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# STANDING WAVE SOLUTIONS FOR THE PLANER CHERN-SIMONS GAUGED NONLINEAR SCHRÖDINGER EQUATION WITH AN EXTERNAL ELECTROMAGNETIC FIELD

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Abstract. In this paper we construct a standing solitary wave solution with prescribed total electric charge to the planer Chern-Simons gauged nonlinear Schrödinger equation with an external electromagnetic field by using a variational method.

## 1. Introduction and main result

The Chern-Simons theories exist in three-dimensional space-time and believed to be relevant to quantum Hall effect and high-$T_c$ superconductivity(see, e.g., [13], [7] and the references cited therein). The dynamics of the planer nonrelativistic Chern-Simons model is governed by the following gauged nonlinear Schrödinger equation ([11]):

$$iD_t\psi = -\frac{1}{2}\mathbf{D}^2\psi - g|\psi|^2\psi \quad in \quad \mathbf{R}^2 \times (-\infty, +\infty), \tag{1}$$

where $\psi$ is a scalar field, $D_t = \partial_t + iA^0 + iV_e, \mathbf{D} = \nabla - i\mathbf{A} - i\mathbf{A}_e$ with

$$\mathbf{A}(x,t) = \mathbf{A}(\psi)(x,t) = \frac{1}{\kappa}\int_{\mathbf{R}^2} \mathbf{G}(x-y)\rho(y,t)\,dy \tag{2}$$

$$A^0(x,t) = A^0(\psi)(x,t) = \frac{1}{\kappa}\int_{\mathbf{R}^2} \mathbf{G}(x-y)\cdot \mathbf{j}(y,t)\,dy \tag{3}$$

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and

$$\rho(x,t) = |\psi(x,t)|^2, \quad \mathbf{j}(y,t) = \mathrm{Im}(\psi^* \mathbf{D}\psi). \tag{4}$$

Here $g > 0$ and $\kappa > 0$ are positive constants, $\psi^*$ is the complex conjugate of $\psi$, $\mathbf{D}_A = \nabla - i\mathbf{A}$, $V_e(x)$ and $\mathbf{A}_e$ are external electromagnetic potentials, and $\mathbf{G}(x)$ is defined by

$$\mathbf{G}(x) = (G^1(x), G^2(x)), \quad G^i = \frac{1}{2\pi}\epsilon^{ij}\nabla_j \log|x|,$$

where $\epsilon^{ij}$ is the totally skew-symmetric tensor with $\epsilon^{12} = 1$. Namely,

$$G^1(x) = \frac{x_2}{2\pi|x|^2}, \qquad G^2(x) = -\frac{x_1}{2\pi|x|^2}.$$

Note that for a function $\rho$ with a nice decay property,

$$B = \nabla \times \mathbf{A}(\phi) = \partial_{x_1} A^2(\phi) - \partial_{x_2} A^1(\phi) = -\kappa^{-1}\rho$$

holds. The equation (1) comes from the Euler-Lagrange equation of the Lagrangian density:

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + i\psi^* D_t \psi - \frac{1}{2}|\mathbf{D}\psi|^2 + \frac{g}{2}|\psi|^4$$

under the relativistic notation with the background Minkowski metric $(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1)$, where $F_{\beta\gamma} = \partial_\beta A_\gamma - \partial_\gamma A_\beta$ and $\epsilon^{\alpha\beta\gamma}$ is the totally skew-symmetric tensor with $\epsilon^{012} = 1$. See [13] for more details on this non-relativistic Chern-Simons model.

Throughout this paper, we also use the following notations:

$$\mathbf{D}_e = \nabla - i\mathbf{A}_e, \quad \mathbf{D}_A = \nabla - i\mathbf{A}, \quad \mathbf{A}_e = (A_e^1, A_e^2), \quad B_e = \partial_{x_1} A_e^2 - \partial_{x_2} A_e^1.$$

We study a standing wave solution to (1). Hence, substituting $\psi(x,t) = e^{-iEt}\phi(x)$, we obtain the time-independent equation for static solution $\phi$:

$$-\frac{1}{2}\mathbf{D}^2\phi + A^0\phi + V_e\phi - g|\phi|^2\phi = E\phi, \tag{5}$$

where $\mathbf{A}$ and $A^0$ are defined by (2)-(3) with

$$\rho(x) = |\phi(x)|^2, \quad \mathbf{j}(x) = \mathrm{Im}(\phi^* \mathbf{D}\phi). \tag{6}$$

The static solutions to (1) without external electromagnetic fields was studied on the self-dual case, namely $g = 1/\kappa$( see [11]). The speciality of the self-dual case comes from the identity (see [13]):

$$|\mathbf{D}\phi|^2 = |(D_1 - iD_2)\phi|^2 - ((B + B_e)\rho + \nabla \times \mathbf{j}).$$

The energy function for static solutions is written by

$$J(\phi) = \frac{1}{4}\int_{\mathbf{R}^2} |\mathbf{D}\phi|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^2} V_e|\phi|^2\,dx - \frac{g}{4}\int_{\mathbf{R}^2} |\phi|^4\,dx.$$

Then, we can write $J(\phi)$ as

$$J(\phi) = \frac{1}{4}\int_{\mathbf{R}^2}\left(|(D_1 - iD_2)\phi|^2 - (g - \frac{1}{\kappa})|\phi|^4\right)dx + \int_{\mathbf{R}^2}(\frac{1}{2}V_e - \frac{1}{4}B_e)|\phi|^2\,dx,$$

where we used $\nabla \times \mathbf{A}(\phi) = -\kappa\rho = -\kappa|\phi|^2$ and $\int_{\mathbf{R}^2}\nabla\times\mathbf{j}\,dx$ is dropped with the hypothesis that $\mathbf{j}$ decays sufficiently at infinity, e.g., $|\mathbf{j}(x)| \le C|x|^{-(1+\delta)}$ for some $\delta > 0$. So, if there exists no external field and $g < 1/\kappa$, then the minimal energy configuration is only a trivial one $\phi \equiv 0$. On the other hand, in the self-dual case $g = 1/\kappa$, the energy minimizing configuration requires $\phi$ to satisfy

$$(D_1 - iD_2)\phi = 0$$

with $\nabla \times \mathbf{A}(\phi) = -\kappa|\phi|^2$. It is known there are non-trivial zero-energy self-dual solutions which are explicitly solved (see [11, 13]).

By the continuity equation $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$, it follows that

$$\frac{d}{dt}\int_{\mathbf{R}^2}\rho(x,t)\,dx = 0.$$

Hence, the total electric charge $\int|\psi(x,t)|^2\,dx$ is conserved in this model. So, it is natural to consider under the constraint

$$\int_{\mathbf{R}^2}|\phi(x)|^2\,dx = N. \tag{7}$$

The purpose of this paper is to construct standing wave solutions with prescribed total electric charge $\int|\phi|^2\,dx = N$ under certain external electromagnetic fields $V_e$ and $\mathbf{A}_e$, not necessary in the self-dual case, by using a variational method.

First we note that by its definition $\mathbf{A} = \mathbf{A}(\phi)$ satisfies

$$\nabla\cdot\mathbf{A}(\phi) = 0 \tag{8}$$

in the distribution sense. Under the constraint (7) and the hypothesis of nice decay of functions $\rho$ and $\mathbf{j}$, it follows

$$\lim_{|x|\to\infty}|x|A^i(x) = \frac{1}{2\pi\kappa}\epsilon^{ij}\hat{x}^jN, \tag{9}$$

$$\lim_{|x|\to\infty}A^0(x) = 0, \tag{10}$$

where $\hat{x}^j = x^j/|x|$. For external electromagnetic fields $V_e(x)$ and $\mathbf{A}_e$, throughout this paper, we assume

$$V_e(x) \geq -\alpha, \quad \mathbf{A}_e \in C^1, \quad B_e(x) \leq 0 \tag{11}$$

for some positive constant $\alpha$. We denote by $X$ the completion of $C_0^\infty(\mathbf{R}^2)$ with respect to the norm

$$\|\phi\|_X^2 = \int_{\mathbf{R}^2} (|\mathbf{D}_e u|^2 + (V_e + \alpha + 1)|\phi|^2)\, dx.$$

Note that since it is known the pointwise inequality( see, e.g., [19])

$$|\nabla|\phi|(x)| \leq |\mathbf{D}_e \phi(x)| \ a.e.\ x, \tag{12}$$

$\phi \in X$ implies that $|\phi|$ belongs to the usual Sobolev space $H^1(\mathbf{R}^2)$ because of the assumption (11). We consider the energy

$$J(\phi) = \frac{1}{4}\int_{\mathbf{R}^2} |\mathbf{D}\phi|^2\, dx + \frac{1}{2}\int_{\mathbf{R}^2} V_e|\phi|^2\, dx - \frac{g}{4}\int_{\mathbf{R}^2} |\phi|^4\, dx \tag{13}$$

and consider the minimizing problem

$$\Sigma = \inf_{\phi \in \mathcal{A}_N} J(\phi), \tag{14}$$

where

$$\mathcal{A}_N = \{\phi \in X; \int_{\mathbf{R}^2} |\phi|^2\, dx = N\}.$$

We denote by $C_0$ the best constant of the Gagliardo-Nirenberg inequality:

$$\|u\|_4^4 \leq C_0 \|u\|_2^2 \|\nabla u\|_2^2 \tag{15}$$

for real-valued $u \in H^1(\mathbf{R}^2)$, where $\|u\|_p = \|u\|_{L^p(\mathbf{R}^2)}$ for $1 \leq p < +\infty$. We state our result.

THEOREM 1. *Fix* $N > 0$. *Assume* (11) *and either*
(a) *Assume* $gC_0N < 1$ *and* $\lim_{|x|\to\infty}(V_e(x) - B_e(x)) = +\infty$
*or*
(b) $g\kappa < 1$ *and* $\lim_{|x|\to\infty}(V_e(x) - (g\kappa/2)B_e(x)) = +\infty$.
*Then there exists a minimizer* $\phi \in \mathcal{A}_N$ *to* $\Sigma$.

We do not know uniqueness of the minimizer $\phi \in \mathcal{A}_N$ for a fixed $N$. The following proposition yields the existence of the standing solitary wave solution to (1).

THEOREM 2. *Let* $\phi \in \mathcal{A}_N$ *be a minimizer to* $\Sigma$. *Then,*

(a) *$\phi$ satisfies for some $E \in \mathbf{R}$ the equation* (5);

(b) *$\phi$ is Hölder continuous function and satisfies $|\phi(x)| \to 0$ as $|x| \to \infty$;*

(c) *$\phi$ satisfies the following formulas:*

$$\int_{\mathbf{R}^2} B\,dx = -N/\kappa,$$

$$\lim_{|x|\to\infty} |x| A^i(\phi)(x) = \frac{1}{2\pi\kappa}\epsilon^{ij}\hat{x}^j N,$$

*where $\hat{x}_j = x_j/|x|$.*

The first formula in Theorem 2 (c) is relevant in this model. It says that total magnetic flux is comparable to the total electric charge.

For the self-dual case, in [6], [12] some time-dependent soliton solutions was constructed in the presence of an external harmonic force (i.e. $V_e(x) = \omega|x|^2$ for some constant $\omega > 0$) or an external constant magnetci field $B_e$ by using static solutions and special transformations.

For general case (not necessary the self-dual case) with no external field, Berge-de Bouard-Saut studied in [6] the existence of time-dependent solution to (1) locally and globally in time and blow-up phenomenon in certain cases. Among them, they showed global existence of $H^1$ solution in time to (1) with initial data $\psi_0$ satisfying $\|\psi_0\|_2^2 < 1/gC_0$ which is the same condition as the case (a) in Theorem 1.

However, so far there seems no work dealing with existence of standing solitary wave solutions with prescribed total electric charge under external fields even for the self-dual case, as far as I know.

REMARK 1. The assumptions on the growth of external fields in Theorem 1 can be relaxed slightly (see, e.g., [3]). Because this assumption is used only to assure compactness property for a bounded sequence of $X$. Moreover, we can allow degenerately growing potentials, i.e. potentials which do not tend to infinity in all direction as $|x| \to \infty$, if we use the inequality as in [20, 21] ( see also [16, 17]).

REMARK 2. In this paper, I have not tried to obtain the global boundedness and its decay property of $|\mathbf{j}(x)|$. If one know its global boundedness and $L^1$ integrability or $L^4$-integrability of $|\mathbf{D}\phi|$, then one can show $A^0(x) \to 0$ as $|x| \to \infty$ and an exponential decay property for $|\phi|$ by the standard subsolution estimate and the comparison argument. Moreover, that would also implies an exponential decay property of $|\mathbf{j}(x)|$. This should be treated rigorously in the future.

Throughout this paper the integration is understood over $\mathbf{R}^2$.

## 2. Proof of Theorem 1

To show Theorem 1, first we collect and prepare several inequalities. For $r > 1$, define the fractional integral operator $I_r$:

$$(I_r f)(x) = \int_{\mathbf{R}^2} \frac{f(y)}{|x-y|^{2/r}}\, dy.$$

The following inequality is well-known as Hardy-Littlewood-Sobolev's inequality. See, e.g., [So, Theorem 0.3.2] for its proof.

LEMMA 1. *Suppose $r > 1, 1 < p < q < +\infty$ and $1/r = 1-(1/p-1/q)$. Then there exists a constant $C$ depending only on $p$ and $q$ such that*

$$\|I_r f\|_q \leq C\|f\|_p.$$

LEMMA 2. *Assume $A_e \in C^1, B_e(x) \leq 0$ and let $\phi \in X$. Then we have*

$$\|\frac{1}{|x|} * |\phi|^2\|_4 \leq C_1\|\phi\|_{8/3}^2; \tag{16}$$

$$\|\phi\|_4^4 \leq C_0\|\phi\|_2^2\|\mathbf{D}_e\phi\|_2^2; \tag{17}$$

$$\int_{\mathbf{R}^2} |\mathbf{D}_e\phi|^2\, dx \geq \int_{\mathbf{R}^2} (-B_e)|\phi|^2\, dx; \tag{18}$$

*Proof.* (16) is a consequence of Lemma 1. (17) is a consequence of the Gagliardo-Nirenberg inequality and pointwise estimate (12). (18) is due to [1, Theorem 2.9]. □

LEMMA 3. *Assume $A_e \in C^1, B_e(x) \leq 0$ and let $\phi \in X$. Then for $\mathbf{A}(\phi)$ we have following inequalities:*

$$\|\mathbf{A}(\phi)\|_4^2 \leq \left(\frac{C_1}{2\pi\kappa}\right)^2 \|\phi\|_4^3\|\phi\|_2, \quad \|\mathbf{A}(\phi)\phi\|_2^2 \leq (\frac{C_1}{2\pi\kappa})^2\|\phi\|_4^5\|\phi\|_2; \tag{19}$$

$$\|\mathbf{A}(\phi_1)-\mathbf{A}(\phi_2)\|_4^2 \le C(\max(\|\phi_1\|_X,\|\phi_2\|_X)^3\|\phi_1-\phi_2\|_2; \tag{20}$$

$$\|\phi\|_4^4 \le C_0\|\phi\|_2^2\|\mathbf{D}\phi\|_2^2; \tag{21}$$

$$\int_{\mathbf{R}^2}|\mathbf{D}\phi|^2\,dx \ge \int_{\mathbf{R}^2}\left(-B_e+\frac{|\phi|^2}{\kappa}\right)|\phi|^2\,dx. \tag{22}$$

*Proof.* By Hölder's inequality we have $\|\phi\|_{8/3}^2 \le \|\phi\|_4\|\phi\|_2$. By using the definition of $\mathbf{A}(\phi)$ and Lemma 2 we obtain the estimate for $\|\mathbf{A}(\phi)\|_4$. The estimate for $\|\mathbf{A}(\phi)\phi\|_2$ follows from Hölder's inequality. The proof of (21) is the same as in (17), since we know (12) also holds since $\mathbf{A}(\phi)\in L^4(\mathbf{R}^2)$. Note that

$$\begin{aligned}
\|\mathbf{A}(\phi_1)-\mathbf{A}(\phi_2)\|_4 &\le \frac{1}{2\pi\kappa}\|\int\frac{1}{|x-y|}||\phi_1|^2(y)-|\phi_2|^2(y)|\,dy\|_4\\
&\le C\|\frac{1}{|x|}*||\phi_1|-|\phi_2||(|\phi_1|+|\phi_2|)\|_4\\
&\le C\|(|\phi_1|-|\phi_2|)(|\phi_1|+|\phi_2|)\|_{4/3}\\
&\le C\|\phi_1-\phi_2\|_{8/3}(\|\phi_1\|_{8/3}+\|\phi_2\|_{8/3}).
\end{aligned}$$

By using Hölder's inequality and Lemma 2, we have $\|\phi_1-\phi_2\|_{8/3}\le C\|\phi_1-\phi_2\|_X^{1/2}\|\phi_1-\phi_2\|_2^{1/2}$ and $\|\phi_j\|_{8/3}\le C\|\phi_j\|_X, j=1,2$. Thus we obtain the estimate (20). To show (22), first we take a sequence $\eta_j\in C_0^\infty(\mathbf{R}^2)$ such that $\eta_j\to|\phi|$ in $H^1(\mathbf{R}^2)$. Let $\mathbf{A}_j=(1/\kappa)\int\mathbf{G}(x-y)\eta_j^2(y)\,dy$ for each $j=1,2,\cdots$. Then $\mathbf{A}_j\in C^1$. Hence, by Lemma 2 we have

$$\int|(\mathbf{D}_e-i\mathbf{A}_j)\phi|^2\,dy \ge \int(-B_e+\frac{\eta_j^2}{\kappa})|\phi|^2\,dy.$$

Now it is easy to see from Gagliardo-Nirenberg inequality and (20) that

$$\begin{aligned}
\int\eta_j^2|\phi|^2\,dy &\to \int|\phi|^4\,dy\\
\int|(\mathbf{D}_e-i\mathbf{A}_j)\phi|^2\,dy &\to \int|(\mathbf{D}_e-i\mathbf{A}(\phi))\phi|^2\,dy
\end{aligned}$$

Thus we complete the proof of (22). □

Although we do not know whether the weak lower semicontinuity of $J(\phi)$ on $X$ holds or not, we have the following lemma which plays an important role in the proof of Theorem 1.

LEMMA 4. *Assume $\phi_j \in X$ converges to $\phi \in X$ weakly in $X$ and strongly in $L^4(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$. Then the lower semicontinuity of $J(\phi)$:*

$$J(\phi) \leq \liminf_{k\to\infty} J(\phi_{j_k}) = \Sigma \tag{23}$$

*holds.*

*Proof.* First note that

$$\begin{aligned}
&\int |(\mathbf{D}_e - i\mathbf{A}(\phi_j))\phi_j|^2\,dx \\
= &\int |\mathbf{D}_e\phi_j|^2\,dx - i\int \mathbf{A}(\phi_j)(\phi_j\cdot(\mathbf{D}_e\phi_j)^* - \phi_j^*\cdot\mathbf{D}_e\phi_j)\,dx + \int |\mathbf{A}(\phi_j)\phi_j|^2\,dx.
\end{aligned}$$

We have

$$\begin{aligned}
&\int |(\mathbf{A}(\phi_j)\phi_j - \mathbf{A}(\phi)\phi)\cdot(\mathbf{D}_e\phi_j)^*|\,dx \\
\leq &\int |(\phi_j-\phi)\mathbf{A}(\phi_j)\cdot(\mathbf{D}_e\phi_j)^*|\,dx + \int |\phi(\mathbf{A}(\phi_j)-\mathbf{A}(\phi))\cdot(\mathbf{D}_e\phi_j)^*|\,dx \\
\leq &\ \|\mathbf{A}(\phi_j)\|_4\|\phi_j-\phi\|_4\|\mathbf{D}_e\phi_j\|_2 + \|\mathbf{A}(\phi_j)-\mathbf{A}(\phi)\|_4\|\phi\|_4\|\mathbf{D}_e\phi_j\|_2 \\
\leq &\ C\|\phi_j-\phi\|_4 + C\|\phi_j-\phi\|_2 \to 0.
\end{aligned}$$

By the weak convergence in $X$, we also have

$$\int \mathbf{A}(\phi)\phi\cdot(\mathbf{D}_e\phi_j)^*\,dx \to \int \mathbf{A}(\phi)\phi\cdot(\mathbf{D}_e\phi)^*\,dx.$$

Hence, it follows that

$$\int \mathbf{A}(\phi_j)\phi_j\cdot(\mathbf{D}_e\phi_j)^*\,dx \to \int \mathbf{A}(\phi)\phi\cdot(\mathbf{D}_e\phi)^*\,dx. \tag{24}$$

In a similar way, we also obtain

$$\begin{aligned}
&\left|\int |\mathbf{A}(\phi_j)\phi_j|^2\,dx - \int |\mathbf{A}(\phi)\phi|^2\,dx\right| \\
\leq &\int |\mathbf{A}(\phi_j)\phi_j - \mathbf{A}(\phi)\phi|(|\mathbf{A}(\phi_j)\phi_j| + |\mathbf{A}(\phi)\phi|)\,dx \\
\leq &\ \|\mathbf{A}(\phi_j)\phi_j - \mathbf{A}(\phi)\phi\|_2(\|\mathbf{A}(\phi_j)\phi_j\|_2 + \|\mathbf{A}(\phi)\phi\|_2) \\
\leq &\ C(\|\mathbf{A}(\phi_j)(\phi_j-\phi)\|_2 + \|(\mathbf{A}(\phi_j)-\mathbf{A}(\phi))\phi\|_2) \to 0.
\end{aligned}$$

Here we used (20) and the boundedness of $\phi_j$ in $X$. It is easy to see

$$\int |\phi_j|^4\,dx \to \int |\phi|^4\,dx.$$

These and weak convergence in $X$ imply the desired estimate (23). □

*Proof of Theorem 1.* First, we consider the case (a). We note that, by (21) it follows that

$$J(u) \geq \frac{1}{4}(1 - gC_0 N) \int |\mathbf{D}\phi|^2 \, dx + \int V_e |\phi|^2 \, dx$$

on $\mathcal{A}_N$. Take a minimizing sequence $\phi_j \in \mathcal{A}_N$, that is

$$J(\phi_j) \to \Sigma, \quad \int |\phi_j|^2 \, dx = N.$$

By the assumption $gC_0 N < 1$ we obtain

$$\int |\mathbf{D}\phi_j|^2 + V_e |\phi_j|^2 \, dx \leq C$$

for some constant $C$. By using (22) and the definition of $\mathbf{A}$ we have

$$\int |\mathbf{D}\phi_j|^2 \, dx \geq \frac{1}{\kappa} \int |\phi_j|^4 \, dx - \int B_e |\phi_j|^2 \, dx,$$

and hence we also obtain the boundedness of $\{\|\phi_j\|_4\}$ by the assumption $B_e \leq 0$. Since $|\mathbf{D}_e \phi_j|^2 \leq 2(|\mathbf{D}\phi_j|^2 + 2|\mathbf{A}(\phi_j)\phi_j|^2$, it follows from (19) and $\int |\phi_j|^2 \, dx = N$ that

$$\int |\mathbf{D}_e \phi_j|^2 \, dx \leq 2 \int |\mathbf{D}\phi_j|^2 \, dx + 2 \int |\mathbf{A}(\phi_j)\phi_j|^2 \, dx \leq C$$

for some constant $C$. Therefore, $\{\phi_j\}$ is a bounded sequence in $X$, and hence we can take a subsequence $\phi_{j_k}$ and $\phi \in X$ such that $\phi_{j_k}$ converges weakly to $\phi$ in $X$ and strongly in $L^2_{loc}$. We also have

$$\int (V_e(x) - B_e(x)) |\phi_j|^2 \, dx \leq C.$$

By using the assumption $V_e(x) - B_e(x) \to \infty$ as $|x| \to \infty$, we can obtain the strong convergence in $L^p(\mathbf{R}^2)$ for any $+\infty > p \geq 2$ (see, e.g., [3], [Ku]). Thus we have $\phi \in \mathcal{A}_N$. Now, by Lemma 4 we have

$$J(\phi) \leq \liminf_{k \to \infty} J(\phi_{j_k}) = \Sigma \tag{25}$$

which conclude that $\phi$ is a minimizer to $\Sigma$.

For the case $g\kappa < 1$, we first use (22) and get

$$J(\phi) \geq \frac{1}{4}(1 - g\kappa) \int |\mathbf{D}\phi|^2 + \frac{1}{2} \int V_e |\phi|^2 - \frac{g\kappa}{4} \int B_e |\phi|^2 \, dx.$$

Then as in the case of (a) we obtain the boundedness of $\{\phi_j\}$ in $X$ and

$$\int (V_e - \frac{g\kappa}{2} B_e)|\phi_j|^2\, dx \leq C.$$

Thus, we can conclude in the same way. □

## 3. Proof of Theorem 2

To show Theorem 2, first we recall the following lemma.

LEMMA 5 (Kato's inequality). *Assume $n \geq 2$, $a \in L^4_{loc}(\mathbf{R}^n)$, div $a \in L^2_{loc}(\mathbf{R}^n)$. Then for $u \in L^2_{loc}(\mathbf{R}^n)$ satisfying $\nabla u \in L^{4/3}_{loc}(\mathbf{R}^n)$ and $(\nabla - ia)^2 u \in L^1_{loc}(\mathbf{R}^n)$, we have*

$$\Delta|u| \geq Re\left(sgn(u)(\nabla - ia)^2 u\right)$$

*in the distribution sense, where*

$$sgn(u) = \begin{cases} \frac{\bar{u}}{|u|} & if\, u \neq 0 \\ 0 & if\, u = 0. \end{cases}$$

When $a \in C^1$, this was first proved by T. Kato ( [14]). This lemma is due to [10, Lemma 2.1]. See also [22, Theorem 2] for slightly different assumptions to assure Kato's inequality.

*Proof of Theorem 2.* (a) Although this is basically known, without detail computation (see [13]), we present the computation in details for the sake of completeness. Let $\psi \in C_0^\infty(\mathbf{R}^2)$. We introduce the notation:

$$\mathbf{A}(\phi, \psi)(x) = \frac{1}{\kappa}\mathrm{Re}\left(\int \mathbf{G}(x-y)\phi(y)\psi^*(y)\, dy\right).$$

Then for $t \in \mathbf{R}$ we have

$$\begin{aligned} \mathbf{A}(\phi + t\psi)(x) &= \frac{1}{\kappa}\int \mathbf{G}(x-y)|\phi(y) + t\psi^*(y)|^2\, dy \\ &= \mathbf{A}(\phi)(x) + 2t\mathbf{A}(\phi,\psi)(x) + t^2\mathbf{A}(\psi)(x). \end{aligned}$$

Thus it follows that

$$\begin{aligned} & (\mathbf{D}_e - i\mathbf{A}(\phi + t\psi))(\phi + t\psi) \\ = \ & \mathbf{D}\phi + t\mathbf{D}\psi - 2it\mathbf{A}(\phi,\psi)\phi + O(t^2), \end{aligned}$$

where $\mathbf{D} = \mathbf{D}_e - i\mathbf{A}(\phi)$. Therefore we have

$$\begin{aligned} & |(\mathbf{D}_e - i\mathbf{A}(\phi + t\psi))(\phi + t\psi)|^2 \\ = \; & |\mathbf{D}\phi|^2 + t\mathrm{Re}(\mathbf{D}\phi(\mathbf{D}\psi)^*) + 2it\mathbf{A}(\phi,\psi)(\mathbf{D}\phi\phi^* - (\mathbf{D}\phi)^*\phi) + O(t^2). \end{aligned}$$

Now we obtain

$$\begin{aligned} \frac{d}{dt}J(\phi + t\psi)\Big|_{t=0} &= \frac{1}{2}\mathrm{Re}\int \mathbf{D}\phi(\mathbf{D}\psi)^*\,dx \\ &+ \frac{i}{2}\int \mathbf{A}(\phi,\psi)(\mathbf{D}\phi\phi^* - (\mathbf{D}\phi)^*\phi)\,dx \\ &+ \int V_e\mathrm{Re}(\phi\psi^*)\,dx - g\int \mathrm{Re}(|\phi|^2\phi\psi^*)\,dx = 0. \end{aligned}$$

We also note

$$2i\mathbf{j} = (\mathbf{D}\phi\phi^* - (\mathbf{D}\phi)^*\phi)$$

and by (3)

$$\begin{aligned} & -\int \mathbf{A}(\phi,\psi)\cdot\mathbf{j}\,dx \\ = \; & -\int\left(\frac{1}{\kappa}\mathrm{Re}\int \mathbf{G}(x-y)\phi(y)\psi(y)^*\,dy\right)\cdot\mathbf{j}(x)\,dx \\ = \; & \mathrm{Re}\int A^0(y)\phi(y)\psi(y)^*\,dy. \end{aligned}$$

This concludes that there exists a constant $E$ such that

$$\mathrm{Re}\left[\frac{1}{2}\int \mathbf{D}\phi\cdot(\mathbf{D}\psi)^* + A^0\phi\psi^* + V_e\phi\psi^* - g|\phi|^2\phi\psi^*\right]dx = E\mathrm{Re}\int \phi\psi^*\,dx$$

for every $\psi \in C_0^\infty(\mathbf{R}^2)$. This implies that $\phi$ satisfies (5) in the weak sense. In particular, $E$ is determined by

$$EN = \frac{1}{2}\int |\mathbf{D}\phi|^2 + \int A^0|\phi|^2 + \int V_e|\phi|^2 - g\int |\phi|^4.$$

(b) First note that $\mathbf{A}(\phi), A^0(\phi) \in L^4(\mathbf{R}^2)$, since

$$\|A^0(\phi)\|_4 \le C\|\phi\|_4\|\mathbf{D}\phi\|_2$$

for some constant $C$ by using Lemma 1. Then the standard elliptic regularity theorem (see, e.g., [18], [16]) implies the Hölder continuity of $\phi$. By noting (8) and using Lemma 4, we have

$$\begin{aligned} \Delta|\phi| &\ge (A^0 + V_e - g|\phi|^2 - E)|\phi| \\ &\ge (A^0 - \alpha - g|\phi|^2 - E)|\phi| \end{aligned}$$

in the distribution sense. Noting $A^0 \in L^4(\mathbf{R}^2)$, $|\phi|^2 \in L^2(\mathbf{R}^2)$, and $|u| \in H^1(\mathbf{R}^2)$, there exist constants $C$ and $r_0$ such that

$$|\phi(x)| \le C\left(\int_{B(x,r_0)} |\phi(y)|^2\, dy\right)^{1/2}$$

uniformly on $x \in \mathbf{R}^2$ (see, e.g., [9]). Since $|\phi| \in L^2(\mathbf{R}^2)$, this implies $|\phi(x)| \to 0$ as $|x| \to \infty$.

(c) We already know $|\phi|^2 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ and $|\phi|^2$ is Hölder continuous. Hence, by the argument as in [8], one can show $\mathbf{A} \in C^1(\mathbf{R}^2)$ and

$$B = \nabla \times \mathbf{A} = -\frac{|\phi|^2}{\kappa}.$$

This also yields immediately the first formula of the part (c) of Theorem 2. The second formula of the part (c) of Theorem 2 can be proved as in [5] and [4], for example. We omit the details. □

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## References

[1] Avron J., Herbst I., and Simon B., *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), 847–883.

[2] Bergé L., de Bouard A., and Saut J.C., *Blowing up time dependent solutions of the planer*, Chern-Simons gauged nonlinear Schrödinger equation, Nonlinearity, **8** (1995), 235–253.

[3] Bartsch T. and Wang Z.-Q., *Existence and multiplicity results for some superlinear elliptic problems on* $\mathbf{R}^n$, Comm. Partial Diff. Eq. **20** (1995), 1725–1741.

[4] Chanillo, S. and Kiessing, M.K.-H., *Conformal invariant systems of nonlinear PDE of Liouville type*, Geometric and Fourier Anal. **6** (1995), 924–947.

[5] Chen, W. and Li, C., *Qualitative properties of solutions to some nonlinear elliptic equations in* $\mathbf{R}^2$, Duke Math. J. **71** (1993), 427–439.

[6] Ezawa Z.F., Hotta M., and Iwasaki A., *Nonrelastivstic Chern-Simons vortex solitons in external magnetic field*, Phys. Rev. D. **44** (1991), 452–463.

[7] ———, *Anyon field theory and fractional quantum Hall statics*, Prog. Theo. Phys. Suppl. **107** (1992), 185–194.

[8] Gilbarg D. and Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grund. Math. Wiss., Springer-Verlag, **224** (1983).

[9] Hinz, A.M. and Kalf, H., *Subsolution estimates and Harnack's inequality for Schrödinger operators*, J.Reine Angew. Math. **404** (1990), 118–134.
[10] Hinz, A.M. and Stolz, G., *Polynomial boundedness of eigenfunctions and the spectrum of Schrödinger operators*, Math. Ana. **294** (1992), 195–211.
[11] Jackiw R. and Pi S.-Y., *Classical and quantal nonrelativistic Chern-Simons theory*, Phys. REv. D. **42** (1990), 3500–3513.
[12] ———, *Time-dependent Chern-Simons solitons and their quantization*, Phys. Rev. D. **44** (1991), 2524–2532.
[13] ———, *Self-Dual Chern-Simons Solitons*, Prog. Theo. Phys. Suppl. **107** (1992), 1–40.
[14] Kato, T., *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148.
[15] Kurata, K., *Local boundedness and continuity for weak solutions of* $-(\nabla - ib)^2u + Vu = 0$, Math. Z. **224** (1997), 641–653.
[16] ———, *Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields*, Nonlinear Analysis **41** (2000), 763–778.
[17] ———, *An estimate on the Heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials*, to appear in J. London Math. Soc.
[18] Ladyzhenskaya, O. A. and Ural'tseva, N. N., *Linear and Quasilinear Elliptic Equations*, Academic Press, 1968.
[19] Reed M., Simon B., *Methods of Modern Mathematical Physics II*, Academic Press, New York, 1972.
[20] Shen, Z., *$L^p$ estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier **45** (1995), 513–546.
[21] Shen, Z., *Estimates in $L^p$ for magnetic Schrödinger operators*, Indiana Univ. Math. J. **45** (1996), 817–841.
[22] Simon, B., *Schrödinger operators with sigular magnetic vector potentials*, Math. Z. **131** (1973), 361–370.
[23] Sogge, C. D., *Fourier Integrals in Classical Analysis*, Cambridge University Press, 1993.

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