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REGULARITY AND SINGULARITY OF WEAK SOLUTIONS TO OSTWALD-DE WAELE FLOWS

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ABSTRACT. We find a regularity criterion for the Ostwald-de Waele models like Serrin's condition to the Navier-Stokes equations. Moreover, we show short time existence and estimate the Hausdorff dimension of the set of singular times for the weak solutions.

1. Introduction

In this paper, we study the regularity of the weak solutions of the pseudo-plastic Ostwald-de Waele non-Newtonian models:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \mu \dfrac{\partial \Gamma_{ij}}{\partial x_j} + f_i, \\ \dfrac{\partial u_j}{\partial x_j} = 0, \\ \Gamma_{ij} = |E(\nabla u)|^r E_{ij}(\nabla u), \\ E_{ij}(\nabla u) = \dfrac{1}{2}\Big(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\Big), \end{cases} \tag{1.1}$$

in $Q_T = \Omega \times (0,T)$, with the initial condition $u(x,0) = u_0(x)$ for $x \in \Omega$ and the periodic boundary condition, where $\Omega = [0,1]^3$ and $T > 0$ is a fixed number. If $r = 0$, then the models become the Navier-Stokes equations. If $r < 0$ then it is a pseudo-plastic fluid, and if $r > 0$ then it is a dilant fluid (see Böhme [2]). The values of the parameters μ, r of some of the pseudo-plastic Ostwald-de Waele models are given in Whitaker [11]. For example, for paper pulp $\mu = 0.418$, $r = -0.425$, and

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for carboxymethyl cellulose in water $\mu = 0.194$, $r = -0.434$. Since the viscosity can be treated by scaling, we simply assume that $\mu = 1$. Also for the simplicity we assume that f is a smooth function in Q_T.

We assume that any weak solution u,

$$u \in L^{r+2}(0,T;W^{1,r+2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$

satisfies

$$\iint u\cdot\phi_t - |E(\nabla u)|^r E(\nabla u)\cdot E(\nabla\phi) - (u\cdot\nabla)u\cdot\phi + p\nabla\cdot\phi + f\cdot\phi \, dx\, dt = 0$$

for all $\phi \in C^\infty(Q_T)$. Moreover we assume that u satisfies the energy estimate:

$$(1.2) \quad \sup_{0<t<T} \|u(t)\|^2 + \iint |\nabla u|^{r+2}\, dx\, dt \\ \leq C\|u_0\|^2 + C\int_0^T \|f\|_{W^{-1,\frac{r+2}{r+1}}(\Omega)}^{\frac{r+2}{r+1}}\, dt.$$

The existence of weak solutions of bipolar fluid for $-\frac{1}{5} < r < 0$ is given in Málek, Nečas, Rokyta and Růžička [6]. For $r > 0$, the existence of weak solutions is still open. For any $r > -1$, the regularity problem is also open.

From Sobolev's embedding theorem we know that the solution space $L^2(0,\infty;H^1(\Omega)) \cap L^\infty(0,\infty;L^2(\Omega))$ of weak solutions for the Navier-Stokes equations ($r = 0$) is continuously embedded in $L^{\frac{10}{3}}_{loc}(\Omega \times (0,\infty))$. But we do not know yet how to bound L^∞-norm of u in terms of $L^{\frac{10}{3}}$-norm of u. On the other hand, it is proved by Serrin [9] that any weak solution u of the Navier-Stokes equations ($r = 0$) on a cylinder $B \times (a,b)$ satisfying

$$\int_a^b \Big(\int_B |u|^\alpha\, dx \Big)^{\frac{\beta}{\alpha}} dt < \infty \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} < 1, \quad \alpha \geq 3$$

is necessarily L^∞ function on any compact subsets of the cylinder. Observe that when $\alpha = \beta = 5$, 5 is the critical number for the homogeneous Lebesgue space. The limiting case $3/\alpha + 2/\beta = 1, \alpha > 3$ for the initial value problem was considered by Fabes, Jones and Riviere, [4]. For more details on Serrin's condition, refer to Choe [3].

We define $L^{\alpha,\beta}$ as the set of measurable functions f satisfying

$$||f||_{L^{\alpha,\beta}} = \left(\int \left[\int |f|^{\alpha} dx \right]^{\frac{\beta}{\alpha}} dt \right)^{\frac{1}{\beta}} < \infty.$$

In Section 2 we find a regularity criterion to the pseudo-plastic Ostwald-de Waele models:
If $u \in L^{\alpha,\beta}(Q_T)$ for some (α, β) satisfying

$$\frac{3}{\alpha} + \frac{5r+4}{2\beta} \leq \frac{5r+2}{2},$$

with $\alpha > \frac{6}{2+5r}$, or if $u \in L^{\frac{6}{2-5r},\infty}(Q_T)$ and $\|u\|_{L^{\frac{6}{2-5r},\infty}} \leq \varepsilon_0$ for some small ε_0, then we have

$$u \in L^{r+2}(0,T; W^{2,r+2}(\Omega)) \cap L^{\infty}(0,T; W^{1,2}(\Omega)).$$

Observe that when $\alpha = \beta = \frac{5r+10}{5r+2}$ is the critical number for the homogeneous Lebesgue space. From Sobolev's embedding theorem we know that the solution space $L^{r+2}(0,T; W^{1,r+2}(\Omega)) \cap L^{\infty}(0,T; L^2(\Omega))$ of weak solutions is continuously embedded in $L^{\frac{5r-10}{3}}(Q_T)$. Theorem 2.3 is our main result in this section.

In Section 3, we find that there is a strong solution locally in time for $-\frac{1}{5} < r \leq 0$. Moreover, like in the case of the Navier-Stokes equations, we estimate the Hausdorff dimension of singular times. As is well known in the case of the Navier-Stokes equations, that is, $r = 0$, the dimension of singular times is less than or equal to $\frac{1}{2}$. Here we show that

$$\text{dimension of singular time } \leq \frac{2-5r}{4+5r}.$$

We note that when $r = 0$, our result agrees with known result for the Navier-Stokes equations(see [10]). Theorem 3.1 and Theorem 3.2 are our main results in this section.

2. Serrin's criterion for regularity

In this section, we show that a weak solution is strong if the velocity u satisfies a Serrin type condition. We let

$$\mathcal{V} \stackrel{\text{def}}{=} \{v \in C_0^\infty(\Omega)^3 : \text{ div } v = 0\},$$
$$\mathbf{V}_q \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } W^{1,q}(\Omega)^3,$$
$$\mathbf{H} \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3.$$

Let $\langle \cdot, \cdot \rangle$ is the usual inner product of $\mathbf{H}$. We will use the notations:

$$\|u\| \stackrel{\text{def}}{=} \|u\|_2 \stackrel{\text{def}}{=} \langle u, u\rangle^{1/2},$$
$$\|u\|_q \stackrel{\text{def}}{=} \|u\|_{L^q(\Omega)} \stackrel{\text{def}}{=} \Big(\int_\Omega |u|^q dx \Big)^{1/q}.$$

If $q = r + 2$, then $q' = \frac{r+2}{r+1}$ satisfies $1/q' + 1/q = 1$.

We remind Korn's inequality given in Nečas and Hlaváček [8] for $s = 2$, and in Mosolov and Mjasnikov [7] for $1 < s < \infty$,

$$\|\nabla v\|_s \leq C\Big(\int_\Omega |E_{ij}(\nabla v) E_{ij}(\nabla v)|^{s/2}\, dx \Big)^{1/s}. \tag{2.1}$$

The Korn's inequality is used for the proof of the energy estimate (1.2). We also recall the generalized form of Korn's inequality in Bellout, Bloom and Nečas [1]

$$\|u\|_{2,s}^s \leq C \int_\Omega \left| \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \right|^{s/2} dx \qquad \text{for } 1 < s < \infty. \tag{2.2}$$

Although the proof is a straightforward computation, the following lemma will be useful for the proof of the existence of weak solutions.

Lemma 2.1. *Let $-1 < r < 0$. Suppose $u \in W^{2,s}$ for $1 < s < 2$, then*

$$\int |\nabla^2 u|^s dx \leq \frac{s}{2(1+r)} \int_\Omega \partial_k(|E(\nabla u)|^r E_{ij}(\nabla u)) \partial_k E_{ij}(\nabla u)\, dx + C \int |\nabla u|^{-\frac{rs}{2-s}} dx$$

for some C.

Proof. Since $0 < s < 2$, from Hölder inequality and Young's inequality we obtain

$$\begin{aligned}
&\int |\nabla E(\nabla u)|^s dx \\
&= \int |E(\nabla u)|^{-\frac{rs}{2}} |E(\nabla u)|^{\frac{rs}{2}} |\nabla E(\nabla u)|^s dx \\
&\le \Big(\int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 dx \Big)^{\frac{s}{2}} \Big(\int |E(\nabla u)|^{-\frac{rs}{2-s}} dx \Big)^{\frac{2-s}{2}} \\
&\le \frac{s}{2} \int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 dx + \frac{2-s}{2} \int |E(\nabla u)|^{-\frac{rs}{2-s}} dx.
\end{aligned}$$

Then, Korn's inequality (2.2) for $1 < s < 2$ implies that

$$\int |\nabla^2 u|^s dx \le \frac{s}{2} \int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 dx + C \int |\nabla u|^{-\frac{rs}{2-s}} dx.$$

From direct calculations, we find that

$$\begin{aligned}
&\partial_k \big(|E(\nabla u)|^r E_{ij}(\nabla u) \big)\, \partial_k E_{ij}(\nabla u) \\
&= \partial_k \big((E_{l\ell}(\nabla u)\, E_{l\ell}(\nabla u))^{\frac{r}{2}}\, E_{ij}(\nabla u) \big)\, \partial_k E_{ij}(\nabla u) \\
&= |E(\nabla u)|^r \partial_k E_{ij}(\nabla u)\, \partial_k E_{ij}(\nabla u) \\
&\qquad + r |E(\nabla u)|^{r-2} E_{l\ell}(\nabla u)\, \partial_k E_{l\ell}(\nabla u)\, E_{ij}(\nabla u)\, \partial_k E_{ij}(\nabla u).
\end{aligned}$$

Hence we obtain that

$$|E(\nabla u)|^r |\nabla E(\nabla u)|^2 \le \frac{1}{1+r} \partial_k \big(|E(\nabla u)|^r E_{ij}(\nabla u) \big)\, \partial_k E_{ij}(\nabla u)$$

and this completes the proof. □

We note that the condition $1 < s < 2$ is crucial in the proof of Lemma 2.1. Taking $s = r + 2$ in Lemma 2.1, we have

$$\begin{aligned}
\int |\nabla^2 u|^{r+2} dx \le C \int_\Omega \partial_k (|E(\nabla u)|^r E_{ij}(\nabla u)) \partial_k E_{ij}(\nabla u)\, dx \\
+ C \int |\nabla u|^{r+2} dx.
\end{aligned}$$

Now we are ready to find an energy estimate for the velocity. By the inner product of (1.1) with Δu formally, we get

$$\frac{d}{dt} \|\nabla u\|^2 + \langle \nabla \big(|E|^r E_{ij}(\nabla u) \big), \nabla E_{ij}(\nabla u) \rangle + b(u, u, \Delta u) = -\langle f, \Delta u \rangle.$$

Hence, considering the identity in the proof of Lemma 2.1, we have

$$\frac{d}{dt}\|\nabla u\|^2 + C\int |E|^r |\nabla E_{ij}(\nabla u)|^2\, dx \le \int (u\cdot\nabla)u\cdot \Delta u\, dx + \big|\langle f, \Delta u\rangle\big|,$$

and integrating with respect to time we get

$$(2.3)\quad \|\nabla u(t)\|^2 + C\iint |E|^r|\nabla E_{ij}(\nabla u)|^2\, dx\, dt \le \iint (u\cdot\nabla)u\cdot\Delta u\, dx + \int \big|\langle f,\Delta u\rangle\big|\, dt + \|\nabla u_0\|^2.$$

From Hölder inequality we have that

$$\int \big|\langle f,\Delta u\rangle\big|\, dt \le \int \Big(\int |f|^{\frac{r-2}{r-1}}\, dx\Big)^{\frac{r-1}{r-2}} \int \Big(\int |\Delta u|^{r+2}\, dx\Big)^{\frac{1}{r-2}}.$$

Also, from the proof of Lemma 2.1 we have

$$\int |\nabla E(\nabla u)|^{r+2} dx \le \Big(\int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 dx\Big)^{\frac{r-2}{2}} \Big(\int |E(\nabla u)|^{r+2} dx\Big)^{-\frac{r}{2}}.$$

Then, by Korn's inequality (2.2), we have

$$(2.4)\quad \int |\nabla^2 u|^{r+2} dx \le \Big(\int |E(\nabla u)|^r|\nabla E(\nabla u)|^2\, dx\Big)^{\frac{r-2}{2}} \Big(\int |\nabla u|^{r+2}\, dx\Big)^{-\frac{r}{2}}.$$

Hence, combining all the previous estimates, we obtain

$$
\begin{aligned}
&\int \left|\langle f, \Delta u\rangle\right| dt \\
&\le \int \left[\left(\int |f|^{\frac{r-2}{r-1}}\, dx \right)^{\frac{r-1}{r-2}} \right. \\
&\qquad \left. \times \left(\int |E(\nabla u)|^r |\nabla E(\nabla u)|^2\, dx \right)^{\frac{1}{2}} \left(\int |\nabla u|^{r+2}\, dx \right)^{\frac{-r}{2(r-2)}} \right] dt \\
&\le \left\{ \int \left[\left(\int |f|^{\frac{r-2}{r-1}}\, dx \right)^{\frac{2(r-1)}{r-2}} \left(\int |\nabla u|^{r+2}\, dx \right)^{\frac{-r}{r-2}} \right] dt \right\}^{\frac{1}{2}} \\
&\qquad \times \left[\iint |E(\nabla u)|^r |\nabla E(\nabla u)|^2\, dx\, dt \right]^{\frac{1}{2}} \\
&\le \left[\iint |f|^{\frac{r-2}{r-1}}\, dx\, dt \right]^{\frac{r-1}{r-2}} \left[\iint |\nabla u|^{r+2}\, dx\, dt \right]^{\frac{-r}{2(r-2)}} \\
&\qquad \times \left[\iint |E(\nabla u)|^r |\nabla E(\nabla u)|^2\, dx\, dt \right]^{\frac{1}{2}} \\
&\le C \left[\iint |f|^{\frac{r-2}{r-1}}\, dx\, dt \right]^{\frac{r-1}{r-2}} \left[\iint |E(\nabla u)|^r |\nabla E(\nabla u)|^2\, dx\, dt \right]^{\frac{1}{2}}.
\end{aligned}
$$

In the last step, the a priori assumption $u \in L^{r+2}(0,T;\mathbf{V}_{r+2})$ is applied. Consequently, by Young's inequality, we have

$$
\begin{aligned}
\int \left|\langle f, \Delta u\rangle\right| dt \le C \left[\iint |f|^{\frac{r-2}{r-1}}\, dx\, dt \right]^{\frac{2(r-1)}{r-2}} \\
+ \varepsilon \iint |E(\nabla u)|^r |\nabla E(\nabla u)|^2\, dx\, dt.
\end{aligned}
\tag{2.5}
$$

We now consider the nonlinear convection term. The Serrin type condition is necessary to have a closed form of inequality of strong norms. Again, the proof is straight forward, but the computations are rather complicated because of inhomogeneous exponents. For the relevant estimates of the Navier-Stokes equations, we recall that Choe[3] considered a similar computation when $r = 0$.

LEMMA 2.2. *We let* $-\frac{2}{5} < r < 0$. *Suppose that* $u \in L^{\alpha,\beta}$ *such that* $\alpha \geq \frac{6}{2+5r}$ *and*

$$\frac{3}{\alpha} + \frac{5r+4}{2\beta} \leq \frac{5r+2}{2}, \tag{2.6}$$

we have

$$\iint |(u\cdot\nabla)u\cdot\Delta u|\,dx\,dt \leq C\|u\|_{L^{\alpha,\beta}} \times \sup\left(\int |\nabla u|^2\,dx\right)^{\frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}} \left(\iint |E|^r |\nabla E(\nabla u)|^2\,dx\,dt\right)^{\frac{(r-2)(\alpha-3)}{\alpha(5r-4)}}.$$

If $u \in L^{\frac{6}{2-5r},\infty}$, *then we have*

$$\iint |(u\cdot\nabla)u\cdot\Delta u|\,dx\,dt \leq C\|u\|_{L^{\frac{6}{2-5r},\infty}} \times \sup\left(\int |\nabla u|^2\,dx\right)^{\frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}} \left(\iint |E|^r |\nabla E(\nabla u)|^2\,dx\,dt\right)^{\frac{(r-2)(\alpha-3)}{\alpha(5r-4)}}.$$

Proof. We note that $\alpha > 1$ and hence we have

$$\int (u\cdot\nabla)u\cdot\Delta u\,dx \leq \left(\int |u|^\alpha\,dx\right)^{\frac{1}{\alpha}} \left(\int |\nabla u|^{\frac{\alpha}{\alpha-1}} |\Delta u|^{\frac{\alpha}{\alpha-1}}\,dx\right)^{\frac{\alpha-1}{\alpha}}.$$

Moreover knowing that $\alpha, \beta > 1$ and integrating in time, we obtain

$$\iint (u\cdot\nabla)u\cdot\Delta u\,dx\,dt \leq \left[\int \left(\int |u|^\alpha\,dx\right)^{\frac{\beta}{\alpha}} dt\right]^{\frac{1}{\beta}} \times \left[\int \left(\int |\nabla u|^{\frac{\alpha}{\alpha-1}} |\Delta u|^{\frac{\alpha}{\alpha-1}}\,dx\right)^{\frac{\alpha-1}{\alpha}\frac{\beta}{\beta-1}} dt\right]^{\frac{\beta-1}{\beta}}.$$

Since we are assuming $-\frac{2}{5} < r < 0$, we find that

$$\frac{r+2}{r+1} < \frac{3r+6}{4r+2} < \frac{6}{2+5r} \leq \alpha.$$

Thus, from careful adjustment of exponents, we get

$$
\begin{aligned}
\int |\nabla u|^{\frac{\alpha}{\alpha-1}} |\Delta u|^{\frac{\alpha}{\alpha-1}} \, dx &= \int |\nabla u|^{\frac{8\alpha r-6r-4\alpha-12}{(\alpha-1)(5r-4)}} |\nabla u|^{\frac{-3\alpha r-6r+12}{(\alpha-1)(5r-4)}} |\Delta u|^{\frac{\alpha}{\alpha-1}} \, dx \\
&\leq \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r-3r-2\alpha-6}{(\alpha-1)(5r-4)}} \left(\int |\nabla u|^{\frac{3(r-2)}{1-r}} \, dx \right)^{\frac{(1-r)(-\alpha r-2r-4)}{(\alpha-1)(5r-4)(r-2)}} \\
&\qquad \times \left(\int |\Delta u|^{r+2} \, dx \right)^{\frac{\alpha}{(\alpha-1)(r-2)}} \\
&\leq \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r-3r-2\alpha-6}{(\alpha-1)(5r-4)}} \left(\int |\nabla^2 u|^{r+2} \, dx \right)^{\frac{3(-\alpha r-2r-4)}{(\alpha-1)(5r-4)(r-2)}} \\
&\qquad \times \left(\int |\Delta u|^{r+2} \, dx \right)^{\frac{\alpha}{(\alpha-1)(r-2)}} \\
&\leq \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r-3r-2\alpha-6}{(\alpha-1)(5r-4)}} \left(\int |\nabla^2 u|^{r+2} \, dx \right)^{\frac{2(\alpha-3)}{(\alpha-1)(5r-4)}}.
\end{aligned}
$$

Hence, integrating in time and applying Hölder inequality in time integral, we have

$$
\begin{aligned}
&\left[\int \left(\int |\nabla u|^{\frac{\alpha}{\alpha-1}} |\Delta u|^{\frac{\alpha}{\alpha-1}} \, dx \right)^{\frac{\alpha-1}{\alpha} \frac{\beta}{\beta-1}} dt \right]^{\frac{\beta-1}{\beta}} \\
&\leq \left[\int \left(\int |\nabla u|^2 \, dx \right)^{\frac{\beta}{\beta-1} \frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}} \left(\int |\nabla^2 u|^{r+2} \, dx \right)^{\frac{2\beta}{\beta-1} \frac{\alpha-3}{\alpha(5r-4)}} dt \right]^{\frac{\beta-1}{\beta}} \\
&\leq \sup \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}} \left[\int \left(\int |\nabla^2 u|^{r+2} \, dx \right)^{\frac{2\beta}{\beta-1} \frac{\alpha-3}{\alpha(5r-4)}} dt \right]^{\frac{\beta-1}{\beta}}.
\end{aligned}
$$

If

$$
\frac{2\beta}{\beta-1} \frac{\alpha+3}{\alpha(5r+4)} \leq 1, \tag{2.7}
$$

then, we have

$$
\begin{aligned}
&\left[\int \left(\int |\nabla u|^{\frac{\alpha}{\alpha-1}} |\Delta u|^{\frac{\alpha}{\alpha-1}} \, dx \right)^{\frac{\alpha-1}{\alpha} \frac{\beta}{\beta-1}} dt \right]^{\frac{\beta-1}{\beta}} \\
&\qquad \leq \sup \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}} \left(\iint |\nabla^2 u|^{r+2} \, dx \, dt \right)^{\frac{2(\alpha-3)}{\alpha(5r-4)}}.
\end{aligned}
$$

By (2.4), we have

$$\left[\int \left(\int |\nabla u|^{\frac{\alpha}{\alpha-1}} |\Delta u|^{\frac{\alpha}{\alpha-1}} \, dx \right)^{\frac{\alpha-1}{\alpha} \frac{\beta}{\beta-1}} dt \right]^{\frac{\beta-1}{\beta}}$$
$$\leq \sup \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r - 3r - 2\alpha - 6}{\alpha(5r-4)}}$$
$$\times \left(\iint |E|^r |\nabla E(\nabla u)|^2 \, dx \, dt \right)^{\frac{(r-2)(\alpha-3)}{\alpha(5r-4)}} \left(\iint |\nabla u|^{r+2} \, dx \, dt \right)^{\frac{-r(\alpha-3)}{\alpha(5r-4)}}$$
$$\leq C \sup \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r - 3r - 2\alpha - 6}{\alpha(5r-4)}} \left(\iint |E|^r |\nabla E(\nabla u)|^2 \, dx \, dt \right)^{\frac{(r-2)(\alpha-3)}{\alpha(5r-4)}}.$$

Therefore, we have

$$\text{(2.8)} \quad \iint (u \cdot \nabla) u \cdot \Delta u \, dx \, dt \leq C \|u\|_{L^{\alpha,\beta}} \sup \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r - 3r - 2\alpha - 6}{\alpha(5r-4)}}$$
$$\times \left(\iint |E|^r |\nabla E(\nabla u)|^2 \, dx \, dt \right)^{\frac{(r-2)(\alpha-3)}{\alpha(5r-4)}}.$$

For the case $\alpha = \frac{6}{2+5r}$ and $\beta = \infty$, we may follow a similar way and omit the proof. □

The relation of the exponents can be given

$$\frac{4\alpha r - 3r + 2\alpha - 6}{\alpha(5r+4)} + \frac{(r+2)(\alpha+3)}{\alpha(5r+4)} = 1$$

and the condition (2.7) on α and β is equivalent to

$$\frac{3}{\alpha} + \frac{5r+4}{2\beta} \leq \frac{5r+2}{2}, \qquad \text{for } \alpha \geq \frac{6}{2+5r},$$

which is corresponds to Serrin's condition for the Navier-Stokes equations ($r = 0$):

$$\frac{3}{\alpha} + \frac{2}{\beta} \leq 1.$$

Now we are ready to prove our main result in this section which proves that if the velocity u satisfies a Serrin type condition (2.6), then u is a *strong solution*.

THEOREM 2.3. *Let* $-\frac{2}{5} < r \leq 0$. *Suppose that* $u_0 \in W^{1,2}(\Omega)$. *Let* u *be a weak solution of* (1.1). *If* $u \in L^{\alpha,\beta}(Q_T)$ *for some* (α, β) *satisfying* (2.6) *with* $\alpha > \frac{6}{2+5r}$, *then we have*

$$u \in L^{r+2}(0,T; W^{2,r+2}(\Omega)) \cap L^{\infty}(0,T; W^{1,2}(\Omega)) \tag{2.9}$$

and

(2.10)

$$\sup_t \int_\Omega |\nabla u(t)|^2 \, dx + \iint |\nabla^2 u|^{r+2} \, dx \, dt \leq C \left[\iint |f|^{\frac{r-2}{r-1}} \, dx \, dt \right]^{\frac{2(r-1)}{r-2}} + C \int \|f\|^{\frac{r-2}{r-1}}_{W^{-1,\frac{r-2}{r-1}}(\Omega)} \, dt + C\|\nabla u_0\|^2$$

for some C. *In case* $\alpha = \frac{6}{2+5r}$, *there is a number* ε_0 *such that if* $u \in L^{\frac{6}{2+5r},\infty}(Q_T)$ *and* $\|u\|_{L^{\frac{6}{2+5r},\infty}} \leq \varepsilon_0$, *then we have* (2.9) *and* (2.10).

REMARK. For $r \leq -\frac{1}{5}$, the existence of a weak solution is unknown yet. However, there exists a sequence $\{u^m\}$of Galerkin approximate solutions to (1.1). If $\{u^m\}$ are in $L^{\alpha,\beta}(Q_T)$, then there exist a limit u of a subsequence $\{u^{m'}\}$ of $\{u^m\}$, which becomes a strong solution.

Proof. Assume $\alpha > \frac{6}{2+5r}$. The case $\alpha = \frac{6}{2+5r}$ will be considered later. Combining (2.3), (2.5) and (2.8), we have that for t, $0 \leq t \leq T$,

$$\begin{aligned}
&\|\nabla u(t)\|^2 + C \iint |E|^r |\nabla E_{ij}(\nabla u)|^2 \, dx \, dt \\
&\quad \leq C\|u\|_{L^{\alpha,\beta}} \sup \left(\int |\nabla u|^2 \, dx \right)^{\frac{4\alpha r - 3r - 2\alpha - 6}{\alpha(5r-4)}} \\
&\qquad \times \left(\iint |E|^r |\nabla E(\nabla u)|^2 \, dx \, dt \right)^{\frac{(r-2)(\alpha-3)}{\alpha(5r-4)}} \\
&\quad + C \left[\iint |f|^{\frac{r-2}{r-1}} \, dx \, dt \right]^{\frac{2(r-1)}{r-2}} + \|\nabla u_0\|^2.
\end{aligned}$$

We know that L^p norm is absolutely continuous with respect to Lebesgue measure. Consequently, for any given ε we can choose a sequence of time

$$\{0, T_1, T_2, \cdots, T_i, T_{i+1}, \cdots, T_m\}$$

such that

$$\int_{T_i}^{T_{i+1}} \left[\int_\Omega |u|^\alpha \, dx \right]^{\frac{\beta}{\alpha}} dt \le \varepsilon$$

for all i, where $T_m = T$. If we consider the time integration on $[T_i, T_{i+1}]$, we have

$$\begin{aligned} \sup_{T_i \le t \le T_{i+1}} \int_\Omega |\nabla u(t)|^2 \, dx &+ C \int_{T_i}^{T_{i+1}} \int_\Omega |E|^r |\nabla E_{ij}(\nabla u)|^2 \, dx \, dt \\ &\le \varepsilon C \left[\sup_{T_i \le t \le T_{i+1}} \int |\nabla u|^2 \, dx + \int_{T_i}^{T_{i+1}} \int |E|^r |\nabla E(\nabla u)|^2 \, dx \, dt \right] \\ &\quad + C \left[\int_{T_i}^{T_{i+1}} \int |f|^{\frac{r-2}{r-1}} \, dx \, dt \right]^{\frac{2(r-1)}{r-2}} + \|\nabla u_0\|^2. \end{aligned}$$

If ε is small enough, then we get

$$\begin{aligned} (2.11) \quad \sup_{T_i \le t \le T_{i+1}} \int_\Omega |\nabla u(t)|^2 \, dx &+ C \int_{T_i}^{T_{i+1}} \int_\Omega |E|^r |\nabla E_{ij}(\nabla u)|^2 \, dx \, dt \\ &\le C \left[\int_{T_i}^{T_{i+1}} \int |f|^{\frac{r-2}{r-1}} \, dx \, dt \right]^{\frac{2(r-1)}{r-2}} + \|\nabla u_0\|^2. \end{aligned}$$

Also, from Lemma 2.1, we have

$$\begin{aligned} &\sup_{T_i \le t \le T_{i+1}} \int_\Omega |\nabla u(t)|^2 \, dx + C \int_{T_i}^{T_{i+1}} \int_\Omega |\nabla^2 u|^{r+2} \, dx \, dt \\ &\le C \left[\int_{T_i}^{T_{i+1}} \int |f|^{\frac{r-2}{r-1}} \, dx \, dt \right]^{\frac{2(r-1)}{r-2}} + C \int_{T_i}^{T_{i+1}} \int_\Omega |\nabla u|^{r+2} \, dx \, dt + \|\nabla u_0\|^2. \end{aligned}$$

Therefore, iterating on i and using the energy estimate (1.2), we have

$$\begin{aligned} &\sup_t \int_\Omega |\nabla u(t)|^2 \, dx + \iint |\nabla^2 u|^{r+2} \, dx \, dt \\ &\qquad \le C \left[\iint |f|^{\frac{r-2}{r-1}} \, dx \, dt \right]^{\frac{2(r-1)}{r-2}} + C \int \|f\|_{W^{-1,\frac{r-2}{r-1}}(\Omega)}^{\frac{r-2}{r-1}} \, dt + C \|\nabla u_0\|^2 \end{aligned}$$

for some C.

The case $\in L^{\frac{6}{2-5r},\infty}$ can be treated from the estimate

$$\begin{aligned}
&\|u\|_{L^{\alpha,\beta}} \sup\Big(\int |\nabla u|^2\,dx\Big)^{\frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}}\\
&\qquad\times\Big(\iint |E|^r|\nabla E(\nabla u)|^2\,dx\,dt\Big)^{\frac{(r-2)(\alpha+3)}{\alpha(5r-4)}}\\
&\le \varepsilon_0 \sup\Big(\int |\nabla u|^2\,dx\Big)^{\frac{4\alpha r-3r-2\alpha-6}{\alpha(5r-4)}}\\
&\qquad\times\Big(\iint |E|^r|\nabla E(\nabla u)|^2\,dx\,dt\Big)^{\frac{(r-2)(\alpha+3)}{\alpha(5r-4)}}.
\end{aligned}$$

□

3. Strong solutions and time singularity

We now show the short time regularity. Suppose that f is independent of time t. Let u be a weak solution. We follow the idea used in Foias, Guillopé and Temam [5], and in Bellout, Bloom and Nečas [1]. Consider the inner product of (1.1) with

$$\frac{-\partial_k^2 u}{(1+\|\nabla u\|^2)^\lambda},$$

where $\lambda > 1$ will be a number determined later. Therefore, from integration by parts, we have

$$\begin{aligned}
&\frac{1}{2(1-\lambda)}\partial_t(1+\|\nabla u\|^2)^{1-\lambda} + \frac{C}{(1+\|\nabla u\|^2)^\lambda}\int \partial_k(|E|^r E)\partial_k E\,dx\\
&\qquad\le \frac{1}{(1+\|\nabla u\|^2)^\lambda}\Big(\int |\nabla u|^3 + |f|^{\frac{r-2}{r-1}} + \varepsilon|\nabla^2 u|^{r+2}\,dx\Big).
\end{aligned}$$

Integrating the previous inequality with respect to t, we obtain

$$\begin{aligned}
&\frac{1}{2(1-\lambda)}\Big((1+\|\nabla u(T)\|^2)^{1-\lambda} - (1+\|\nabla u(0)\|^2)^{1-\lambda}\Big)\\
&\qquad + C\iint \frac{\partial_k(|E|^r E)\partial_k E}{(1+\|\nabla u\|^2)^\lambda}\,dx\,dt\\
&\qquad\le \iint \frac{1}{(1+\|\nabla u\|^2)^\lambda}\Big(|\nabla u|^3 + |f|^{\frac{r-2}{r-1}} + \varepsilon|\nabla^2 u|^{r+2}\Big)\,dx\,dt.
\end{aligned}$$

Now we need to estimate the second derivative term. Considering Lemma 2.1 and taking ε small, we have

$$\begin{aligned}\frac{1}{2(1-\lambda)}\Big((1+\|\nabla u(T)\|^2)^{1-\lambda}-(1+\|\nabla u(0)\|^2)^{1-\lambda}\Big)\\ +C\iint\frac{\partial_k(|E|^r E)\partial_k E}{(1+\|\nabla u\|^2)^\lambda}\,dx\,dt\\ \le\iint\frac{|\nabla u|^3+|f|^{\frac{r-2}{r-1}}+\varepsilon|\nabla u|^{r+2}}{(1+\|\nabla u\|^2)^\lambda}\,dx\,dt\\ \le\iint\frac{|\nabla u|^3}{(1+\|\nabla u\|^2)^\lambda}\,dx\,dt+\varepsilon\int\frac{\left(\int|\nabla u|^2\,dx\right)^{\frac{r-2}{2}}}{(1+\|\nabla u\|^2)^\lambda}\,dx\,dt+Ct\\ \le\iint\frac{|\nabla u|^3}{(1+\|\nabla u\|^2)^\lambda}\,dx\,dt+Ct \qquad (3.1)\end{aligned}$$

since $\lambda > 1$ and f is smooth. In fact, with a little careful computations we can allow less regularity on f, but the proof will be natural.

We now consider the convection term. We restrict r to the case $-1/5 < r \le 0$. Since we are considering periodic functions, ∇u is average free and hence we have Sobolev inequality controlled by second derivatives only. With this observation and Hölder inequality, we obtain

$$\begin{aligned}\int|\nabla u|^3\,dx&=\int|\nabla u|^{\frac{12r-6}{5r-4}+\frac{3r-6}{5r-4}}\,dx\\ &\le\Big(\int|\nabla u|^2\,dx\Big)^{\frac{6r-3}{5r-4}}\Big(\int|\nabla u|^{\frac{3(r-2)}{1-r}}\,dx\Big)^{\frac{1-r}{5r-4}}\\ &\le\Big(\int|\nabla u|^2\,dx\Big)^{\frac{6r-3}{5r-4}}\Big(\int|\nabla^2 u|^{r+2}\,dx\Big)^{\frac{3}{5r-4}}.\end{aligned}$$

By Lemma 2.1, we can estimate L^{r+2} norm of the second derivatives in terms of nonlinear energy functions and we have for any given small ε,

$$\begin{aligned}&\int|\nabla u|^3\,dx\\ &\le\Big(\int|\nabla u|^2\,dx\Big)^{\frac{6r-3}{5r-4}}\Big(\int|E|^r|\nabla E(\nabla u)|^2\,dx\Big)^{\frac{6-3r}{10r-8}}\Big(\int|\nabla u|^{r+2}\,dx\Big)^{\frac{-3r}{10r-8}},\end{aligned}$$

and

$$\int |\nabla u|^3\,dx \le \frac{C}{\varepsilon}\Big(\int |\nabla u|^2\,dx\Big)^{\frac{12r-6}{7r-2}}\Big(\int |\nabla u|^{r+2}\,dx\Big)^{\frac{-3r}{7r-2}} + \varepsilon\int |E|^r|\nabla E(\nabla u)|^2\,dx. \tag{3.2}$$

For $0 \ge r > -1/5$, one has $0 \le \frac{-3r}{7r+2} < 1$.

We define

$$\lambda \overset{\text{def}}{=} \frac{12r+6}{7r+2},$$

and set

$$A(t) \overset{\text{def}}{=} 1 + \|\nabla u(t)\|^2.$$

If we take ε small, from (3.1) we obtain

$$\begin{aligned}
&-\frac{7r+2}{10r+8}\Big(A(t)^{-\frac{5r-4}{7r-2}} - A(0)^{-\frac{5r-4}{7r-2}}\Big) \\
&\qquad + C\int \frac{1}{A^\lambda}\int |E|^r|\nabla E(\nabla u)|^2\,dx\,dt \\
&\le C\int_0^t \Big(\int |\nabla u|^{r+2}\,dx\Big)^{\frac{-3r}{7r-2}} dt + Ct \\
&\le C\Big(\iint |\nabla u|^{r+2}\,dx\,dt\Big)^{\frac{-3r}{7r-2}} t^{\frac{10r-2}{7r-2}} + Ct \\
&\le C\big(t^{\frac{10r-2}{7r-2}} + t\big).
\end{aligned}$$

Hence we can estimate $A(t)$ in terms of time so that

$$A(t) \le \Big[A(0)^{-\frac{5r-4}{7r-2}} - C\frac{10r+8}{7r+2}\big(t^{\frac{10r-2}{7r-2}} + t\big)\Big]^{-\frac{7r-2}{5r-4}}.$$

Hence, there is a time T_0 depending on u_0, f and δ such that

$$\sup_{0\le t\le T_0} \|\nabla u(t)\| \le C(u_0, f, \delta),$$

and

$$\int_0^{T_0}\int |\nabla^2 u|^{r+2}\,dx\,dt \le C(u_0, f, \delta),$$

where $C(u_0, f, \delta)$ depends on u_0, f and δ. Indeed, we can explicitly compute T_0 in terms of $A(0)$ and we state this fact as a theorem of the short time regularity.

THEOREM 3.1. *Let $-\frac{1}{5} < r \leq 0$. Suppose that*

$$A(0) \stackrel{\text{def}}{=} \int_{\Omega} \left(1 + |\nabla u(0)|^2\right) dx < \infty,$$

then there is a strong solution on $(0, T_0)$, for all T_0 satisfying

$$T_0^{\frac{10r-2}{7r-2}} + T_0 \leq C A(0)^{-\frac{5r-4}{7r-2}}$$

for some C.

We now estimate Hausdorff dimension of the set of singular times. When $r = 0$, it is known that the dimension of set of singular times is less than or equal to $\frac{1}{2}$. In particular, we refer to Ch. 5 of Temam [10] for a detailed proof for the case of $r = 0$.

We may assume that $T < 1$. Once we estimated the life time interval of strong solution, we may follow a known method for Navier-Stokes equations to estimate the Hausdorff dimension of singular time. We let

$$\mathcal{O} \stackrel{\text{def}}{=} \{t < T : \int |\nabla u|^2 \, dx(t) < \infty\},$$

then $\mathcal{O}$ is right open from Theorem 3.1. So $\mathcal{O}$ is the countable union of semi-open intervals, say,

$$\mathcal{O} = \cup[a_i, b_i).$$

In particular, we set the open set

$$\mathcal{O}_1 = \cup(a_i, b_i),$$

then $\mathcal{S} \stackrel{\text{def}}{=} [0, T] \setminus \mathcal{O}_1$ is closed and has Lebesgue measure zero. Let $t \in (a_i, b_i)$. Then, by Theorem 3.1, we have

$$(b_i - t)^{\frac{10r-2}{7r-2}} + (b_i - t) \geq \frac{C}{\left(1 + \|u(t)\|^2(t)\right)^{\frac{4-5r}{7r-2}}}.$$

Since $0 < \frac{10r+2}{7r+2} < 1$, we have $(b_i - t)^{\frac{10r-2}{7r-2}} \geq (b_i - t)$, and

$$2(b_i - t)^{\frac{10r-2}{5r-4}} \geq \frac{C}{1 + \|u(t)\|^2(t)}.$$

Hence,

$$(b_i - t)^{-\frac{10r-2}{5r-4}} \leq C\left(1 + \|u(t)\|^2(t)\right).$$

Integrating this from a_i to b_i, we have

$$(b_i - a_i)^{\frac{2-5r}{5r-4}} \le C\Big(b_i - a_i + \int_{a_i}^{b_i} \|u(t)\|^2\,dt\Big).$$

Therefore, we have

$$\sum (b_i - a_i)^{\frac{2-5r}{5r-4}} \le C\Big(T + \int_0^T \|u(t)\|^2\,dt\Big) < \infty.$$

For every $\varepsilon > 0$, we can find a finite part I_ε of I such that

$$\sum_{i \notin I_\varepsilon} (\beta_i - \alpha_i) \le \varepsilon, \qquad \sum_{i \notin I_\varepsilon} (\beta_i - \alpha_i)^{\frac{2-5r}{4-5r}} \le \varepsilon.$$

The set $[0,T] \setminus \cup_{i \in I_\varepsilon}(a_i, b_i)$ is the union of finite number of mutually disjoint closed interval, say, B_j, $j = 1, \cdots, N$. It is clear that $\cup_{j=1}^N B_j \supset \mathcal{S}$. Since (α_i, β_i) are mutually disjoint, (α_i, β_i) is contained in one and only one interval B_i. We denote I_j the set of i's such that $B_j \supset (\alpha_i, \beta_i)$. It is clear that I_ε, $I_i, \cdots, I_N$ is a partition of I and that

$$B_j = \Big(\cup_{i \in I_j} (\alpha_i, \beta_i)\Big) \cup \Big(B_j \cap \mathcal{S}\Big), \qquad \text{for all } j.$$

Hence

$$\operatorname{diam} B_j = \sum (\beta_i - \alpha_i) \le \varepsilon,$$

and

$$\begin{aligned}
\Big(dH^{\frac{2-5r}{4-5r}}\Big)(\mathcal{S}) &\le \sum_{j=1}^{N} \Big(\operatorname{diam} B_j\Big)^{\frac{2-5r}{4-5r}} \\
&\le \sum_{j=1}^{N} \Big(\sum_{i \in I_j} (\beta_i - \alpha_i)\Big)^{\frac{2-5r}{4-5r}} \\
&\le \sum_{i \notin I_\varepsilon} (\beta_i - \alpha_i)^{\frac{2-5r}{4-5r}} \\
&\le \varepsilon,
\end{aligned}$$

where dH^k is the k-dimensional Hausdorff measure. Since ε is chosen arbitrarily, we conclude:

THEOREM 3.2. *Let u be a weak solution, $r \in (-\frac{1}{5}, 0]$. Then there exists a closed set $\mathcal{S} \subset [0,T]$, whose $\frac{2-5r}{4+5r}$-dimensional* Hausdorff *measure vanishes, such that u is continuous from $[0,T] \setminus \mathcal{S}$ into $\mathbf{V}_2$.*

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