0)Lu(t) + \int_{+0}^t \dot{k}(t-s)Lu(s)ds.
\]

By Fubini's theorem and integration by parts,

\[
\int_{\epsilon}^t \int_{s}^{\sigma} k(\sigma - s)Lu(s) ds d\sigma r(t-\sigma)
\]

\[
= \int_{\epsilon}^t \int_{s}^{t} k(\sigma - s)d\sigma r(t-\sigma)Lu(s) ds
\]

\[
= \int_{\epsilon}^t \left\{ \left[ k(\sigma - s)r(t-\sigma) \right]_s^\epsilon - \int_{s}^{t} \dot{k}(\sigma - s)r(t-\sigma)d\sigma \right\} Lu(s) ds
\]

\[
= \int_{\epsilon}^t \left\{ k(t-s)r(0) - k(0)r(t-s) - (\dot{k} * r)(t-s) \right\} Lu(s) ds
\]

\[
= \int_{\epsilon}^t \left\{ k(t-s)r(0) + \dot{k}(t-s) \right\} Lu(s) ds
\]

\[
= r(0) \int_{\epsilon}^t k(t-s)Lu(s) ds + \int_{\epsilon}^t \dot{k}(t-s)Lu(s) ds.
\]
Hence \( \int_0^t \dot{k}(t - s)Lu(s)\,ds \) is uniformly bounded and the limit \( \int_0^t \dot{k}(t - s)Lu(s)\,ds \) exists with
\[
\int_0^t \int_0^\sigma k(\sigma - s)Lu(s)\,dsd\sigma r(t - \sigma) = r(0) \int_0^t k(t - s)Lu(s)\,ds + \int_0^t \dot{k}(t - s)Lu(s)\,ds.
\]
Letting \( \epsilon \to 0 \) in
\[
\int_\epsilon^{t'} k(t' - s)Lu(s)\,ds - \int_\epsilon^t k(t - s)Lu(s)\,ds = \int_t^{t'} \frac{d}{d\sigma} \int_\epsilon^{\sigma} k(\sigma - s)Lu(s)\,dsd\sigma
\]
\[
= \int_t^{t'} \left\{ k(0)Lu(\sigma) + \int_\epsilon^{\sigma} \dot{k}(\sigma - s)Lu(s)\,ds \right\} d\sigma,
\]
we get
\[
\int_0^{t'} k(t' - s)Lu(s)\,ds - \int_0^t k(t - s)Lu(s)\,ds = \int_t^{t'} \left\{ k(0)Lu(\sigma) + \int_0^{\sigma} \dot{k}(\sigma - s)Lu(s)\,ds \right\} d\sigma
\]
for \( 0 < t < t' \). This shows that (23) is true.

**Case where \( k(0) = 0 \) or \( k \) is singular at \( t = 0 \)**

Suppose that \( k(\cdot) \) is Laplace transformable, its Laplace transform \( \hat{k}(\lambda) \) has a meromorphic extension to the sector
\[
\Sigma \left( 0, \theta_0 + \frac{\pi}{2} \right) = \left\{ \lambda \in \mathbb{C} ; |\arg \lambda| < \theta_0 + \frac{\pi}{2} \right\}
\]
and \( \hat{k}(\lambda) \neq 0 \) for \( \lambda \in \Sigma \left( 0, \theta_0 + \frac{\pi}{2} \right) \) for some \( \theta_0 \in \left[ 0, \frac{\pi}{2} \right] \). Suppose also that there exists an angle \( \theta_1 \in \left[ \frac{\pi}{2}, \pi \right] \) such that
\[
\| L(zM + L)^{-1} \| \leq \text{const for any } z \in \Sigma = \{ 0 \} \cup \Sigma(0, \theta_1),
\]
\[
\frac{1}{\hat{k}(\lambda)} \in \Sigma \text{ for all } \lambda \in \Sigma \left( 0, \theta_0 + \frac{\pi}{2} \right).
\]
Then we can apply the result on analytic resolvents of J. Prüss [7: Theorem 2.1] to solve the integral equation (5). Equation (6) has a unique solution given by

\[ Puv(t) = \int_0^t \omega(t-s)(Pf)'(s)ds \]

for any \( f \) such that \( Pf \in C^1([0, \tau]; N(T)) \), \( Pf(0) = 0 \), where \( \omega \) is the inverse Laplace transform of \( 1/\lambda \hat{k}(\lambda) \).

**Example 4.** Let \( k(t) = t^\alpha, -1 < \alpha < 0 \). Then \( \omega(t) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)\Gamma(\alpha+1)} \).

**Case of \( \lambda = 0 \) is a pole for \((\lambda L + M)^{-1}\)**

Here \( X \) is a complex Banach space, not necessarily reflexive. Suppose that \( \lambda = 0 \) is a pole for the resolvent of \( T = ML^{-1} \). Then \( R(T) \) is closed and \( X \) has the direct decomposition representation \( X = N(T) \oplus R(T) \). Since \( \hat{T}^{-1} \) is bounded in this case, we can easily establish the following results.

**Theorem 5.** Let \( z = 0 \) be a simple pole for the resolvent of \( ML^{-1} \). If \( k \in C^1([0, \tau]), k(0) > 0, \) and \( f \in C^1([0, \tau]; X), f(0) \in R(T), \) then (1) has a unique strict solution.

**Theorem 6.** Let \( \lambda = 0 \) be a simple pole for \((\lambda + T)^{-1}, T = ML^{-1} \). Let \( k \in L^{1}_{loc}([0, \infty]), \) with \( 1/(\lambda \hat{k}(\lambda)) = \hat{\omega}, \omega \in L^1_{loc}([0, \infty]). \) Then, for all \( f \in C^1([0, \tau]; X), f(0) \in R(T), \) problem (1) has a strict solution.

**Theorem 7.** Let \( \lambda = 0 \) be a simple pole for \((\lambda + T)^{-1}, T = ML^{-1} \). Let \( k \in L^{1}_{loc}([0, \infty]) \cap C^m([0, \tau]) \cap C^{m+1}((0, \tau]), k^{(j)}(0) = 0, j = 0, 1, \ldots, m, m \in \mathbb{N} \cup \{0\}. \) If \( f \in C^{(m+2)}([0, \tau]; X), Pf^{(j)}(0) = 0, j = 0, 1, \ldots, m + 1, \) then problem (P) has a strict solution provided that \( 1/(\lambda^{m+2}\hat{k}(\lambda)) \) is the Laplace transform of a function in \( L^{1}_{loc}(0, \infty). \)

Examples of the case where \( \lambda = 0 \) is a simple pole for \((\lambda L + M)^{-1}\) are found in section 5 of A. Favini, L. Pandolfi and H. Tanabe[4]. Analogous results also hold in the case of multiple pole.
References


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