THE CONVERGENCE THEOREMS FOR THE McSHANE-STIELTJES INTEGRAL

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ABSTRACT. In this paper, we define the uniformly sequence for the vector valued McSane-Stieltjes integrable functions and prove the dominated convergence theorem for the McShane-Stieltjes integrable functions.

1. Introduction and Preliminaries

It is well known that the Riemann-Stieltjes integral is not adequate for advanced mathematics, since there are many functions that are not Riemann-Stieltjes integrable, and since the integral does not possess sufficiently strong convergence theorems. In the late 1960’s, McShane [8] proved that the Lebesgue integral is indeed equivalent to a modified version of the Henstock integral (cf. Henstock [5]). Yoon, Eun and Lee [9] defined the McShane-Stieltjes integral for real-valued function which is the generalization of the McShane integral and investigated some properties of this integral. Gordon [3] generalized the definition of the McShane integral for real-valued functions to functions taking values in Banach spaces and investigated some of its properties. Many authors have studied McShane integral (cf. [3], [4]).

In this paper, we define the uniformly sequence for the Banach-valued McShane-Stieltjes integrable functions and prove the dominated convergence theorem for the McShane-Stieltjes integrable functions. Throughout this paper, $X$ is a Banach space and we always assume that $\alpha$ is an increasing function on $[a, b]$ unless otherwise stated. We begin with some definitions.

Definition 1.1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A free tagged interval $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$ and a point $x \in [a, b]$. 

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The free tagged interval \((x, [c, d])\) is subordinate to \(\delta\) if
\[
[c, d] \subseteq (x - \delta(x), x + \delta(x)).
\]

The letter \(\mathcal{P}\) will be used to denote finite collections of non-overlapping free tagged intervals. Let \(\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}\) be such a collection in \([a, b]\). We adopt the following terminology:

1. The points \(\{x_i : 1 \leq i \leq n\}\) are the tags of \(\mathcal{P}\) and the intervals \(\{[c_i, d_i] : 1 \leq i \leq n\}\) are the intervals of \(\mathcal{P}\).
2. If \((x_i, [c_i, d_i])\) is subordinate to \(\delta\) for each \(i\), then \(\mathcal{P}\) is subordinate to \(\delta\).
3. If \(\mathcal{P}\) is subordinate to \(\delta\) and \([a, b] = \bigcup_{i=1}^{n}[c_i, d_i]\), then \(\mathcal{P}\) is a free tagged partition of \([a, b]\) that is subordinate to \(\delta\).

Let \(\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}\) be a finite collection of non-overlapping free tagged intervals in \([a, b]\), let \(f : [a, b] \to \mathbb{R}\), and let \(\alpha\) be an increasing function on \([a, b]\). We will use the following notations:
\[
f(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(d_i - c_i)
\]
and
\[
f^\alpha(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(\alpha(d_i) - \alpha(c_i)).
\]

**Definition 1.2.** The function \(f : [a, b] \to X\) is the McShane integrable on \([a, b]\) if there exists a vector \(z\) in \(X\) with the following property:

For each \(\varepsilon > 0\) there exists a positive function \(\delta\) on \([a, b]\) such that
\[
\|f(\mathcal{P}) - z\| < \varepsilon
\]
whenever \(\mathcal{P}\) is subordinate to \(\delta\) on \([a, b]\).

The function \(f\) is McShane integrable on a measurable set \(E \subseteq [a, b]\) if the function \(f\chi_E\) is McShane integrable on \([a, b]\).

**Definition 1.3.** The function \(f : [a, b] \to X\) is the McShane-Stieltjes integral function with respect to \(\alpha\) if for each \(\varepsilon > 0\) there exists a positive function \(\delta\) on \([a, b]\) such that
\[
|f^\alpha(\mathcal{P}) - z| < \varepsilon
\]
enever \(\mathcal{P} = \{(t_i, [a_i, b_i]) : 1 \leq i \leq n\}\) is a McShane partition of \([a, b]\) that is subordinate to \(\delta\). In this case, we write \(z = \int_{a}^{b} f d\alpha\).

A function \(f : [a, b] \to X\) is McShane-Stieltjes integrable with respect to \(\alpha\) on a measurable set \(E \subseteq [a, b]\) if \(f\chi_E\) is McShane-Stieltjes integrable with respect to \(\alpha\) on \([a, b]\).
2. The Convergence Theorems for the McShane-Stieltjes Integral

We now mention Henstock's Lemma for real-valued McShane integrable functions. For the proof, see Gordon [4].

**Lemma 2.1 (Saks-Henstock Lemma).** Let $f : [a, b] \to \mathbb{R}$ be McShane integrable on $[a, b]$. Let $F(x) = \int_a^x f$ for each $x \in [a, b]$, and let $\varepsilon > 0$. Suppose that $\delta$ is a positive function on $[a, b]$ such that $|f(\mathcal{P}) - F(\mathcal{P})| < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to $\delta$. If $\mathcal{P}_0 = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is subordinate to $\delta$, then

$$\sum_{i=1}^{n} \left| f(x_i)(d_i - c_i) - (F(d_i) - F(c_i)) \right| \leq 2\varepsilon.$$

We state a weak version of Saks-Henstock Lemma which holds for the real-valued McShane-Stieltjes integrable functions, whose proof is identical to the real-valued McShane-integrable function case (See [4, Theorem 3.7]).

**Lemma 2.2 (Weak Saks-Henstock Lemma).** Let $f : [a, b] \to \mathbb{R}$ be McShane-Stieltjes integrable on $[a, b]$ with respect to $\alpha$. Let $F^\alpha(x) = \int_a^x f \, d\alpha$ for each $x \in [a, b]$, and let $\varepsilon > 0$. Suppose that $f$ is a positive function on $[a, b]$ such that $|f^\alpha(\mathcal{P}) - F^\alpha(\mathcal{P})| < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to $\delta$.

If $\mathcal{P}_0 = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is any collection of non-overlapping free tagged intervals that is subordinate to $\delta$, then

$$\sum_{i=1}^{n} \left| f(x_i)(\alpha(d_i) - \alpha(c_i)) - (F^\alpha(b) - F^\alpha(a)) \right| \leq 2\varepsilon.$$

We define the uniform McShane-Stieltjes integrability for a sequence of McShane-Stieltjes integrable functions.

**Definition 2.3.** Let $\alpha$ be an increasing function on $[a, b]$ and let $\{f_n\}$ be a sequence of vector-valued McShane-Stieltjes integrable functions on $[a, b]$ with respect to $\alpha$. The sequence $\{f_n\}$ is uniformly McShane-Stieltjes integrable functions on $[a, b]$ with respect to $\alpha$ if for each $\varepsilon > 0$, there exists a positive function $\delta$ defined on $[a, b]$ such that $\|f_n^\alpha(\mathcal{P}) - \int_a^b f_n d\alpha\| < \varepsilon$ for all $n$ whenever $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to $\delta$.

**Theorem 2.4.** Let $\{f_n\}$ be a sequence of vector-valued McShane-Stieltjes integrable functions defined on $[a, b]$ and suppose that $\{f_n\}$ converges pointwise to $f$ on $[a, b]$. 
If \( \{f_n\} \) is uniformly McShane-Stieltjes integrable on \([a, b]\) with respect to \(\alpha\), then \(f\) is McShane-Stieltjes integrable on \([a, b]\) with respect to \(\alpha\) and

\[
\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.
\]

**Proof.** Let \(\alpha\) be an increasing function on \([a, b]\). Since \(\{f_n\}\) is uniformly McShane-Stieltjes integrable on \([a, b]\) with respect to \(\alpha\), there exists a free tagged partition \(\mathcal{P}_0\) of \([a, b]\) such that \(|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)| < \varepsilon\) for all \(n\). Since \(\{f_n\}\) converges pointwise to \(f\) on \([a, b]\), for a free tagged partition \(\mathcal{P}_0 = \{(x_i, (c_i, d_i)) : i = 1, \ldots, k\}

\[
\|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| = \left\| \sum_{i=1}^k f_n(x_i)(\alpha(d_i) - \alpha(c_i)) - \sum_{i=1}^k f_m(x_i)(\alpha(d_i) - \alpha(c_i)) \right\|
\]

\[
\leq \sum_{i=1}^k \left\| f_n(x_i) - f_m(x_i) \right\| (\alpha(b) - \alpha(a))
\]

\[
= (\alpha(b) - \alpha(a)) \sum_{i=1}^k \left\| f_n(x_i) - f_m(x_i) \right\|.
\]

For each \(x_i\), there exists a positive integer \(K_i(x_i)\) such that

\[
\|f_n^\alpha(x_i) - f_m^\alpha(x_i)\| \leq \frac{\varepsilon}{k} \quad \text{for all } n, m \geq K_i.
\]

Set \(N = \max\{K_i : 1 \leq i \leq k\}\). Then

\[
\|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| < \varepsilon \quad \text{for all } n, m \geq N.
\]

There exists a positive integer \(N\) such that \(\|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| < \varepsilon\) for all \(m, n \geq N\). Then

\[
\left\| \int_a^b f_n d\alpha - \int_a^b f_m d\alpha \right\|
\]

\[
= \left\| \int_a^b f_n d\alpha - f_n^\alpha(\mathcal{P}_0) \right\| + \left\| f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0) \right\| + \left\| f_m^\alpha(\mathcal{P}_0) - \int_a^b f_m d\alpha \right\|
\]

\[
< 3\varepsilon
\]

for all \(m, n \geq N\). It follows that \(\{\int_a^b f_n d\alpha\}\) is a Cauchy sequence in Banach space \(X\).

Let \(L = \lim_{n \to \infty} \int_a^b f d\alpha\). We need to show that \(\int_a^b f d\alpha = L\). Hence, it is sufficient to show that \(\int_a^b f d\alpha = L\). Let \(\varepsilon > 0\). By hypothesis, there exists a positive function \(\delta\) on \([a, b]\) such that \(\|f_n^\alpha(\mathcal{P}) - \int_a^b f_n d\alpha\| < \varepsilon\) for all \(n\) whenever \(\mathcal{P}\) is a free tagged partition of \([a, b]\) that is subordinate to \(\delta\). Since \(\{f_n\}\) converges pointwise to \(f\), there
exists \( k \geq N \) such that \( \| f^\alpha(\mathcal{P}) - f^\alpha_k(\mathcal{P}) \| < \varepsilon \). Hence

\[
\| f^\alpha(\mathcal{P}) - L \| \leq \| f^\alpha(\mathcal{P}) - f^\alpha_k(\mathcal{P}) \| + \| f^\alpha_k(\mathcal{P}) - \int_a^b f_k \, d\alpha \| + \| \int_a^b f_k \, d\alpha - L \| < 3\varepsilon.
\]

It follows that \( f \) is McShane-Stieltjes integrable on \([a, b]\) with respect to \( \alpha \) and

\[
\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.
\]

Now, we will prove the Dominated Convergence Theorem for McShane-Stieltjes integrable functions.

**Theorem 2.5** (Dominated Convergence Theorem). Let \( \alpha \) be an increasing function on \([a, b]\). Let \( \{f_n\} \) be a sequence of vector-valued McShane-Stieltjes integrable functions on \([a, b]\) with respect to \( \alpha \) and suppose that \( \{f_n\} \) converges pointwise to \( f \) on \([a, b]\). Let \( F^\alpha_n(x) = \int_a^x f_n \, d\alpha \). If there exists a real-valued McShane-Stieltjes integrable function \( g \) on \([a, b]\) such that \( \| f_n \| \leq g \) for all \( n \) and if \( \{F^\alpha_n\} \) is a Cauchy sequence in \( X \), then \( f \) is McShane-Stieltjes integrable on \([a, b]\) and

\[
\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.
\]

**Proof.** Let \( \varepsilon > 0 \), and \( G^\alpha(x) = \int_a^x g \, d\alpha \). Then \( G \) is absolutely continuous on \([a, b]\), and there exists \( \eta > 0 \) such that

\[
\left\| \sum_{i=1}^k (G^\alpha(d_i) - G^\alpha(c_i)) \right\| < \varepsilon
\]

whenever \( \{[c_i, d_i] : 1 \leq i \leq k\} \) is a finite collection of non-overlapping intervals in \([a, b]\) that satisfy \( \sum_{i=1}^k (d_i - c_i) < \eta \). By Egoroff's Theorem, there exists an open set \( O \) with the Lebesgue measure \( \mu(O) < \eta \) such that \( \{f_n\} \) convergence uniformly to \( f \) on \([a, b] - O\). Choose a positive integer \( N \) such that

\[
\left\| \int_a^b f_n \, d\alpha - \int_a^b f_m \, d\alpha \right\| < \varepsilon \quad \text{and} \quad \| f_n(x) - f_m(x) \| < \varepsilon
\]

for all \( m, n \geq N \) and for all \( x \in [a, b] - O \). Let \( \delta_g \) be a positive function on \([a, b]\) such that

\[
|g^\alpha(\mathcal{P}) - \int_a^b g \, d\alpha| < \varepsilon \quad \text{and} \quad \left\| f^\alpha_n(\mathcal{P}) - \int_a^b f_n \, d\alpha \right\| < \varepsilon
\]

for \( 1 \leq n \leq N \) whenever \( \mathcal{P} \) is a free tagged partition of \([a, b]\) that is subordinate to \( \delta_g \). Define a positive function \( \delta \) on \([a, b]\) by

\[
\delta(x) = \begin{cases} 
\delta_g(x), & \text{if } x \in [a, b] - O \\
\min\{\delta_g(x), \rho(x, O^c)\}, & \text{if } x \in O
\end{cases}
\]
where \( \rho(x, O^c) = \inf \{|x - y| : y \in O^c \} \). Suppose that \( P \) is a free tagged partition of \([a, b]\) that is subordinate to \( \delta \) and fix \( n > N \). Let \( P_1 \) be the subset of \( P \) that had tags in \([a, b] - O \) and let \( P_2 = P - P_1 \). Using the weak Saks-Henstock lemma (Lemma 2.2) and \( \mu(P_2) < \delta \)

\[
|f_n^\alpha(P) - f_n^\alpha(P)| \leq |f_n^\alpha(P_1) - f_n^\alpha(P_1)| + |f_n^\alpha(P_2) - f_n^\alpha(P_2)| \\
\leq \epsilon(\alpha(b) - \alpha(a)) + g^\alpha(P_2) \\
\leq \epsilon(\alpha(b) - \alpha(a)) + |g^\alpha(P_2) - G^\alpha(P_2)| + |G^\alpha(P_2)| \\
\leq \epsilon(\alpha(b) - \alpha(a)) + 2\epsilon + \epsilon \\
= \epsilon(\alpha(b) - \alpha(a) + 3).
\]

Hence,

\[
\left| f_n^\alpha(P) - \int_a^b f_n d\alpha \right| \\
\leq \left| f_n^\alpha(P) - \int_a^b f_n d\alpha \right| + \left| \int_a^b f_n d\alpha \right| + \left| \int_a^b f_n d\alpha - \int_a^b f_n d\alpha \right| \\
< \epsilon(\alpha(b) - \alpha(a) + 2) + \epsilon + \epsilon.
\]

Hence \( \{f_n\} \) is uniformly McShane-Stieltjes integrable on \([a, b]\) with respect to \( \alpha \).

By Theorem 2.4, \( f \) is McShane-Stieltjes integrable on \([a, b]\) with respect to \( \alpha \) and \( \int_a^b f d\alpha = \lim_{n \to \infty} \int_a^b f d\alpha \).

\[ \square \]

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