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Motivating Curls

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We motivate the curl of the velocity field of a fluid in three ways: from a calculation of the velocity of a rotating fluid relative to a coordinate system rotating with the fluid and from two calculations of a vector form of circulation on small circles or spheres suspended in a moving fluid.

1. INTRODUCTION

> ⋯ I propose, but with great diffidence, to call this vector the *Curl* or *Version* of the original vector function. ⋯ It represents the direction and magnitude of the rotation of the subject matter carried by the vector [function]. ⋯ I have sought for a word which shall neither, like Rotation, Whirl, or Twirl, connote motion, nor, like Twist, indicate a helical or screw structure which is not of the nature of vector at all.
>
> — J. C. Maxwell (1871)

These words were written by James Clerk Maxwell in 1871, some twenty years after Stokes' Theorem and the curl of a vector field were discovered. Special cases of curl had been discussed a little earlier in Britain, France, and Germany. William Thompson first stated what is now called "Stokes' Theorem." Much of the history of curl and of Stokes' Theorem can be found in Maxwell (1871) and Stokes (1880-1905, pp. 320-321; 1990).

Among calculus books including a motivation of the curl of a vector field, most use either a limiting argument involving Stokes' Theorem or a calculation of the angular velocity of a simple, planar flow. Physics and engineering books often infer the curl by restricting the flow to a small square and then calculating one component of the curl by linearizing the circulation (the line integral occurring in Stokes' Theorem). For these arguments see, for example, Brand (1957), Davis and Snider (1979), Feynman, Leighton and Sands (1964), and Kemmer (1977). In this note we infer the form of curl in three other ways. First we show that the form of curl emerges from a calculation of the velocity of a rotating fluid relative to a coordinate system rotating with the fluid. This calculation is part of intermediate mechanics but is rarely seen in beginning calculus courses. Our discussion is based on that in Symon (1971). Secondly, we show that the form of curl emerges from two calculations of a vector form of circulation on small circles or spheres suspended in a moving fluid.

2. A ROTATING COORDINATE SYSTEM

Suppose that a fluid is rotating with constant angular speed ω about a line L through the origin, so that, looking downwards along L, the fluid is flowing counterclockwise. The angular velocity of the fluid is the vector $\boldsymbol{\omega}$ in the direction of L and with length ω (see Figure 1). A coordinate system relative to which the fluid is stationary can be described as follows. Letting $\mathbf{u}$ and $\mathbf{v}$ be fixed, perpendicular unit vectors such that $\mathbf{u}\times\mathbf{v} = (1/\omega)\boldsymbol{\omega}$, define, for any time t, the mutually perpendicular unit vectors $\mathbf{e}_1(t) = \cos(\omega t)\mathbf{u} + \sin(\omega t)\mathbf{v}$, $\mathbf{e}_2(t) = -\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{v}$, and $\mathbf{e}_3(t) = (1/\omega)\boldsymbol{\omega}$. It is easy to show that $d\mathbf{e}_1(t)/dt = \omega\mathbf{e}_2(t)$ and $d\mathbf{e}_2(t)/dt = -\omega\mathbf{e}_1(t)$.

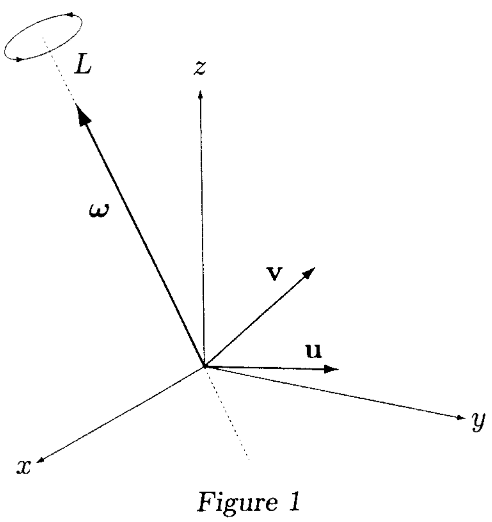

Figure 1

If at time t the position of a fluid particle is $\mathbf{r}(t) = \sum_i r_i(t)\mathbf{e}_i(t)$, its velocity is

$$\mathbf{v} = \frac{d}{dt}\sum_i r_i\mathbf{e}_i = \sum_i \left(\frac{dr_i}{dt}\,\mathbf{e}_i + r_i\frac{d\mathbf{e}_i}{dt}\right)$$
$$= \sum_i \frac{dr_i}{dt}\,\mathbf{e}_i + \omega\left(-r_2\mathbf{e}_1 + r_1\mathbf{e}_2\right).$$

Writing $\mathbf{v}^* = \sum_i (dr_i/dt)\,\mathbf{e}_i$, which is the velocity of the particle as observed in the rotating system, and noting that the remaining terms are $\boldsymbol{\omega}\times\mathbf{r}$, the relationship between $\mathbf{v}$ and $\mathbf{v}^*$ is

(1) $$\mathbf{v} = \mathbf{v}^* + \boldsymbol{\omega}\times\mathbf{r}.$$

We use this equation—now regarding it as an equation in the variables x, y, and z—in showing that the vector $\nabla\times\mathbf{v}$ is a measure of rotation of the fluid. From (1) and the fact that $\nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = 2\boldsymbol{\omega}$,

$$\nabla\times\mathbf{v} = \nabla\times\mathbf{v}^* + \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = \nabla\times\mathbf{v}^* + 2\boldsymbol{\omega}.$$

If we assume that the fluid is at rest with respect to the rotating system then $\nabla\times\mathbf{v}^* = \mathbf{0}$ and

$$\nabla\times\mathbf{v} = 2\boldsymbol{\omega}.$$

Hence, $\frac{1}{2}\nabla\times\mathbf{v}$ is the angular velocity of the fluid.

3. THE "MOMENT OF CIRCULATION" ON A CIRCLE

The circulation $\mathcal{C}$ of a fluid around a closed curve C is $\mathcal{C} = \int_C \mathbf{v}\cdot\mathbf{T}\,ds$, where $\mathbf{T}$ is a unit tangent to C (see Figure 2(a)). As noted earlier, this integral appears in Stokes' Theorem. We define the *moment of circulation on C relative to* $\mathbf{r}_0$ as

(2) $$\mathcal{T}_a = \int_C \left((\mathbf{r} - \mathbf{r}_0)\times\mathbf{v}_\mathbf{T}\right)\,ds,$$

where $\mathbf{v}_\mathbf{T} = (\mathbf{v}\cdot\mathbf{T})\,\mathbf{T}$.

Because implicit in the notion of rotation is an axis about which the rotation occurs, we replace C by a circle C_a of radius a and centered at $\mathbf{r}_0$. We parameterize C_a by $\mathbf{r} = \mathbf{r}(\theta) = \mathbf{r}_0 + (a\cos\theta)\mathbf{e}_1 + (a\sin\theta)\mathbf{e}_2$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors in the plane of C_a. Denoting the circulation and moment of circulation on C_a by $\mathcal{C}_a$ and $\mathcal{T}_a$, it follows easily that

(3) $$\mathcal{T}_a = a\mathcal{C}_a\,\mathbf{e}_3,$$

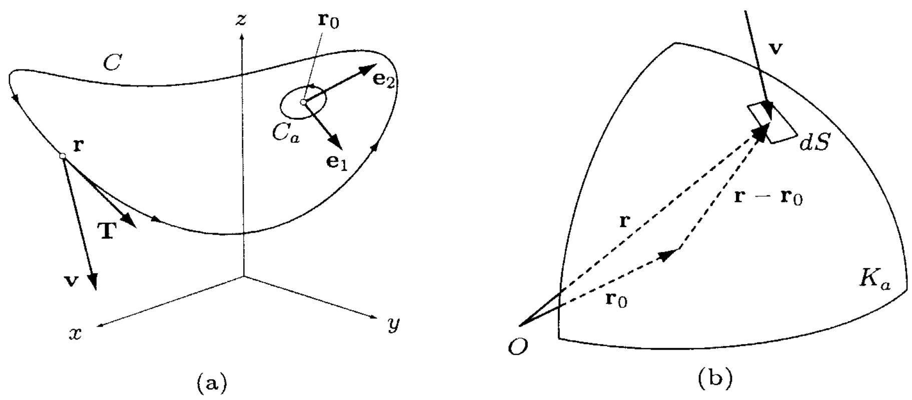

Figure 2

where $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. This relates the circulation and the moment of circulation. Informally, we have a "force" of magnitude $\mathcal{C}_a$ acting on C_a, with lever arm a. This gives a "torque" $\mathcal{T}_a$.

Because we are interested in a measure of the rotation of the fluid at $\mathbf{r}_0$, we think of a as small and approximate $\mathcal{T}_a$ with the linearization

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{r}_0 + (\mathbf{r} - \mathbf{r}_0)) \approx \mathbf{v}(\mathbf{r}_0) + \mathbf{J}_0(\mathbf{r} - \mathbf{r}_0), \tag{4}$$

where $\mathbf{r}$ is on C_a and $\mathbf{J}_0$ is the Jacobian matrix of $\mathbf{v}$ at $\mathbf{r}_0$. Using (4) in (2), a straightforward calculation shows that

$$\mathcal{T}_a \approx \pi a^3 (\nabla \times \mathbf{v}(\mathbf{r}_0) \cdot \mathbf{e}_3)\ \mathbf{e}_3. \tag{5}$$

Thus, to maximize the magnitude of the moment of circulation per unit area we orient the plane of C_a so that its normal is in the direction of $\nabla \times \mathbf{v}(\mathbf{r}_0)$.

4. THE "MOMENT OF CIRCULATION" ON A SPHERE

Discussions of the rotation of a fluid often include mention or sketches of a small paddle wheel, suspended in the liquid but free to rotate about its axle. Instead, we suspend a ball K_a of radius a and calculate its moment of circulation $\mathcal{K}_a$. We show that, for small a, the direction of this "torque vector" is approximately that of the direction of the curl of the velocity field.

Referring to Figure 2(b), which shows part of the sphere K_a, we define the *moment of circulation on K_a relative to $\mathbf{r}_0$* as

$$\mathcal{K}_a = \iint_{K_a} ((\mathbf{r} - \mathbf{r}_0) \times \mathbf{v})\ dS,$$

where $\mathbf{r}_0$ is the center of K_a and dS is a surface element. Resolving the velocity $\mathbf{v}$ at $\mathbf{r}$ into normal and tangential components $\mathbf{v_N}$ and $\mathbf{v_T}$, we note that because $\mathbf{r}-\mathbf{r}_0$ is parallel to $\mathbf{v_N}$,

$$\mathcal{K}_a = \iint_{K_a} ((\mathbf{r}-\mathbf{r}_0)\times\mathbf{v_T})\, dS.$$

From this we see that the moment of circulation on K_a has the same structure as the moment of circulation on C_a.

Parameterizing K_a by

$$\mathbf{r} = \mathbf{r}(\phi,\theta) = \mathbf{r}_0 + a\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi,$$

We have

$$(6) \qquad \mathcal{K}_a = a\int_0^{2\pi}\int_0^{\pi} \sin\phi\big(\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\times\mathbf{v}(\mathbf{r}(\phi,\theta)\big)\, d\phi\, d\theta.$$

If, for points $\mathbf{r}$ near $\mathbf{r}_0$, we use the linear approximation (4), a straightforward calculation shows that

$$\mathcal{K}_a \approx \tfrac{4}{3}\pi k a^4 \nabla\times\mathbf{v}(\mathbf{r}_0).$$

We note that the moment of circulation on K_a per unit volume is $a\nabla\times\mathbf{v}(\mathbf{r}_0)$, a result similar in form to the moment of circulation on C_a per unit area.

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