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Some Asymptotic Properties of Conditional Covariance in the Item Response Theory[1]

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Abstract

A dimensionality assessment procedure DETECT uses the property of being near zero of conditional covariances as an indication of unidimensionality. This study provides the convergent properties to zero of conditional covariances when the data is unidimensional, with which DETECT extends its theoretical grounds.

Keywords : conditional covariance, item response theory, unidimensionality, local independence, DETECT

1. INTRODUCTION

A dimensionality assessment procedure DETECT (Kim (1994) and Kim (2000)) makes use of the conditional covariance of being near zero as an indication of unidimensionality of the data in the Item Response Theory (IRT). Under the assumptions of the traditional IRT, conditional covariance of a pair of items is zero when data is unidimensional. However, due to the unreliability of the condition for the finite length test, the estimated conditional covariances often turn out to be non-zero even with unidimensional data. This paper studies convergence to zero of the conditional covariance under unidimensionality as the test length goes to infinity, which strengthens the theoretical background of DETECT.

The next section briefly explains the IRT framework and DETECT and the following states theorems with their proofs. Then the final section concludes by presenting the possible and related use of the results and future work relevant to this study.

2. AN ITEM RESPONSE THEORY MODELING

The IRT modeling attempts to describe the manifest probabilities of an observable variable X, by an underlying model for a possibly vector valued latent variable Θ. Given enough data we observe the probability of a particular response pattern x of a test having n items,

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$P(X=x) = P(X_1=x_1,\ldots,X_n=x_n)$. Then the model tries to reconstruct these manifest probabilities through an informative model detailing how responses depend upon an examinee's latent ability Θ. We denote this latent variable model as (X,Θ) and θ as a realization of Θ. For convenience, we consider this modeling only for the case of dichotomously scored $(0/1)$ items, i.e., for $i=1,\ldots,n$, $x_i=0$ for the incorrect answer or 1 for the correct one. A complete model for the manifest distribution of X is given by specifying for all values of θ, the density function $f(\theta)$, and the joint vector response functions $P(X=x\mid\theta)$ such that for each x the following equation holds;

$$P(X=x) = \int\ldots\int P(x\mid\theta)f(\theta)\,d\theta.$$

Moreover, for a given θ, the probability $P_i(\theta)$ of answering an item X_i correct is denoted $P_i(\theta) = P(X_i=1\mid\theta) = E[X_i\mid\theta]$ and called the item characteristic curve.

In order to make the model statistically illuminating and psychologically meaningful, definitions of local independence, monotonicity, and dimension are included.

Definitions 1. A model (X,Θ) is locally independent (LI) if for every given θ and every response vector x

$$\begin{aligned} P(X=x\mid\theta) &= P(X_1=x_1,\ldots,X_n=x_n\mid\theta)\\ &= \prod_{i=1}^{n} P(X_i=x_i\mid\theta). \end{aligned}$$

2. A model (X,Θ) is said to be monotone (M) if $P(X_{i_1}=1,\ldots,X_{i_k}=1\mid\theta)$ is nondecreasing in θ for each subset $(i_1,\ldots,i_k)$ of $(1,\ldots,n)$. The model is said to be strictly monotone if nondecreasing is replaced by strictly increasing.
3. A model (X,Θ) is said to be d dimensional ($d=1$) if Θ is a d dimensional random vector.

Testing the LI assumption is equivalent to testing the assumption of unidimensionality when LI is assumed. If a $d=1$, LI, and M model yields rejection (lack of fit), and if we insist on M, either LI or $d=1$ must go. It is the modeler's choice to decide which way to go on this.

One straightforward and theoretically sound approach to investigate the possibility of multidimensionality is to consider covariances of disjoint subsets of items after conditioning on a reliable substitute for the latent variable. For instance, one might consider the conditional covariance of a pair of items. For a pair of items, X_i and X_j, under the local independence

$$Cov(X_i,X_j\mid\theta) = P(X_i=1,X_j=1\mid\theta) - P(X_i=1\mid\theta)\,P(X_j=1\mid\theta) = 0$$

due to the factoring of joint conditional probabilities into products of marginal conditional probabilities. Thus, requiring that $Cov(X_i,X_j\mid\theta)=0$ is a necessary but not sufficient condition for LI, and consequently any investigation that reveals that for some θ, $Cov(X_i, X_j | \theta) \neq 0$ succeeds in discovering a violation of LI. Of course, the latent variable θ cannot be observed and one is forced to examine conditional covariances where the conditioning variable is a suitable substitute for θ. A large body of theoretical research has focused on the adequacy of observable statistics such as total score as substitutes for θ in examining conditional covariances.

It is assumed that a test is composed of disjoint clusters that are dimensionally distinctive from each other and each cluster is dimensionally homogeneous. Then it is conjectured that the conditional covariance estimate of an item pair given score of the remaining test is positive or negative subject to whether two items in the pair belong to the same cluster or not, respectively. The index DETECT by Kim (1994) combines non-zero second order conditional covariances of item pairs evidencing violation of unidimensionality by adding conditional covariances when two items come from the same cluster and subtracting conditional covariances when two items come from different clusters. That is, for an (i, j) item pair

$$DETECT = \frac{1}{\binom{n}{2}} \sum_{(i,j)} (-1)^{\delta_{ij}} \left(\widehat{Cov}_{ij} - \overline{Cov} \right)$$

where summation extends over all item pairs and δ_{ij} is determined to be 0 when two items belong to the same cluster or 1 otherwise. Note that the average $\overline{Cov}$ is the overall mean of covariances and the subtraction is introduced to extract off the bias occurred possibly in the unidimensional case (Kim, 1994). Also it is obvious that the maximum DETECT occurs if the correct cluster formation is utilized.

A primary objective of dimensionality assessment is to identify dimensionally homogeneous item clusters if exist in a test. DETECT makes an attempt to identify such clusters and to quantify the amount of the lack of unidimensionality present in the test. A large body of research has shown good performance of DETECT. For details, see Kim (1994). More theoretical and applied consideration relating to DETECT is also given in Zhang and Stout (1999).

The following section focuses on the convergence properties of conditional covariances employed as building blocks of DETECT especially when the data is unidimensional.

3. ASYMPTOTIC PROPERTIES OF CONDITIONAL COVARIANCE WHEN $d = 1$

The aforementioned equality $Cov(X_i, X_j | \theta) = 0$ holds under the conditioning on the true ability θ. Holland and Rosenbaum (1986) examined the conditional association as an observable stand-in for θ is utilized for the condition. Making use of Holland and

Rosenbaum's theorem, the following inequality is directly resulted when conditioned on an observed proportion correct score S out of $n-2$ items after deleting items X_i and X_j. That is, under the condition of LI and M

$$Cov(X_i, X_j \mid S) \geq 0$$

for $i \neq j$.

Strict inequality is given when the model is strictly monotone; $Cov(X_i, X_j \mid S) > 0$ for $i \neq j$, under the condition of LI and strictly monotone. This strict inequality is unfortunate, because zero or very close to zero conditional covariance is more desirable, but of vital importance. However, as the test length approaches infinity, the sum score on the remaining items becomes more reliable and a better ordinal estimate of θ ensuring that items X_i and X_j are asymptotically uncorrelated given the score on the growing number of remaining items.

Assume that there is an infinite item pool $\{X_i, i \geq 1\}$. Suppose that for fixed n the all item characteristic functions $P_i(\theta)$, $i = 1, \ldots, n$, are differentiable. Also, suppose that there exist an interval $[a, b]$, and real numbers $\delta > 0$ and $C < \infty$ satisfying, for all $\theta \in [a, b]$ and i,

$$f(\theta) \geq \delta, \quad (1)$$

$$P_i'(\theta) \geq \delta, \quad (2)$$

$$P_i'(\theta) \leq C, \quad (3)$$

$$P_i(\theta)(1 - P_i(\theta)) \geq \delta. \quad (4)$$

For fixed n, let J_k, $k = 0, \ldots, n-2$, be the number of examinees in the k_{th} cell of a partition of examinees based on the score S. In particular, assume that each partition contains ''enough'' examinees so that asymptotic properties are applicable. Let $\theta_k^{(n)}$ be defined by $E[S \mid \Theta = \theta_k^{(n)}] = \frac{k}{n-2}$ for fixed n and k. Suppose, for all k, that

$$\theta_k^{(n)} \in (a - \delta, b + \delta). \quad (5)$$

Suppose there exists $c > 0$ satisfying

$$\frac{\min_{0 \leq k \leq n-2} J_k}{\max_{0 \leq k \leq n-2} J_k} \geq c \quad (6)$$

for all n, and

$$\min_{0 \leq k \leq n-2} J_k \rightarrow \infty \quad (7)$$

as n goes to infinity. Suppose

$$\frac{\max_{0 \leq k \leq n-2} J_k}{n} \rightarrow 0 \quad (8)$$

as n goes to infinity. Let $J^{(n)}$ represent the total number of examinees taking an n item test and define $s_k = \frac{k}{n-2}$. The above equations (1)-(8) and conditions can be considered as the regularity conditions in the IRT under which the coming convergence has meaning practically.

Proposition (Stout, 1987) Fix k and n and assume the conditions (1)-(8) hold. Let constants a_n satisfy

$$2n^{-1} \le a_n \le Cn^{-1/2}.$$

Then

$$P(|\Theta - \theta^{(n)}| \ge \frac{2x}{\delta} \mid S = s_k) \le \frac{C\exp(-nx^2/4)}{a_n(n^{1/2}a_n - n^{-1/2})}$$

for all $a_n \le x \le (\log n)^{-1/2}$ and δ.

The next theorem assumes the properties given above, and concerns the convergence of conditional covariances as the test length and sample size approach infinity.

Theorem 1 If a model (X, Θ) is unidimensional and locally independent and if (1)-(8) hold, then for any fixed k and for fixed $i \neq j$

$$(J_k\, n)^{1/2}\, Cov(X_i, X_j \mid S = s_k) \to 0$$

as n goes to infinity.

Proof By observing that for any k, $k = 0, \ldots, n-2$,

$$\begin{aligned} Cov(X_i, X_j \mid S = s_k) &= E[Cov(X_i, X_j \mid \theta) \mid S = s_k] \\ &\quad + Cov(E[X_i \mid \theta], E[X_j \mid \theta] \mid S = s_k] \\ &= Cov(P_i(\theta), P_j(\theta) \mid S = s_k) \end{aligned}$$

by LI and

$$Cov(P_i(\theta), P_j(\theta) \mid S = s_k) \le \{Var(P_i(\theta) \mid S = s_k)\, Var(P_j(\theta) \mid S = s_k)\}^{1/2},$$

it suffices to consider $Var(P_i(\theta) \mid S = s_k)$.

Let $\theta^{(n)} = \theta_k^{(n)}$ and $f^{(n)}(\theta)$ be the density function of $\theta^{(n)}$ for fixed n and k, and

$$E^{(n)}[P_i(\theta)] = E[P_i(\theta) \mid S = s_k] = \int P_i(\theta) f^{(n)}(\theta)\, d\theta.$$

Let $x' = \frac{2x}{\delta}$ and $\Omega^{(n)} = [\theta^{(n)} - x', \theta^{(n)} + x']$. Then

$$\begin{aligned} Var(P_i(\theta) \mid S=s_k) &= \int (P_i(\theta)-E^{(n)}[P_i(\theta)])^2 f^{(n)}(\theta)\, d\theta \\ &= \int_{\Omega^{(n)}} (P_i(\theta)-E^{(n)}[P_i(\theta)])^2 f^{(n)}(\theta)\, d\theta \\ &\quad + \int_{\Omega^{(n)\,c}} (P_i(\theta)-E^{(n)}[P_i(\theta)])^2 f^{(n)}(\theta)\, d\theta \\ &\equiv V + W. \end{aligned}$$

By Proposition given above, for any $a_n \le x \le (\log n)^{-1/2}$

$$W \le \frac{C\exp(-nx^2/4)}{a_n(n^{1/2}a_n - n^{-1/2})} \equiv Q_n.$$

Consider V. Now using the mean value theorem, there exists θ_0 satisfying $|\theta_0 - \theta^{(n)}| \le x'$ such that

$$\begin{aligned} |P_i(\theta) - E^{(n)}[P_i(\theta)]| &= |P_i(\theta) - \int_{\Omega^{(n)}} P_i(\theta')f^{(n)}(\theta')\, d\theta' - \int_{\Omega^{(n)\,c}} P_i(\theta')f^{(n)}(\theta')\, d\theta'| \\ &\le |P_i(\theta) - \frac{\int_{\Omega^{(n)}} P_i(\theta')f^{(n)}(\theta')\, d\theta'}{\int_{\Omega^{(n)}} f^{(n)}(\theta')\, d\theta'} \int_{\Omega^{(n)}} f^{(n)}(\theta')\, d\theta'| + Q_n \\ &\le |P_i(\theta) - P_i(\theta_0)\int_{\Omega^{(n)}} f^{(n)}(\theta')\, d\theta'| + Q_n \\ &\le |P_i(\theta) - P_i(\theta_0)| \int_{\Omega^{(n)}} f^{(n)}(\theta)\, d\theta + 2Q_n \\ &\le Cx' + 2Q_n. \end{aligned}$$

Thus $Var(P_i(\theta) \mid S=s_k) \le (Cx' + 2Q_n)^2 + Q_n$.

Now let $h_n = (J_k n)^{1/2}\{(Cx' + 2Q_n)^2 + Q_n\}$ for each n. In order to complete the proof it is needed to show that $h_n \to 0$ as n goes to infinity uniformly in k. It suffices to observe that

$$(J_k n)^{1/2}(x')^2 \to 0, \qquad (J_k n)^{1/2} Q_n \to 0$$

as n goes to infinity, uniformly in k. Recall $x' = \frac{2x}{\delta}$. Taking $x' = \frac{\log n}{n^{1/2}}$ works, for example. I.e.,

$$(J_k n)^{1/2}\, Cov(X_i, X_j \mid S=s_k) \le h_n \to 0$$

as n goes to infinity, uniformly in k. □

Informally under the assumption of unidimensionality, as the test length goes to infinity the conditional covariance between two items vanishes more quickly than the reciprocal of the square root of the product of the test length and the number of examinees in the score cell.

Now consider the asymptotic behavior of

$$\widehat{Cov}(X_i, X_j \mid S=s_k) = \frac{1}{J_k}\sum_{t=1}^{J_k}(X_{it} - \overline{X_i} \mid S=s_k)(X_{jt} - \overline{X_j} \mid S=s_k)$$

as an estimate of $Cov(X_i, X_j \mid S=s_k)$ when test length n approaches infinity. Here X_{it} denotes the t_{th} examinee's response for the i_{th} item. For fixed k, it only utilizes examinees in the subgroup of k items correct. Here $\overline{X_l}$, $l=i,j$, denotes the average score on the item X_l, $l=i,j$, among the examinees in the subgroup, respectively, and the summation extends over the examinees in the score subgroup. Theorem 2 implies that the expected value of a weighted sum of conditional covariances, under the assumption of unidimensionality and local independence, converges to zero fairly rapidly as the test length increases.

Theorem 2 If a model (X, Θ) is unidimensional and locally independent and if (1)-(8) hold, then for a fixed (i,j) item pair

$$E\left[\sum_{k=0}^{n-2} \frac{J_k}{J^{(n)}} \widehat{Cov}(X_i, X_j \mid S=s_k)\right] \rightarrow 0$$

as n goes to infinity.

Proof First, consider the conditional covariance only in one partition of examinees for fixed k. Let $Cov_{ij} = Cov(X_i, X_j \mid s_k = k)$.

$$\begin{aligned} E[\widehat{Cov}_{ij}] &= E\left[\frac{1}{J_k} \sum_{t=1}^{J_k} (X_{it} - \overline{X_i} \mid S=s_k)(X_{jt} - \overline{X_j} \mid S=s_k)\right] \\ &= \frac{1}{J_k}\left\{ \sum_{t=1}^{J_k} E[X_{it} X_{jt} \mid S=s_k] - J_k(\overline{X_i} \mid S=s_k)(\overline{X_j} \mid S=s_k) \right\} \\ &= \frac{1}{J_k}\left\{ \sum_{t=1}^{J_k} E[X_{it} X_{jt} \mid S=s_k] - J_k E[X_i \mid S=s_k]\, E[X_j \mid S=s_k] \right\} \\ &= Cov_{ij}, \end{aligned}$$

by the unbiasedness of $\overline{X_l}$, $l=i,j$, and it shows that $\widehat{Cov}(X_i, X_j \mid S=s_k)$ is an unbiased estimator of $Cov(X_i, X_j \mid S=s_k)$.

Then

$$E\left[\sum_{k=0}^{n-2} \frac{J_k}{J^{(n)}} \widehat{Cov}(X_i, X_j \mid S=s_k)\right] = \frac{1}{J^{(n)}} \sum_{k=0}^{n-2} J_k\, Cov(X_i, X_j \mid S=s_k),$$

and it can be expressed as

$$\frac{1}{J^{(n)}} \sum_{k=0}^{n-2} \left(\frac{J_k}{n}\right)^{1/2} (J_k n)^{1/2} Cov(X_i, X_j \mid S=s_k).$$

Now, using h_n defined in the proof of Theorem 1, the above is bounded by

$$\begin{aligned} \frac{1}{J^{(n)}} \sum_{k=0}^{n-2} \left(\frac{J_k}{n}\right)^{1/2} h_n &= h_n \sum_{k=0}^{n-2} \frac{1}{n^{1/2}} \frac{J_k}{J^{(n)}} \\ &= \frac{h_n}{n^{1/2}} \sum_{k=0}^{n-2} \frac{J_k}{J^{(n)}} \\ &= \frac{h_n}{n^{1/2}} \\ &\rightarrow 0 \end{aligned}$$

quickly as n approaches infinity, by Theorem 1 and $1/n$, and it completes the proof. □

It easily follows that under the unidimensionality and local independence, the expected value of DETECT converges to zero as the test length approaches infinity when only one cluster is applied. Equivalently, one can expect when the test is unidimensional and locally independent the expected value of DETECT to converge to zero as the test length approaches infinity for whatever cluster formation is employed. For instance, when a test is unidimensional, the cluster have no significant meaning because they are formed primarily from random statistical error.

4. CLOSING REMARKS

Two convergence theorems are proved under the conditions of unidimensionality and local independence. Under the assumption of unidimensionality, as the test length grows the conditional covariance between two items vanishes more quickly than the reciprocal of the square root of the product of the test length and the number of examinees in the score cell. Also the expected value of a weighted sum of conditional covariances, under the assumption of unidimensionality and local independence, converges to zero fairly rapidly as the test length increases. These results can be served as a bench mark in the dimensionality assessment using DETECT. Similar study for the more than one dimension case is possible even though much more complicated work needed.

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