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REMARKS ON UNIQUENESS AND BLOW-UP CRITERION TO THE EULER EQUATIONS IN THE GENERALIZED BESOV SPACES

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ABSTRACT. In this paper, we disscuss a uniqueness problem for the Cauchy problem of the Euler equation. We give a sufficient condition on the vorticity to show the uniqness of a class of generalized solution in terms of the generalized Besov space. The condition allows the iterated logarithmic singularity to the vorticity of the solution. We also disscuss the break down (or blow up) condition for a smooth solution to the Euler equation under the related assumption.

1. Introduction

We discuss on the uniqueness and blow-up problem for the Euler equations:

$$(1.1)\qquad \begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p, & t>0, x\in\mathbb{R}^n,\\ \quad \operatorname{div} u = 0, & t>0, x\in\mathbb{R}^n,\\ \quad u(0,x)=u_0(x). \end{cases}$$

Here $u=u(t,x)=(u^1(t,x),u^2(t,x),\cdots,u^n(t,x))$ and $p=p(t,x)$ denote the unknown velocity and the unknown pressure of the incompressible ideal fluid, respectively, while $u_0(x)=(u_0^1(x),u_0^2(x),\cdots,u_0^n(x))$ is a given initial data.

We first consider the uniqueness of generalized solutions of the Euler equations (1.1). We call a measurable function u as a generalized solution of (1.1) if u satisfies the following conditions:

(i) $u\in L^\infty(0,T;H^1_\sigma)$;

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(ii) $\int_s^t \{-(u, \partial_\tau \Phi) + (u \cdot \nabla u, \Phi)\}\, d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s))$ for every $0 \le s \le t < T$ and every $\Phi \in C_0^\infty([0,T] \times \mathbb{R}^n)$ with div $\Phi = 0$;

(iii) $\|u(t)\|_2 \le \|u_0\|_2$ for almost all $0 \le t \le T$.

Here $C_{0,\sigma}^\infty$ denotes the set of all C^∞ vector functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with compact support in $\mathbb{R}^n$, such that div $\phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. (Note that L^r stands for the vector-valued spaces.) H_σ^s is the closure of $C_{0,\sigma}^\infty$ with respect to the H^s-norm $\|\phi\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}}\phi\|_2$ for $s \ge 0$ (if u belongs to $L^\infty(0,T;L^n)$, the condition (iii) is derived from (i) and (ii). Hence (iii) is not necessary when $2 \le n \le 4$). There are many results for the existence of the generalized solution to (1.1). See, for example, Chae [5] under more general setting.

Yudovich [18] proved the uniqueness of generalized solutions under vorticity $\omega \equiv$ rot $u \in L^1(0,T;L^\infty)$. Then he extended the uniqueness result to some classes of flows with *unbounded* vorticity, where the vorticity may have iterated logarithm singularity ([19]). The basic argument in proving his uniqueness result is to employ the Perron-Nagumo type uniqueness criterion (more specifically, Osgood's uniqueness theorem) to the ordinary differential equations. Therefore in the assumption for proving Yudovich's result, it is required a continuity of solutions in time variable.

On the other hand, in the whole space case, we showed in [12] a different kind of uniqueness theorem under the condition $\omega \in L(\log L)^{1/2}(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^n))$, where we only used Gronwall's theorem. It is, then, expected that by applying Osgood's theorem, one can obtain a slightly better uniqueness condition on generalized solutions than the condition we claimed in [12] (see also [16] for a solvability of (1.1) in the Besov space).

In order to specify the assumption on the solution, we introduce a new class of the solution that is written in the terminology of the Besov spaces (for the Besov space, c.f. [2]). The class $\dot{B}^{-\log^{k'}}_{\infty,\infty}$ (defined in the later) essentially includes the functions of the bounded mean oscillation (BMO) and therefore includes the logarithmic function. Moreover the space also includes a finitely iterated logarithmic function such as

$$\underbrace{\log(e+|x|^{-1})\log(e+\log(e+|x|^{-1}))\cdots\underbrace{\log(e+\log(e+\cdots\log}_{k+1\text{-times iterated}}(e+|x|^{-1}))\cdots)}_{k+1\text{-times multiplied}},$$

$|x|<1$.

By the aid of the logarithmic Sobolev inequality of the Beale-Kato-Majda type, we shall show the uniqueness of generalized solutions with $\omega \in L^1(0,T;BMO)$, or under more general condition.

The second purpose in this paper is to discuss the continuation of smooth solutions for the Euler equations, that is the blow-up problem. In [1], a smooth solution of the 3D Euler equation is shown to be regular under rot $u(t) \in L^1(0,T;L^\infty)$. (We note that this corresponds with the class which guarantees the uniqueness of generalized solution.) This result is extended into the slightly larger class of condition in Kozono-Taniuchi [11]. In this paper we prove the continuation of smooth solutions under the same condition as in our uniqueness result [12].

After completing this work, the authors are noticed that an analogous uniqueness result for the ideal fluid flow is obtained by Vishik [17]. This result is also based on Yudovich's argument in the borderline of the Besov space. However the detailed argument seems different from ours. The authors are grateful to Misha Vishik for sending us the preprint.

2. Preliminaries

Before presenting our results, we recall some notations and definition of the Besov spaces (c.f., [15]). Let ϕ_j, $j=0,\pm1,\pm2,\pm3,\cdots$ be the Littlewood-Paley dyadic decomposition satisfying $\hat{\phi}_j(\xi)=\phi(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty}\hat{\phi}_j(\xi)=1$ except $\xi=0$. We put a smooth cut off $\psi \in \mathcal{S}(\mathbb{R}^n)$ to fill the origin, where $\hat{\psi}(\xi)\in C_0^\infty(B_1)$ and $B_1=\{x\in\mathbb{R}^n : |x|<1\}$ such that $\hat{\psi}+\sum_{j=0}^{\infty}\hat{\phi}_j(\xi)=1$.

Definition. The homogeneous Besov space $\dot{B}^s_{p,\rho} = \{f \in \mathcal{S}'; \|f\|_{\dot{B}^s_{p,\rho}} < \infty\}$ is introduced by

$$\|f\|_{\dot{B}^s_{p,\rho}} = (\sum_{j=-\infty}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho}$$

for $s \in \mathbb{R}$, $1 \le p, \rho \le \infty$.

Now we introduce a generalization of the Besov space. Let

$$\log^+ t \equiv \max(0, \log t),$$

$$(\log^+)^k t \equiv \underbrace{\log^+ \log^+ \cdots \log^+}_{k\text{-times iteration}} t;$$

$$e_k \equiv \underbrace{\exp \exp \cdots \exp}_{k\text{-times iteration}} 1,$$

$$\text{for } k = 1, 2, \cdots$$

and

$$\log^{(0!)} t \equiv 1,$$

$$\log^{(k!)} t \equiv (\log^+ t)(\log^+ \log^+ t)(\log^+ \log^+ \log^+ t) \cdots (\log^+)^k t, \qquad \text{for } t > e_k.$$

Note that the series $\{(n \log^{(k!)} n)^{-1}\}_{n \ge e_k}^{\infty} \notin l^1$.

Definition. $\dot{B}^{-\log^{k!}}_{p,\rho} = \{f \in \mathcal{S}'; \|f\|_{\dot{B}^{-\log^{k!}}_{p,\rho}} < \infty\}$ is introduced by

$$\|f\|_{\dot{B}^{-\log^{k!}}_{p,\rho}} = (\sum_{j=-\infty}^{\infty} (\log^{(k!)}(|j| + e_k))^{-\rho} \|\phi_j * f\|_p^\rho)^{1/\rho} \quad \text{for } \rho < \infty,$$

$$\|f\|_{\dot{B}^{-\log^{k!}}_{p,\infty}} = \sup_{-\infty < j < \infty} (\log^{(k!)}(|j| + e_k))^{-1} \|\phi_j * f\|_p.$$

It is easy to see the inequality

$$\|f\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \le \|f\|_{\dot{B}^0_{\infty,\infty}} \le C\|f\|_{BMO}$$

holds for $f \in BMO$, where BMO is characterized by

$$f \in L^1_{loc}, \quad \sup_{r>0, x\in\mathbb{R}^n} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - \bar{f}_{B_r}| dy < \infty.$$

Therefore $\log|x|^{-1}$ is included in $\dot{B}^{-\log^{k!}}_{\infty,\infty}$. Moreover the singular function; $\log^{(k!)}|x|^{-1}$ is also included in $\dot{B}^{-\log^{(k-1)!}}_{\infty,\infty}$.

We next recall Osgood's uniqueness theorem which plays important role in proving new uniqueness result for the Euler equations. We state a slightly general case.

LEMMA 2.1 (Osgood). *Let Φ be nonnegative function on $(0,T)$ with $\int_{+0}\Phi(t)dt<\infty$, let $\Psi(t)$ be continuous for $t\geq 0$ and nondecreasing near $t=+0$ and $\Psi(0)=0$, $\Psi(v)>0$ if $v>0$ and let $\int_{+0}dv/\Psi(v)=\infty$. Assume that $v\in L^\infty(0,T)$ and*

$$0\leq v(t)\leq\int_0^t\Phi(\tau)\Psi(v(\tau))d\tau \text{ a.e. } 0\leq t<T.$$

Then $v(t)\equiv 0$ for almost all $t\in[0,T]$.

Lemma 2.1 is simply proved by setting $V(t)=\int_0^t\Phi(\tau)\Psi(v(\tau))d\tau$. If we assume $V(t)>0$ for $t\in(0,T]$, then since $V(t)$ solves $V'(t)\leq\Phi(t)\Psi(V(t))$, we reach a contradiction to the assumption. For more detailed proof and related theorem, see for example, Hartman [7] p.33.

We should note that t, $t\log(1/t+e)$, $t\log(1/t+e)\log\log(1/t+e^e),\cdots$ satisfy the hypotheses on Ψ.

We have the following the logarithmic Sobolev inequality originally due to Brezis-Gallouet [3], Brezis-Wainger[4] and Beale-Kato-Majda [1] (see for some generalization [6], [14], [11], [10]).

LEMMA 2.2. *Let $s>n/p$, $p\in(1,\infty)$, and $k=0,1,2,\cdots$. Then there exist positive constants $C_1=C_1(n,k,p,s)$ and $C_2=C_2(n,k,p,s)$ such that*

$$\|f\|_\infty\leq C_1\bigl(1+\|f\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}\log^{((k+1)!)}(\|f\|_{W^{s,p}}+C_2)\bigr) \tag{2.1}$$

for any $f\in W^{s,p}$.

We prove this lemma in Appendix.

COROLLARY 2.3. *Let $s>n/p+1$, $p\in(1,\infty)$, and $k=0,1,2,\cdots$ Then there exist positive constants $C_1=C_1(n,k,p,s)$ and $C_2=C_2(n,k,p,s)$ such that*

$$\|\nabla u\|_\infty\leq C_1\bigl(1+\|\mathrm{rot}\ u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}\log^{((k+1)!)}(\|u\|_{W^{s,p}}+C_2)\bigr) \tag{2.2}$$

for any n-dimensional vector function $u\in W^{s,p}$ with div $u=0$.

Corollary 2.3 is derived from the following equalities:

$$\begin{aligned}\partial_i u =& R_i\vec{R}\times \text{rot } u \quad \text{(Biot-Savart law)};\\ \|\phi_j * R_i f\|_\infty =& \|(R_i(\phi_{j-1}+\phi_j+\phi_{j+1}))*\phi_j * f\|_\infty,\end{aligned}$$

where $\vec{R}=(R_1,R_2,\cdots,R_i,\cdots,R_n)$ denotes the Riesz transform.

3. Uniqueness

We prove uniqueness theorem similar to Yudovich [19]. Our main result now reads:

THEOREM 3.1 (Uniqueness). *Let u and $\tilde{u}$ be generalized solutions for the* Euler *equation. Suppose that one of the solutions satisfies* rot $u \in L^1(0,T;\dot{B}^{-\log^{k!}}_{\infty,\infty})$, *then $u\equiv\tilde{u}$ on $[0,T]$. Here $\dot{B}^{-\log^{k!}}_{\infty,\infty}$ is defined by*

$$\|f\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} = \sup_{-\infty<j<\infty}\frac{\|\phi_j * f\|_\infty}{\log^{(k!)}(|j|+e_k)}.$$

In particular, there exists no more than one generalized solution for the Euler equation with vorticity $\omega\in L^1(0,T;BMO)$, since $BMO\subset \dot{B}^0_{\infty,\infty}\subset\dot{B}^{-\log^{k!}}_{\infty,\infty}$.

Proof of Theorem 3.1. Let $u,\tilde{u}$ be two generalized solutions to the Euler equations with same initial data and let $w=u-\tilde{u}$. Now we decompose the solution u into the three parts in the phase variables such as

$$\begin{aligned}u(x) =& \psi_{-N}*u(x)+\sum_{|j|\le N}\phi_j*u(x)+\sum_{j>N}\phi_j*u(x)\\ =& u_l(x)+u_m(x)+u_h(x),\end{aligned}\tag{3.1}$$

where $\psi_{-N}=\sum_{j<-N}\phi_j$. Then by the Hausdorff-Young inequality, the low frequency part is estimated as

$$\begin{aligned}|(w\cdot\nabla u_l,w)| =& |(w\cdot\nabla w,u_l)|\\ \le& \|\psi_{-N}*\nabla(w\otimes w)\|_2\|u\|_2\\ \le& C\|\nabla\psi_{-N}\|_2\|w\|_2^2\|u\|_2\\ \le& C2^{-N/2}\|w\|_2^2\|u\|_2.\end{aligned}\tag{3.2}$$

The second term is estimated as

$$
\begin{aligned}
&|(w\cdot\nabla u_m, w)| \\
&\le \|w\|_2^2\|\sum_{|j|\le N}\phi_j * \nabla u\|_\infty \\
&\le C\|w\|_2^2\sum_{|j|\le N}\|\phi_j * \nabla u\|_\infty \qquad (3.3)\\
&\le C\|w\|_2^2 N\log^{(k!)}(N+e_k)\sup_{|j|\le N}(\log^{(k!)}(|j|+e_k))^{-1}\|\phi_j * \nabla u\|_\infty \\
&\le C\|w\|_2^2 N\log^{(k!)}(N+e_k)\|\nabla u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}
\end{aligned}
$$

for any $N \ge 1$. The third term in the right hand side of (3.1) is simply estimated by the Hausdorff-Young inequality that

$$
\begin{aligned}
&|(w\cdot\nabla u_h, w)| \\
&= |(w\cdot\nabla w, u_h)| \\
&\le \|w\|_2\|\nabla w\|_2\|(\sum_{j>N}(\phi_{j-1}+\phi_j+\phi_{j+1}) * \phi_j * u)\|_\infty \\
&\le \|w\|_2\|\nabla w\|_2\sum_{j>N}\|\{(-\Delta)^{-1/2}(\phi_{j-1} \qquad (3.4)\\
&\qquad\qquad + \phi_j+\phi_{j+1})\} * (-\Delta)^{1/2}\{\phi_j * u\}\|_\infty \\
&\le C\|w\|_2\|\nabla w\|_2\sum_{j>N}2^{-j}\|\phi_j * (-\Delta)^{1/2}u\|_\infty \\
&\le C\|w\|_2\|\nabla w\|_2\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}\sum_{j>N}2^{-j}(\log^{(k!)}(|j|+e_k)) \\
&\le C2^{-N/2}\|w\|_2\|\nabla w\|_2\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}.
\end{aligned}
$$

Now we consider the case $\|w\|_2^2 < C_k^{-1} \le 1$. Gathering the estimates (3.2)-(3.4) with (3.1) and choosing N properly large satisfying $2^{-N/2} \simeq$

$\|w\|_2$, we see by $u, w \in L^\infty(0,T;H^1_\sigma)$ (also c.f. (5.6) in Appendix) that

(3.5)

$$
\begin{aligned}
&|(w \cdot \nabla u, w)| \\
&\le C\|w\|_2^2\|u\|_2 \\
&\quad + CN\log^{(k!)}(N+e_k)\|w\|_2^2\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \\
&\quad + C2^{-N/2}\|w\|_2\|\nabla w\|_2\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \\
&\le C\|w\|_2^2\|u\|_2 \\
&\quad + C\left\{1+\log(\|w\|_2^{-2})\log^{(k!)}(\log^+(\|w\|_2^{-2})+e_k)\right\}\|w\|_2^2\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \\
&\quad + C\|w\|_2^2\|\nabla w\|_2\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \\
&\le C(1+\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}})\|w\|_2^2\left\{1+\log^{((k+1)!)}(\|w\|_2^{-2}+e_{k+1})\right\}.
\end{aligned}
$$

When $\|w\|_2^2 \ge C_k^{-1}$, we choose $N = 1$ and it follows

$$|(w \cdot \nabla u, w)| \le C(1+\|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}})\|w\|_2^2.$$

Hence in both cases, (3.5) holds.

Next we shall show the following estimate:

$$\text{(3.6)} \qquad \|w(t)\|_2^2 \le 4\int_0^t |(w \cdot \nabla u, w)(\tau)|d\tau \text{ a.e. on } 0 \le t < T.$$

Since $B^0_{\infty,1} \subset L^\infty$, we observe that

$$
\begin{aligned}
\|u\|_\infty \le& \|\psi * u\|_\infty + \sum_{j\ge1}\|\phi * u\|_\infty \\
\le& C\|u\|_2 + \sum_{j\ge1}\|\{(-\Delta)^{-1/2}(\phi_{j-1}+\phi_j+\phi_{j+1})\} * \phi_j * (-\Delta)^{1/2}u)\|_\infty \\
\le& C\|u\|_2 + \sum_{j\ge1}2^{-j}\|\phi_j * (-\Delta)^{1/2}u\|_\infty \\
\le& C\|u\|_2 + \|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}\sum_{j\ge1}2^{-j}(\log^{(k!)}(|j|+e_k)) \\
\le& C(\|u\|_2 + \|\text{rot } u\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}),
\end{aligned}
$$

which implies

$$u \in L^1(0,T;L^\infty) \cap L^\infty(0,T;H^1). \tag{3.7}$$

By (3.7) and (i) in the definition of the generalized solution, we have

$$u \cdot \nabla u \in L^1(0,T;L^2) \text{ and } u \in L^{n/(n-1)}(0,T;L^{2n}). \tag{3.8}$$

On the other hand, $\tilde{u}$ satisfies

$$\tilde{u} \cdot \nabla \tilde{u} \in L^\infty(0,T;L^{2n/(2n-1)}) \text{ and } \tilde{u} \in L^\infty(0,T;L^2), \tag{3.9}$$

since $\|\tilde{u} \cdot \nabla \tilde{u}\|_{2n/(2n-1)} \le \|\tilde{u}\|_{2n/(n-1)} \|\nabla \tilde{u}\|_2 \le C\|\tilde{u}\|_2^{1/2}\|\nabla \tilde{u}\|_2^{3/2}$. Then using the standard mollifier argument, the regularities (3.8), (3.9) and the definition of the generalized solution ((i), (ii) and (iii)) yield (3.6) (c.f., [13]).

We then substitute (3.5) to (3.6) to have

$$\|w(t)\|_2^2 \le C\int_0^t \|w(\tau)\|_2^2 \left\{1 + \log^{((k+1)!)}(\|w(\tau)\|_2^{-2} + e_{k+1})\right\} \times (1 + \|\text{rot } u(\tau)\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}})d\tau \quad \text{a.e. for } 0 \le t < T.$$

Thus Lemma 2.1 (Osgood's theorem) implies

$$w \equiv 0 \text{ a.e. on } [0,T].$$

This proves Theorem 3.1. □

4. Smoothness extension

We also have the following continuation theorem for smooth solutions.

THEOREM 4.1 (Continuation of smooth solutions). *Let $s > n/2 + 1$ and u be a solution for the* Euler *equation in the class $C([0,T);H^s) \cap C^1([0,T);H^{s-1})$. Suppose that*

$$\text{rot } u \in L^1(0,T;\dot{B}^{-\log^{k!}}_{\infty,\infty}),$$

then u can be continued to the solution in $C([0,T');H^s) \cap C^1([0,T');H^{s-1})$ for some $T' > T$.

Proof of Theorem 4.1. Now we prove Theorem 4.1 as Beale-Kato-Majda [1]. It is proved by Kato-Lai [8] that for the given initial data $a \in H^s$ there exist $T^* > 0$ and a unique solution u to (1.1) in the class $C([0,T^*); H^s) \cap C^1([0,T^*); H^{s-1})$. Here the local existence time interval T^* can be estimated from below as follows

$$T^* \geq C/\|a\|_{H^s}.$$

Hence by the standard argument of continuation of local solutions, it suffices to establish the following apriori estimate

$$\sup_{0<t<T} \|u(t)\|_{H^s} < \infty. \tag{4.1}$$

It follows from the commutator estimate in given by Kato-Ponce [9, Proposition 4.2] that

$$\|u(t)\|_{H^s} \leq \|a\|_{H^s} \exp\left(C\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right), \quad 0<t<T, \tag{4.2}$$

where $C = C(n,s)$. Substituting (2.2) to (4.2), we have

$$\begin{aligned}
&\|u(t)\|_{H^s} + C_2 \\
\leq &(\|a\|_{H^s} + C_2) \\
&\times \exp\left(C\int_0^t \{1 + \|\omega(\tau)\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \log^{((k+1)!)}(\|u(\tau)\|_{H^s} + C_2)\} d\tau\right)
\end{aligned}$$

for all $0<t<T$. Here ω denotes rot u. Defining $z_1(t) \equiv \log^+(\|u(t)\|_{H^s} + C_2)$, we obtain from the above estimate

$$z_1(t) \leq z_1(0) + CT + C\int_0^t \|\omega(\tau)\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} z_1(\tau) \log^{(k!)} z_1(\tau) d\tau, \quad 0<t<T.$$

Then Gronwall's inequality yields

$$z_1(t) \leq (z_1(0) + CT) \exp\left(C\int_0^t \|\omega(\tau)\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} \log^{(k!)} z_1(\tau) d\tau\right) \tag{4.3}$$

for all $0<t<T$ with $C = C(n,s)$. Again defining $z_2(t) \equiv \log^+ z_1(t)$, we obtain from the above estimate

$$z_2(t) \leq \log^+(z_1(0) + CT) + C\int_0^t \|\omega(\tau)\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} z_2(\tau) \log^{((k-1)!)} z_2(\tau) d\tau,$$
$$0<t<T.$$

Repeating this procedure, we have

$$z_{k+1}(t) \le (\log^+)^k(z_1(0) + CT) + C\int_0^t \|\omega(\tau)\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}} d\tau, \quad 0 < t < T,$$

where $z_k(t) \equiv (\log^+)^{k-1}\log(\|u(t)\|_{H^s} + C_2)$. This implies (4.1). □

5. Appendix

Here we give the proof of the logarithmic Sobolev inequality, Lemma 2.2 (c.f. [14]).

Proof of Lemma 2.2. We decompose f into three parts such as

$$\begin{aligned} f(x) =& \psi_{-N} * f(x) + \sum_{|j|\le N} \phi_j * f(x) + \sum_{j>N} \phi_j * f(x) \\ =& f_l(x) + f_m(x) + f_h(x). \end{aligned} \tag{5.1}$$

We first estimate $f_l(x)$ and $f_h(x)$. We easily show that

$$|f_l(x)| \le \|\psi_{-N}\|_{p'}\|f\|_p \le C2^{-nN/p}\|f\|_p, \tag{5.2}$$

and

$$\begin{aligned} |f_h(x)| \le& \sum_{j>N} \|\phi_j * f\|_\infty \\ \le& C\sum_{j>N} 2^{nj/p}\|\phi_j * f\|_p \\ =& C\sum_{j>N} 2^{s'j}\|\phi_j * f\|_p 2^{-(s'-n/p)j} \quad (s > s' > n/p) \\ \le& C2^{-(s'+n/p)N}\|f\|_{B^{s'}_{p,1}} \\ \le& C2^{-\kappa N}\|f\|_{W^{s,p}} \quad (\kappa = s' - p/n). \end{aligned} \tag{5.3}$$

Next, we consider $f_m(x)$. As in (3.3), we have

$$\begin{aligned} |f_m(x)| \le& \sum_{|j|\le N} \|\phi_j * f\|_\infty \\ \le& \sum_{|j|\le N} \log^{(k!)}(|j| + e_k)\frac{\|\phi_j * f\|_\infty}{\log^{(k!)}(|j| + e_k)} \\ \le& N\log^{(k!)}(N + e_k)\|f\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}. \end{aligned} \tag{5.4}$$

Gethering (5.2)-(5.4) with (5.1), we obtain

$$\|f\|_\infty \le C2^{-\gamma N}\|f\|_{W^{s,p}} + CN\log^{(k!)}(N+e_k)\|f\|_{\dot{B}^{-\log^{k!}}_{\infty,\infty}}, \tag{5.5}$$

where $\gamma = \min(\kappa, n/p, 1/\log 2)$. Now we take

$$N = \left[\frac{\log^+\|f\|_{W^{s,p}} + C_2'}{\gamma\log 2}\right] + 1,$$

where $[\cdot]$ denotes Gauss symbol and

$$C_2' = \underbrace{\exp\exp\cdots\exp}_{k\text{-times iteration}}\{(\gamma\log 2)^{\gamma\log 2/(\gamma\log 2-1)}\}.$$

Then (5.5) becomes the desired estimate (2.1), since

$$(\log^+)^j(ax) \le a(\log^+)^j x \tag{5.6}$$

for $x \ge \underbrace{\exp\exp\cdots\exp}_{(j-1)\text{-times iteration}}\{a^{1/(a-1)}\}$ if $a \ge 1$. □

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