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ANALYTIC SMOOTHING EFFECT AND SINGLE POINT SINGULARITY FOR THE NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We show that a weak solution of the Cauchy problem for the nonlinear Schrödinger equation,

$$\begin{cases} i\partial_t u + \partial_x^2 u = f(u, \bar{u}), & t \in (-T,T), x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases}$$

in the negative sobolev space H^s has a smoothing effect up to real analyticity if the initial data only have a single point singularity such as the Dirac delta measure. It is shown that for $H^s(\mathbb{R})$ $(s > -3/4)$ data satisfying the condition

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

the solution is analytic in both space and time variable.

The argument is based on the recent progress on the well-posedness result by Bourgain [2] and Kenig-Ponce-Vega [18] and previous work by Kato-Ogawa [12]. We give an improved new argument in the regularity argument.

1. Introduction

We study the smoothing effect for a general form of the following nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = f(u, \bar{u}), & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}$$

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where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $\bar{u}$ is the complex conjugate of u. This equation arises in a various fields in the physics, optics and the water wave theory. Our problem here is the following: On what condition of the initial data ϕ, the solution has regularizing property up to analytic for $t \neq 0$?

Large interest is devoted to the so called the smoothing effect of the solution. When we consider the well-posedness of those type of equation, L^2 based (Sobolev) space is considered and the regularity of solution is then, derived as much as the same order of regularity of the initial data ϕ. Namely if the initial data $\phi \in H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, then the solution expected to be at most $H^s(\mathbb{R})$, since singularities of the solution come from infinity and regularity is never to be gained by the Schrödinger time evolution.

However, it is studied by many cases that the local or some restricted version of smoothing effect holds. Among others, smoothing effect from the low initial regularity solution to the analyticity is our main concern. Especially, to the weak solution constructed in the Fourier restriction space $X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \langle i\partial_t + \partial_x^2\rangle^b \langle D_x\rangle^s f \in L^2(\mathbb{R}; L^2(\mathbb{R}))\}$, where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$ and $D_x = \mathcal{F}^{-1}|\xi|\mathcal{F}$, it is possible to prove the regularity of solution reaches up to analytic in both space and time variable by an operation of the conformal vector fields. This is proved in the case of Korteweg de Vries equation [12]. Since the structure of the linear part of the nonlinear Schrödinger is related to the KdV case, we may extend this result into the Schrödinger case. The method is the following: We introduce the complimentary linear (variable coefficient) operator $P = 2t\partial_t + x\partial_x$ that plays an role of the compensate part where the main linear operator $L = i\partial_t + \partial_x^2$ can not gain the regularity.

Our goal is to obtain the smoothing effect against a single point singularity. To make it simply, we further restrict the situation as the following. We assume that $f(u, \partial_x u)$ is a polynomial of u and $\bar{u}$ of order p but not depends on $\partial_x u$ nor $\partial_x \bar{u}$. That is the equation we discuss is the following simpler one:

$$(1.2) \qquad \begin{cases} i\partial_t u + \partial_x^2 u = f(u, \bar{u}), & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases}$$

In fact, according to the progress of the result of the well-posedness in the negative index Sobolev spaces, we restrict ourselves to the following (somewhat) exotic nonlinearities:

$$f(u) = u^2, \quad f(\bar{u}) = \bar{u}^2, \tag{1.3}$$

$$f(u, \bar{u}) = |u|^2. \tag{1.4}$$

The following is our main theorem.

THEOREM 1.1. *Let $s > -3/4$ and $f(u, \bar{u}) = u^2, \bar{u}^2$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exist $T > 0$, $b \in (1/2, 1)$ and a unique solution $u \in C((-T, T), H^s) \cap X_b^s$ of the nonlinear dispersive equation (1.1) and for any $(t, x) \in \{(-T, 0) \cup (0, T)\} \times \mathbb{R}$, $u(t, \cdot)$ is a real analytic function in both space and time variable.

If the nonlinearity is $f(u, \bar{u}) = |u|^2$, then the same result holds for $s > -1/2$.

REMARK 1. Due to the work by Kenig-Ponce-Vega [19], it is proved the time local wellposedness in the negative Sobolev space H^s for the each case of the nonlinearity. Our solution is restricted one in those spaces. We should emphasize that the initial data can be taken as a distribution such as the Dirac measure or the principal value of $1/x$ if $-1/2 > s > -3/4$.

REMARK 2. It is well-known that the global in time solution has been obtained (see [4], [10]) to the special dispersive nonlinear equations by the inverse scattering method. Also the analyticity for the inverse scattering solution of KdV equation with a weighted initial data was obtained by Tarama [24]. However, since our method is based on the fact that the solution is in H^s, we do not know if our result is true globally in time.

By a almost similar argument of Theorem 1.1, one can also show the following weaker theorem.

THEOREM 1.2. *Let $s > -3/4$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^2} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exist $T > 0$, $b \in (1/2, 1)$ and a unique solution $u \in C((-T,T), H^s) \cap X_b^s$ of the dispersive equation (1.1) and for any $t \in (-T,0) \cup (0,T)$, $u(t,\cdot)$ is analytic function in space variable and for $x \in \mathbb{R}$, $u(\cdot, x)$ is of Gevrey 2 as a time variable function.

REMARK 3. In both Theorems, the assumption on the initial data implies the analyticity and Gevrey 2 regularity except the origin respectively. In this sense, those results are stating that the singularity at the origin immediately disappears after $t > 0$ or $t < 0$ up to analyticity.

REMARK 4. Some related results are obtained for the linear and nonlinear Schrödinger equations. For linear variable coefficient case, see Kajitani-Wakabayashi [11], Robbiano-Zuily [22] and for nonlinear case, Chihara [3]. They are giving a global weighted uniform estimates of the solution with arbitrary order derivative in space variable. In our case, it is still unknown if the weighted uniform bounds are possible or not.

REMARK 5. The above theorems, we simply assumed that the nonlinearities are quadratic in u or $\bar{u}$. It is also possible to show a similar result for the cubic cases $|u|^2u$, $|u|\bar{u}$, u^3 and $\bar{u}^3$ under some proper regularity setting.

2. Method

Our method is based on the following observation. To make description simple, we consider the following simplest case:

$$\begin{cases} i\partial_t u + \partial_x^2 u = u^2, & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

Firstly, we introduce the generator of the dilation $P = 2t\partial_t + x\partial_x$ for the linear part of the dispersive equation. Noting the commutation relation with the linear dispersive operator $L = i\partial_t + \partial_x^2$:

$$[L, P] = 2L,$$

it follows

$$LP^k = (P+2)^k L, \tag{2.2}$$

for any $k = 1, 2, \cdots$. Applying $P = 2t\partial_x + x\partial_x$ to the equation (2.1), we have

$$i\partial_t(P^k u) + \partial_x^2(P^k u) = (P+2)^k Lu = (P+2)^k(u^2). \tag{2.3}$$

We set $u_k = P^k u$ and $F_k(u, \bar{u}) = (P+2)^k u^2$. Then noting that

$$\begin{aligned}(P+2)^l u &= (P+2)^{l-1}Pu + 2(P+2)^{l-1}u = \cdots \\ &= \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} 2^{l-j} P^j u,\end{aligned} \tag{2.4}$$

we see

$$\begin{aligned}F_k(u,\bar{u}) =&(P+2)^k(u^2) = \sum_{l=0}^{k} \binom{k}{l} (P+2)^l u P^{k-l} u \\ =&\sum_{l=0}^{k}\sum_{j=0}^{l} \binom{k}{l}\binom{l}{j} 2^{l-j} P^j u P^{k-l} u \\ =&\sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!k_1!k_2!} 2^{k_1} u_{k_2} u_{k_3}.\end{aligned}$$

The nonlinear terms $F_k(u, u)$ maintain a similar structure of original nonlinear term. This is because the Leibniz law can be applicable for an operation of P. Thus each of u_k satisfies the following system of equations;

$$\begin{cases} i\partial_t u_k + \partial_x^2 u_k = F_k(u, \bar{u}), & t, x \in \mathbb{R}, \\ v_k(0, x) = (x\partial_x)^k \phi(x). \end{cases} \tag{2.5}$$

Therefore we firstly establish the local well-posedness of the solution to the following infinitely coupled system of dispersive equation in a suitable weak space:

$$\begin{cases} i\partial_t u_k + \partial_x^2 u_k = F_k(u, \bar{u}), & t, x \in \mathbb{R}, \\ u_k(0, x) = \phi_k(x). \end{cases} \tag{2.6}$$

Then taking $\phi_k = (x\partial_x)^k \phi(x)$, the uniqueness and local well-posedness allow us to say $u_k = P^k u$ for all $k = 0, 1, \cdots$.

3. Linear and nonlinear estimates

We firstly consider the corresponding linear equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0, & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{3.1}$$

PROPOSITION 3.1. *For the free evolution $U_k(t)$, we have*

$$\|e^{-it\partial_x^2}\phi\|_{L^\theta(I;L^p)} \le C_0\|\phi\|_2,$$

where

$$\frac{2}{\theta} = \frac{1}{2}\left(1 - \frac{2}{p}\right),$$

and

$$\|\int_0^t e^{-i(t-s)D_x^m} F(s)ds\|_{L^\theta(I;L^p)} \le C_1\|F\|_{L^{\rho'}(I;L^{q'})},$$

where

$$\frac{1}{\theta} + \frac{1}{\rho} = \frac{1}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right).$$

According to the Strichartz type linear estimate Proposition 3.1, we have the bilinear estimate for the nonlinear term:

The following estimates of linear and nonlinear part due to Bourgain [2] and refined by Kenig-Ponce-Vega [17] are our essential tools.

LEMMA 3.2. *Let $s \in \mathbb{R}$, $a, a' \in (0, 1/2)$, $b \in (1/2, 1)$and $\delta < 1$. Then for any $k = 0, 1, 2, \cdots$, we have*

$$\|\psi_\delta \phi_k\|_{X^s_{-a}} \le C\delta^{(a-a')/4(1-a')}\|\phi_k\|_{X^s_{-a'}} \tag{3.2}$$

$$\|\psi_\delta e^{it\partial_x^2}\phi_k\|_{X^s_b} \le C\delta^{1/2-b}\|\phi_k\|_{H^s} \tag{3.3}$$

$$\|\psi_\delta \int_0^t e^{i(t-t')\partial_x^2} F(t')dt'\|_{X^s_b} \le C\delta^{1/2-b}\|F\|_{X^s_{b'}}. \tag{3.4}$$

Proof of Lemma 3.2. See [16]. □

The core part of the nonlinear estimate is to establish the bilinear estimate in the space of X^s_b, which is established by Bourgain and Kenig-Ponce-Vega. The following is the somewhat arranged version of them.

PROPOSITION 3.3 (Kenig-Ponce-Vega[19]). *For* $u \in X_b^s$ *then we have*

$$\|u^2\|_{X_{b-1}^s} \leq C\|u\|_{X_b^s}^2, \quad s > -3/4. \tag{3.5}$$

For the proof and the other cases, see [19].

4. Construction of the solution

According to Bourgain [2], we introduce the Fourier restriction space as

$$X_b^s = \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X_b^s} < \infty\},$$

where

$$\|f\|_{X_b^s}^2 = c\iint \langle \tau + \xi^2\rangle^{2b}\langle\xi\rangle^{2s}|\hat{f}(\tau,\xi)|^2 d\tau d\xi = \|e^{-it\partial_x^2} f\|_{H_t^b(\mathbb{R};H_x^s)}^2.$$

The space where we solve the system is infinite sum of this spaces. Let $f = (f_0, f_1, \cdots, f_k, \cdots)$ denotes the infinite series of distributions and define

$$\mathcal{A}_{A_0}(X_b^s) = \{f = (f_0, f_1, \cdots, f_k, \cdots), f_i \in X_b^s \quad (i = 0, 1, 2, \cdots)$$
$$\text{such that } \|f\|_{\mathcal{A}_{A_0}} < \infty\},$$

where

$$\|f\|_{\mathcal{A}_{A_0}} \equiv \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\|f_k\|_{X_b^s}.$$

The system will be shown to be well-posed in the above space if $s > -3/4$.

The well-posedness is derived by utilizing the contraction principle argument to the corresponding system of integral equations:

$$\psi(t)u_k(t) = \psi(t)e^{it\partial_x^2}\phi_k - \psi(t)\int_0^t e^{i(t-t')\partial_x^2}\psi_T(t')F_k(u,\bar{u})(t')dt'. \tag{4.1}$$

PROPOSITION 4.1. *Let* $f(u)$ *be either* u^2 *or* $\bar{u}^2$ *and* $s > -3/4$. *Suppose that for some* $A_0 > 0$, *the initial data* $\phi \in H^s(\mathbb{R})$ *and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!}\|\phi_k\|_{H^s} < \infty,$$

then there exist $T > 0$, $b \in (1/2, 1)$ and the integral equation (4.1) associated with the nonlinear dispersive equation (1.1) is wellposed in the class [1]$\mathcal{C}((-T,T); H^s) \cap \mathcal{A}_{A_0}(X_b^s)$.

If the nonlinearity is $f(u) = |u|^2$, the corresponding result hold for $s > -1/2$.

REMARK 6. Note that for the usual nonlinear Schrödinger equation case, the same conclusion holds for $s \geq 0$.

The outline of the proof is the following: Let a map $\Phi : \{u_k\}_{k=0}^{\infty} \to \{u_k(t)\}_{k=0}^{\infty}$ such that $\Phi = (\Phi_0, \Phi_1, \cdots)$ and

$$\Phi_k(u) \equiv \psi e^{it\partial_x^2}\phi_k - \psi \int_0^t e^{it\partial_x^2} F_k(u, \bar{u})(t')dt'.$$

Then it is shown that $\Phi_k : \mathcal{A}_{A_0}(H^s) \to \mathcal{A}_{A_1}(X_b^s)$ is a contraction mapping.

In fact, by using Lemma 3.2, we easily see that

$$\begin{aligned}
\|\Phi\|_{\mathcal{A}_{A_1}(X_b^s)} =& \sum_{k=0}^{\infty} \frac{A_1^k}{k!}\|u_k\|_{X_b^s} \\
\leq& C_0 \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\|\phi_k\|_{H^s} \\
&+ C_1 T^{\kappa} \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_0+k_1+k_2} 2^{k_0} \frac{k!}{\prod_{i=0}^2 k_i!} \prod_{i=1}^2 \|u_{k_i}\|_{X_b^s} \\
\leq& C_0\|u\|_{\mathcal{A}_{A_0}(H^s)} + C_1 T^{\kappa} \sum_{k_0=0}^{\infty} 2^{k_0} \frac{A_0^{k_0}}{k_0!} \prod_{i=1}^2 \{\sum_{k_i=0}^{\infty} \frac{A_0^{k_i}}{k_i!}\|u_{k_i}\|_{X_b^s}\}.
\end{aligned}$$

Hence, it follows

$$\|\Phi(u)\|_{\mathcal{A}_{A_1}(X_b^s)} \leq C_0\|\phi\|_{\mathcal{A}_{A_0}(H^s)} + C_1 e^{2A_0} T^{\kappa} \|u\|_{\mathcal{A}_{A_1}(X_b^s)}^p,$$

[1]$\mathcal{C}(I; X)$ denotes a space of a sequence of function $f = \{f_i\}_{i=0}^{\infty}$ with $f_i \in C(I; X)$ for each i.

and also we have the estimate for the difference

$$\begin{aligned}&\|\Phi(u^{(1)})-\Phi(u^{(2)})\|_{\mathcal{A}_{A_1}(X_b^s)}\\&\le C_1e^{2A_0}T^{\kappa}(\|u^{(1)}\|^{p-1}_{\mathcal{A}_{A_1}(X_b^s)}+\|u^{(2)}\|^{p-1}_{\mathcal{A}_{A_1}(X_b^s)})\|u^{(1)}-u^{(2)}\|_{\mathcal{A}_{A_1}(X_b^s)}.\end{aligned}$$

Choosing T small enough, the map Φ is contraction from

$$X_T=\{f=(f_0,f_1,\cdots);f_i\in X_b^s,\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\|f_k\|_{X_b^s}\le 2C_0M_0\}$$

to itself, where $M_0=\|u\|_{\mathcal{A}_{A_0}(H^s)}$. This shows the well-posedness.

5. Bootstrap argument

We have constructed a weak solution to the dispersive equation (1.1) satisfying the following extra conormal regularity:

$$\|P^ku\|_{X_b^s}\le CA_0^kk!\quad k=0,1,\cdots,$$

under the condition to the initial data ϕ:

$$\|(x\partial_x)^k\phi\|_{H^s}\le CA_1^kk!\quad k=0,1,\cdots.$$

Now by the localization argument, the operator P can be regarded as a vector field $P_0=3t_0\partial_t+x_0\partial_x$ where $(t_0,x_0)\in\{(-T,0)\cup(0,T)\}\times\mathbb{R}$ is any fixed point. Since the Fourier restriction norm has originally contains the regularity with the characteristic derivative $L^b=\langle i\partial_t+D_x^2\rangle^b$, we combine the both derivative L^b and P_0^k (and by the localization argument) to derive the regularity. If we set a smooth cut-off $a(t,x)$ whose support are around the point (t_0,x_0) with $\operatorname{supp}a\subset B_{2\varepsilon}$. Then we firstly derive

$$\|aP^ku\|_{L^2_{t,x}(\mathbb{R}^2)}\le CA_2^kk!\quad k=0,1,2,\cdots. \tag{5.1}$$

This estimate is obtained by the following lemma which plays a key role in this bootstrap argument.

LEMMA 5.1. *Let $P=2t\partial_t+x\partial_x$ be the generator of the dilation and $D_{t,x}$ be defined by $\mathcal{F}^{-1}_{t,x}\langle|\tau|+|\xi|\rangle\mathcal{F}_{t,x}$. For a fixed point (t_0,x_0), we suppose that $a(t,x)\in C_0^\infty(B_\varepsilon(t_0,x_0))$ and $f\in H^\nu(\mathbb{R}^2_{t,x})$ with $t\partial_x^2f$, $P^2f\in H^{\nu-2}(\mathbb{R}^2_{t,x})$. Then for $\nu\in\mathbb{R}$, there exists a constant $C>0$ such* *that*

$$\|af\|_{H^\nu(\mathbb{R}^2_{t,x})} \le C\Big\{\|af\|_{H^{\nu-2}(\mathbb{R}^2_{t,x})} + \|t\partial_x^2(af)\|_{H^{\nu-2}(\mathbb{R}^2_{t,x})} + \|P^2(af)\|_{H^{\nu-2}(\mathbb{R}^2_{t,x})}\Big\},$$

where the constant C depends on (t_0, x_0) and ε.

Proof Lemma 5.1. Note that $\langle|\tau|+|\xi|\rangle^2 \le C_1(t_0^{-1}, \langle x_0\rangle^{-1})(1+|t_0\xi^2| + |kt_0\tau + x_0\xi|^2)$, which implies

$$\begin{aligned} &\|\langle D_{t,x}\rangle^\nu(af)\|_{L^2(\mathbb{R}^2)} \\ &\le C_1\{\|\langle D_{t,x}\rangle^{\nu-2}(af)\|_{L^2(\mathbb{R}^2)} + \|\langle D_{t,x}\rangle^{\nu-2}t_0\partial_x^2(af)\|_{L^2(\mathbb{R}^2)} \\ &\quad + \|\langle D_{t,x}\rangle^{\nu-2}P_0^2(af)\|_{L^2(\mathbb{R}^2)}\} \end{aligned} \tag{5.2}$$

for $f \in H^k$ and $P_0 = 2t_o\partial_t + x_0\partial_x$. Since supp$a \subset B_\varepsilon(t_0, x_0)$, the second term of the R.H.S. of (5.2) can be estimated by

$$\varepsilon\|\partial_x^2(af)\|_{H^{\nu-2}} + \|t\partial_x^2(af)\|_{H^{\nu-2}}$$

and the third term by $\varepsilon\|R_2f\|_{H^{\nu-2}} + \|P^2f\|_{H^{\nu-2}}$, where R_2 is a partial differential operators of order 2. Hence by taking ε sufficiently small, we obtain the desired estimate (5.2). □

Based upon the above Lemma 5.1, we proceed to show the regularity. The first step is the following proposition.

PROPOSITION 5.2. *Let u be a solution constructed in Proposition 4.1. For some $a(t,x) \in C_0^\infty(\mathbb{R}^2)$ with $a = 1$ near (t_0, x_0), u satisfies*

$$\|aP^k u\|_{H^1} \le C_3A_3^k k!$$

for all $k = 0, 1, 2, \cdots$.

Proof of Proposition 5.2 . For the simplicity, we illustrate the proof for the case $f(u) = u^2$. The other cases are similarly proved. Taking $\nu = 0$ and $f = u_k = P^k u$ in Lemma 5.1, it follows that

$$\begin{aligned} &\|\langle D_{t,x}\rangle a u_k\|_{H^{-1}(\mathbb{R}^2_{t,x})} \\ &\le C\left\{\|au_k\|_{H^{-2}(\mathbb{R}^2_{t,x})} + \|t\partial_x^2(au_k)\|_{H^{-2}(\mathbb{R}^2_{t,x})} + \|P^2(au_k)\|_{H^{-2}(\mathbb{R}^2_{t,x})}\right\}. \end{aligned} \tag{5.3}$$

The first and third term in the R.H.S. of (5.3) is easily estimated by the terms of u_k and u_{k+l}. The second is the essential part which is estimated by

$$\|a\partial_x^2 u_k\|_{H^{-2}(\mathbb{R}^2)} + \|[\partial_x^2, a]u_k\|_{H^{-2}(\mathbb{R}^2)}.$$

Since the commutator $[\partial_x^2, a]$ is a differential operator of order 1, we see

$$\|[\partial_x^2, a]u_k\|_{H^{-2}(\mathbb{R}^2)} \leq C_3\|au_k\|_{H^{-1}(\mathbb{R}^2)} \leq C_3 A_2^k k!. \tag{5.4}$$

While the first term can be dominated via the relation

$$it\partial_x^2 u_k = \frac{1}{2}Pu_k - \frac{1}{2}x\partial_x u_k + itF_k(u, \bar{u}), \tag{5.5}$$

where $u_k = P^k u$, such that

$$\begin{aligned} &\|ta\partial_x^2 u_k\|_{H^{-1}(\mathbb{R}^2)} \\ &\leq C_4\left\{\|au_{k+1}\|_{H^{-1}(\mathbb{R}^2)} + \|ax\partial_x u_k\|_{H^{-1}(\mathbb{R}^2)} + \|atF_k(u, \bar{u})\|_{H^{-1}(\mathbb{R}^2)}\right\} \end{aligned} \tag{5.6}$$

The first and second terms in the RHS in (5.6) are estimated by $C_3 A_2^k k!$. The term involving the nonlinear interaction is dominated as

$$\begin{aligned} \|atF_k(u, \bar{u})\|_{H^{-2}(\mathbb{R}^2)} \leq & \sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!k_1!k_2!} 2^{k_0} \left\|tau_{k_1}u_{k_2}\right\|_{H^{-2}(\mathbb{R}^2)} \\ \leq & C_5 \sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!k_1!k_2!} 2^{k_0} \prod_{i=1}^{2} \|\tilde{a}u_{k_i}\|_{X_{b-1}^s} \\ \leq & C_5 \sum_{k=k_0+k_1+k_2} 2^{k_0} \frac{k!}{k_0!k_1!k_2!} \prod_{i=1}^{2} C_2 A_2^{k_i} k_i! \\ \leq & C_5 C_2^3 \sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!} 2^{k_0} A_2^{k-k_0} \\ = & C_5 C_2^3 \sum_{k_0=0}^{k} \frac{k! 2^{k_0} A_2^{k-k_0}}{k_0!} \sum_{k_1+k_2=k-k_0} 1 \\ = & C_5 C_2^3 \sum_{k_0=0}^{k} \frac{k! 2^{k_0} A_2^{k-k_0}}{k_0!(k-k_0)!} \frac{(k-k_0+1)!}{1} \\ = & C_2^3 C_5 (A_2+2)^k (k+2)! \leq C_6 A_3^k k!, \end{aligned} \tag{5.7}$$

where we have taken the constants C_5 and C_6 are depending on $|t-t_0|$ and A_3 appropriately large. Hence by gathering the estimates (5.2)-(5.7) and changing the cut off function a if necessary, we have

$$\|au_k\|_{L^2(\mathbb{R}^2)} \le C_7 A_3^k k!, \quad k=0,1,2,\cdots. \tag{5.8}$$

Similar but somewhat tiresome estimates yield that

$$\|\langle D_{t,x}\rangle au_k\|_{L^2(\mathbb{R}^2)} \le C_7 A_4^k k!, \quad k=0,1,2,\cdots.$$

By repeating the above argument once more, we conclude by changing the cut off function into $\tilde{a}$, to have for $\delta > 0$,

$$\|\tilde{a}u_k\|_{H^{1-\delta}} \le C_9 A_5^k k!, \quad k=0,1,2,\cdots. \qquad \square$$

Rest of the proof goes a similar way in [12]. Based on the estimate (5.1), we forward the second step to have

$$\sup_{t_0-\varepsilon<t<t_0+\varepsilon} \|aP^k u(t)\|_{H_x^{1/2-\delta}(B_\varepsilon(x_0))} \le CA_6^k k! \quad k=0,1,2,\cdots.$$

Note that $H^{1/2+\delta}(\mathbb{R}^1)$ is an algebra. Then one can prove by an induction argument, that

$$\sup_t \|a\partial_x^l P^k u(t)\|_{H_x^{1/2-\delta}(B_\varepsilon(x_0))} \le CA_7^{k+l}(k+l)! \quad k,l=0,1,2,\cdots.$$

Finally the operator P can be translated into the time derivative via $t\partial_t = k^{-1}(P - x\partial_x)$;

$$\sup_t \|a(t\partial_t)^{l_1}\partial_x^{l_2} v\|_{H_{t,x}^{1/2-\delta}(\mathbb{R}^2)} \le CA_8^{l_1+l_2}(l_1+l_2)! \quad l_1,l_2=0,1,2,\cdots.$$

This gives the regularity for the solution.

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