A STUDY ON THE BOOLEAN ALGEBRAS IN
THE FRAENKEL-MOSTOWSKI TOPOS

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ABSTRACT. The purpose of this paper is to show that in the Fraenkel-Mostowski topos, the category of the Boolean algebras has enough injectives.

1. Introduction

Banaschewski [1] showed that the category of Boolean algebras in the Blass topos has no non-trivial injectives. But Ebrahimi [4] showed that the category of Boolean algebras in the topos M-Set has enough injectives, also he showed that the relation between injectivity and internal completeness in the topos M-Set. In this paper we show that in the Fraenkel-Mostowski topos the category of Boolean algebras has enough injectives and an injective Boolean algebras object is internally complete. Also we show underlying set functor on $E(\mathcal{F})$ preserves internal complete posets.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results. In original definition of an elementary topos by Lawvere [8] and Tierney [11] it was required by axiom that colimits also exist. But Mikkelsen ([5, p. 84], [10]) discovered that the existence of colimits is implied by the rest of axioms. We use the definition of elementary topos, modified by Mikkelsen.

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Definition 2.1. An elementary topos is a category $E$ that satisfies the following conditions:

(T1) $E$ is finitely complete,

(T2) $E$ has exponentiation, and

(T3) $E$ has a subobject classifier.

Definition 2.2. An object $Q$ in a category $C$ is injective if the contravariant hom functor $\text{hom}_C(-, Q) : C^{\text{op}} \to \text{Set}$ preserves epimorphisms. A category $C$ is said to have enough injectives if every object of $C$ admits a monomorphism to an injective object.


Let $(G, \cdot, e)$ be a group and $\mathcal{F}$ be a normal filter of subgroups of $G$. That is, $\mathcal{F}$ is a nonempty collection of subgroups such that

(1) $G_M, G_N \in \mathcal{F} \implies G_M \cap G_N \in \mathcal{F}$

(2) $G_M \subseteq G_N, G_M \in \mathcal{F} \implies G_N \in \mathcal{F}$ (so, in particular, $G \in \mathcal{F}$)

(3) $G_M \in \mathcal{F}, g \in G \implies gG_Mg^{-1} \in \mathcal{F}$.

By an $\mathcal{F}$-set, we mean a triple $(X, G_X, \psi_X)$ where $X$ is a set, $G_X \in \mathcal{F}$ and $\psi_X : G_X \times X \to X$ is an (left) action of $G_X$ on $X$. That is, $\psi_X : G_X \times X \to X$ is a function such that $\psi_X(e, x) = x$ for all $x \in X$, $\psi_X(m \cdot n, x) = \psi_X(m, \psi_X(n, x))$ for all $m, n \in G_X$ and for all $x \in X$. When no confusion is likely, we just write $X$ for $\mathcal{F}$-set $(X, G_X, \psi_X)$ and we write $(m, x)$ for $\psi_X(m, x)$.

For each $x \in X$ let $\text{fix}(x)$ be the isotropy group $\{m \in G_X \mid \psi_X(m, x) = x\}$ of $x$.

We call an $\mathcal{F}$-set $(X, G_X, \psi_X)$ almost trivial if $\text{fix}(x) \in \mathcal{F}$ for all $x \in X$ and a map $f : X \to Y$ from $\mathcal{F}$-set $(X, G_X, \psi_X)$ to $\mathcal{F}$-set $(Y, G_Y, \psi_Y)$ is almost equivariant if, some $G_N \in \mathcal{F}, G_N \subseteq G_X \cap G_Y$,

$$f(\psi_X(n, x)) = \psi_Y(n, f(x))$$

for all $n \in G_N$ and for all $x \in X$.

The almost trivial $\mathcal{F}$-sets and almost equivalent maps between them constitute a category, denoted by $E(\mathcal{F})$.

Proposition 2.3. $E(\mathcal{F})$ is a topos.

Proof. $(\{\ast\}, G, \psi_{\{\ast\}})$, where $\psi_{\{\ast\}}$ is the trivial action, is terminal object. The pull-back of $(X, G_X, \psi_X)$ and $(Y, G_Y, \psi_Y)$ is $(X \times Y, G_X \cap G_Y, \psi_{X \times Y})$ where

$$\psi_{X \times Y}(n, (x, y)) = (\psi_X(n, x), \psi_Y(n, y))$$.
Let $2 = \{0,1\}$ be a doubleton in $\text{Set}$. Then $\Omega = (2, G, \psi_2)$, where $\psi_2$ is the trivial action of $G$ fixing both elements, is a subobject classifier. For $(X, G_X, \psi_X)$ and $(Y, G_Y, \psi_Y)$, we define an almost trivial object $(Y^X, G_X \cap G_Y, \psi_{YX})$ where $\psi_{YX}(t, f)(x) = \psi_Y(t, f(\psi_X(t^{-1}, x))).$ \hfill \square

**Definition 2.4.** An internal poset $P$ is internally complete if there is an order preserving map $\bigcup: \Omega^P \to P$ which is internally left adjoint to the morphism $\downarrow_{\text{seg}}$.

3. Main Parts

**Theorem 3.1.** The category of Boolean algebras in $E(\mathcal{F})$ has enough injectives.

*Proof.* For any Boolean algebra object $K$ in $E(\mathcal{F})$, Dedekind-MacNeille completion of $K$ is also a Boolean algebra object in $E(\mathcal{F})$ by [1] and there exists an embedding of $K$ into the Dedekind-MacNeille completion of $K$ in $E(\mathcal{F})$. So we only show that complete Boolean algebra object is injective in $E(\mathcal{F})$.

Given Boolean algebra homomorphism $f: A \to C$ where $A$ is a sub Boolean algebra object of a Boolean algebra object $B$ and $C$ is a complete Boolean algebra object in $E(\mathcal{F})$. By definition of the complete Boolean algebra we can define $h: B \to C$ where $h(b) = \bigvee\{f(a) \mid a \in A, a \leq b\}$. Then, for all $k \in G_A \cap G_C$,

$$kh(b) = k\bigvee\{f(a) \mid a \in A, a \leq b\}$$

$$= \bigvee\{kf(a) \mid a \in A, a \leq b\}$$

$$= \bigvee\{f(ka) \mid a \in A, a \leq b\}$$

$$= \bigvee\{f(t) \mid t \in A, t \leq kb\} = h(kb).$$

Thus $h$ is almost equivariant. Also

$$h(b \wedge c) = \bigvee\{f(a) \mid a \in A, a \leq (b \wedge c)\}$$

$$= \bigvee\{f(a) \mid a \in A, a \leq b\} \wedge \bigvee\{f(a) \mid a \in A, a \leq c\}$$

$$= h(b) \wedge h(c).$$

Thus $h$ preserves meets, similarly $h$ preserves joins.

Let $p$ and $q$ be identities on $\bigvee$ and $\wedge$ in $B$ respectively. Then, for all $x \in C$,

$$h(p) \bigvee x = \bigvee\{f(a) \mid a \in A, a \leq p\} \bigvee x = f(p) \bigvee x = x.$$
Thus $h(p)$ is an identity on $\lor$ in $C$, similarly $h(q)$ is an identity on $\land$ in $C$. Also $h$ preserves the complement and extends $f$ since $h(a) = \lor \{ f(d) \mid d \in A, d \leq a \} = f(a)$ for all $a \in A$. □

**Proposition 3.2.** If $A$ is injective Boolean algebra object in $E(F)$ then it is internally complete.

*Proof.* Let $\phi : X \to A$ and $\mu : X \to \Omega^A$ be any two generalized elements. Then, by exponential transpose, for any $\mu : X \to \Omega^A$ there exists a monomorphism $y : Y \to X \times A$ such that

$$
\begin{array}{c}
Y & \longrightarrow & 1 \\
\downarrow & & \downarrow T \\
X \times A & \longrightarrow & \Omega \\
\mu \times i_A & \downarrow & \downarrow i_\Omega \\
\Omega^A \times A & \overset{ev}{\longrightarrow} & \Omega
\end{array}
$$

commutes.

For the morphism $\downarrow \text{seg} : A \to \Omega^A$ where $\Omega = \{2, G, \psi_2\}$ with 2 is a doubleton set and $\psi_2 : G \times 2 \to 2$ is trivial action, since $A$ is injective we obtain $\bigcup : \Omega^A \to A$ such that $\bigcup \downarrow \text{seg} = i_A$.

To prove that $\bigcup$ is internally left adjoint to $\downarrow \text{seg}$, let $\mu \leq \downarrow \text{seg} \phi$. Since $\bigcup$ is order preserving, this implies $\bigcup \mu \leq \phi$. Conversely, let $\bigcup \mu \leq \phi$. Then for any $\downarrow \text{seg} \phi : X \to \Omega^A$, there is a monomorphism $z : Z \to X \times A$ such that

$$
\begin{array}{c}
Z & \longrightarrow & 1 \\
\downarrow z & & \downarrow T \\
X \times A & \longrightarrow & \Omega \\
\downarrow \text{seg} \phi \times i_A & \downarrow & \downarrow i_\Omega \\
\Omega^A \times A & \overset{ev}{\longrightarrow} & \Omega
\end{array}
$$

commutes.

And, for a morphism $(m, \phi m) : M \to X \times A$ where $m : M \to X$ is a monomor-
phism, we get two pullback squares, that is, the squares

\[
\begin{array}{ccc}
W & \xrightarrow{p_1} & Y \\
\downarrow_{p_2} & & \downarrow v \\
M \xrightarrow{(m, \phi m)} & X \times A & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
S & \xrightarrow{q_1} & Z \\
\downarrow_{q_2} & & \downarrow z \\
M \xrightarrow{(m, \phi m)} & X \times A & \\
\end{array}
\]

commute.

Since upper square is pullback and \( \bigcup \downarrow \text{seg} = i_A \), \( \bigcup \mu \leq \phi \) implies \( W \leq M \). Since lower square is pullback, this implies \( W \leq S \). By above two pullback squares, this implies \( Y \leq Z \). By exponential transpose, this implies \( \mu \leq \downarrow \text{seg} \phi \). Therefore we have the desired internal adjunction. \( \square \)

**Proposition 3.3.** Underlying set functor \( W : E(\mathcal{F}) \rightarrow \text{Set} \) preserves internally complete posets.

**Proof.** Let \( A \) be a internally complete poset in \( E(\mathcal{F}) \) and \( \rho : B \rightarrow \Omega^{WA} \) be a generalized element in \( \text{Set} \). Then, by exponential transpose, there is a unique morphism \( (a, b) : X \rightarrow B \times WA \) such that the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & 1 \\
\downarrow_{(a,b)} & & \downarrow T \\
B \times WA & \xrightarrow{\chi_{(a,b)}} & \Omega \\
\downarrow_{\rho \times i_{WA}} & & \downarrow i_{\Omega} \\
\Omega^{WA} \times WA & \xrightarrow{\text{ev}} & \Omega & \\
\end{array}
\]

commutes.

Since \( W \) has a left adjoint \( V \), there is a morphism \( (V a, c) : VX \rightarrow VB \times A \) where \( c \) is the transpose of \( b \). And the morphism \( (V a, c) : VX \rightarrow VB \times A \) has an epi-mono
factorization by the properties of functor and factorization. That is, the diagram

$$
\begin{array}{ccc}
VX & \xrightarrow{(V_{a}, e)} & VB \times A \\
\downarrow e & & \uparrow m \\
I & \xrightarrow{i_{I}} & I
\end{array}
$$

commutes.

And, for above monomorphism $m : I \to VB \times A$ there is a unique morphism $\eta : VB \to \Omega^{A}$ such that the diagram

$$
\begin{array}{ccc}
I & \longrightarrow & 1 \\
\downarrow m & & \downarrow T \\
VB \times A & \xrightarrow{\chi_{m}} & \Omega \\
\eta \times i_{A} & \downarrow & \downarrow i_{\Omega} \\
\Omega^{A} \times A & \xrightarrow{ev} & \Omega
\end{array}
$$

commutes.

Hence, by hypothesis, we get $\cup' \eta : VB \to A$ where $\cup' : \Omega^{A} \to A$ is the internally left adjoint to $\downarrow seg' : A \to \Omega^{A}$ in $E(F)$. Also we obtain a morphism $\cup : \Omega^{WA} \to WA$ because $\cup' \eta$ is the transpose of $B \to WA$. To show that $\cup$ is internally left adjoint to $\downarrow seg$, let $\psi : B \to WA$ be a generalized element in $Set$. Then for $\downarrow seg \psi : B \to \Omega^{WA}$, where $\downarrow seg : WA \to \Omega^{WA}$, there is a unique morphism $n : Y \to B \times WA$ such that the diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & 1 \\
\downarrow n & & \downarrow T \\
B \times WA & \xrightarrow{\chi_{n}} & \Omega \\
\downarrow seg \psi \times i_{WA} & & \downarrow i_{\Omega} \\
\Omega^{WA} \times WA & \xrightarrow{ev} & \Omega
\end{array}
$$

commutes.

Also for a morphism $\downarrow seg \xi : VB \to \Omega^{A}$, where $\downarrow seg'$ and $\xi$ are the transpose of $\downarrow seg'$ and $\psi$, respectively, there exists a unique monomorphism $s : J \to VB \times A$
such that the diagram

\[
\begin{array}{ccc}
J & \rightarrow & 1 \\
\downarrow s & & \downarrow T \\
VB \times A & \xrightarrow{x_s} & \Omega \\
\downarrow \text{seg}'\xi \times \text{i}_A & & \downarrow \text{i}_\Omega \\
\Omega^A \times A & \xrightarrow{\text{ev}} & \Omega \\
\end{array}
\]

commutes.

Hence we get \( \rho \leq \downarrow \text{seg} \psi \) if and only if \( X \leq Y \). Since a morphism \( (V\alpha, c) : VX \rightarrow VB \times A \) factors through a monomorphism \( s : J \rightarrow VB \times A \), we get \( X \leq Y \) if and only if \( I \leq J \). Also, by previous diagrams, we have \( I \leq J \) if and only if \( \eta \leq \downarrow \text{seg}'\xi \). Since \( A \) is internally complete in \( E(\mathcal{F}) \), \( \eta \leq \downarrow \text{seg}'\xi \) if and only if \( \bigcup \eta \leq \xi \). By transposition, we have \( \bigcup' \eta \leq \xi \) if and only \( \bigcup \rho \leq \psi \). Therefore \( W(A) \) is internally complete. \( \square \)

References


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