REGULAR CLOSED BOOLEAN ALGEBRA IN SPACE WITH ONE POINT LINDELÖFFICATION TOPOLOGY

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Abstract. Let \((X^*, \tau^*)\) be the space with one point Lindelöfication topology of space \((X, \tau)\). This paper offers the definition of the space with one point Lindelöfication topology of a topological space and proves that the retraction regular closed function \(f : K(X^*) \to K(X)\) defined by \(f(A^*) = A^*\) if \(p \notin A^*\) or \(f(A^*) = A^* - \{p\}\) if \(p \in A^*\) is a homomorphism. There are two examples in this paper to show that the retraction regular closed function \(f\) is neither a surjection nor an injection.

1. One point Lindelöfication topology

Definition 1. Let \((X, \tau)\) be an arbitrary topological space. A subset \(A\) of \(X\) is called Lindelöf subset if \(A\) as the subspace of \((X, \tau)\) is a Lindelöf space (see Dow and Vermeer [1]). In \((X, \tau)\), the family of closed sets is

\[
\Omega = \{B : B \subset X \text{ and } X - B \in \tau\}
\]

and the family of Lindelöf subsets is

\[
L = \{A : A \subset X \text{ and } A\text{ is a Lindelöf subset in } (X, \tau)\}.
\]

We assume that \(p\) is a point not in \(X\) and \(X^* = X \cup \{p\}\). Suppose that \(\tau^* = \tau \cup \tau_1\) where \(\tau_1 = \{E : E \subset X^* \text{ and } X^* - E \in \Omega \cap L\}\). \((X^*, \tau^*)\) is called the space with one point Lindelöfication topology of topological space \((X, \tau)\).

It is clear that \(X^* \in \tau_1\) since \(\emptyset \in \Omega \cap L\). From Definition 1 one can assert that the family of closed sets in \((X^*, \tau^*)\) is \(\Omega^* = \{A^* : A^* = A \cup \{p\}\}\) where \(A \in \Omega\} \cup \{B^* : B^* \in \Omega \cap L\}.

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For any $x \in X$, we denote the neighborhood system of $x$ in $(X, \tau)$ by $\Xi_x$.

2. Lemmas and propositions

**Lemma A.** For any $x \in X$, $U^* \in \Xi_x$ or $U^* - \{p\} \in \Xi_x$ if and only if $U^*$ is a neighborhood of $x$.

**Proof.** If $U^*$ is a neighborhood of $x$ in $(X^*, \tau^*)$, then $U^* \in \Xi_x$ or $\exists A^* \in \tau_1$ such that $x \in A^* \subset U^*$. In the second case, there is $B \in \Omega \cap L$ satisfying $A^* = X^* - B$ so $x \in X - B \subset A^* \subset U^*$. It follows that $U^* - \{p\} \in \Xi_x$. According to the definition of neighborhood, the converse is obvious. □

**Lemma B.** For any $U^* \subset X^*$, if $p \in U^* \in \Xi_p^*$, then there is $A \in \tau$ such that $A \subset U^* - \{p\}$.

**Proof.** If $U^* \in \Xi_p^*$, then there is $A^* \in \tau_1$ such that $p \in A^* \subset U^*$ so there is $B \in \Omega \cap L$ satisfying $X^* - A^* = B$. It follows that $X - B \in \tau$ and $X - B = A^* - \{p\} \subset U^* - \{p\}$. □

The neighborhood system for any $x \in X$ in $(X^*, \tau^*)$ can be also denoted by $\Xi^*_x = \Xi^0_x \cup \Xi^1_x \cup \Xi^2_x$, where $\Xi^0_x$ is the neighborhood system $\Xi_x$ for $x$ in $(X, \tau)$,

$$\Xi^1_x = \{U \cup \{p\} : U \in \Xi_x\}$$

and

$$\Xi^2_x = \{U^* : \exists A^* \in \tau_1 \text{ such that } x \in A^* \subset U^* \subset X^*\}.$$

The neighborhood system for $p$ is $\Xi^*_p = \{U^* : \exists A^* \in \tau_1 \text{ such that } A^* \subset U^* \subset X^*\}$. That is, for any $U^* \in \Xi^*_p$, there is a set $B \in \Omega \cap L$ such that $X^* - U^* \subset B$.

Throughout this paper, we denote the closure and the interior of set $A$ in $(X, \tau)$ by $c(A)$ and $i(A)$ respectively. The closure and the interior of set $A$ in $(X^*, \tau^*)$ are represented by $c^*(A)$ and $i^*(A)$ respectively. The family of regular closed sets in $(X, \tau)$ and in $(X^*, \tau^*)$ are denoted by $K(X)$ and $K^*(X^*)$ respectively, that is, for any $A \in K(X)$, $c(i(A)) = A$ and for any $A^* \in K^*(X^*)$, $c^*(i^*(A^*)) = A^*$.

For any $A, B \in K(X)$ we define $A \circ B = c(i(A \cap B))$, $A' = c(i(X - A))$ and $A \circ B = (A \circ B') \cup (B \circ A')$. In order to distinguish an operation in $K^*(X^*)$ from in $K(X)$, we write $A \circ B$ for $A, B \in K(X)$ and the operation $\circ$ is operated in $K(X)$, $A \circ^* B$ for $A, B \in K^*(X^*)$ and the operation in $K^*(X^*)$, $A' = c(i(X - A))$,
$A' = c^*(i^*(X^* - A))$, $A \circ B = (A \circ B') \cup (B \circ A')$ and $A \circ B = (A \circ B') \cup (B \circ A')$, etc.

**Proposition 1.** $c^*(A) = c(A) \cup \{p\}$ if $A \subset X$, and, for any $B \in \Omega \cap L$, $A \cap (X - B) \neq \emptyset$.

**Proof.** According to Dugundji [2, p. 77], $c(A) \subset c^*(A)$ since $(X, \tau)$ is a subspace of $(X^*, \tau^*)$. It is clear that $p \notin A$ if $A \subset X$. For any $U^* \in \Xi_p^*$, $U^* \cap A \neq \emptyset$ since $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$. It follows that $p \in c^*(A)$. On the other hand, for any $x \in X$, if $x \notin c(A)$, then there is $U \in \Xi^0_{\tau}$ such that $U \cap A = \emptyset$ so $x \notin c^*(A)$ because $\Xi^0_{\tau} \subset \Xi^*_{\tau}$. From above one can assert that $c^*(A) = c(A) \cup \{p\}$.

**Corollary 1.** For any $A \subset X$, $p \notin A$ and the following two conclusions hold:

1. $c^*(A) = c(A)$ if $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$.
2. $c^*(A) = c(A) \cup \{p\}$ if $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$.

An apparent fact is $c^*(X) = X^*$ if $(X, \tau)$ is not a Lindelöf space. In this case $(X^*, \tau^*)$ is right the one point Lindelöfication of $(X, \tau)$ from Definition 1. So in the rest of this paper we assume that $(X, \tau)$ is always a non-Lindelöf space unless stated otherwise.

**Proposition 2.** If $p \in A$, then $c^*(A) = c(A - \{p\}) \cup \{p\}$.

**Proof.** It is clear that $c^*(\{p\}) = \{p\}$ since $X^* - \{p\} = X \in \tau^*$. So $c^*(A) = c^*(A - \{p\}) \cup c^*(\{p\}) = c(A - \{p\}) \cup \{p\}$ from Corollary 1.

**Corollary 2.** For any $A \subset X^*$, the followings are hold.

1. If $p \notin A$ and $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$, then $c^*(A) = c(A - \{p\})$;
2. If $p \in A$ or $p \notin A$ and $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$, then $c^*(A) = c(A - \{p\}) \cup \{p\}$.

**Proposition 3.** For any $A \subset X$, $i^*(A) = i(A)$.

**Proof.** Since $p \in X^* - A = (X - A) \cup \{p\}$, $i^*(A) = X^* - c^*((X - A) \cup \{p\}) = X^* - (c(X - A) \cup \{p\}) = i(A)$ from Proposition 2.

**Proposition 4.** If $p \in A \subset X^*$, then the followings are hold.

1. $i^*(A) = i(A - \{p\}) \cup \{p\}$ provided that $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$.
2. $i^*(A) = i(A - \{p\})$ provided that $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$.
Proof. It is clear that \( p \not\in X^* - A \) since \( p \in A \) so \( X^* - A = X - (A - \{ p \}) \).

(1) Since there is \( B \in \Omega \cap L \) satisfying \( A \cap (X - B) = \emptyset \), \( i^*(A) = X^* - c^*(X - [A - \{ p \}]) = X^* - c(X - [A - \{ p \}]) = i(A - \{ p \}) \cup \{ p \} \) from (1) of Corollary 1.

(2) Since \( A \cap (X - B) \neq \emptyset \) for any \( B \in \Omega \cap L \),

\[
i^*(A) = X^* - c^*(X - [A - \{ p \}]) = X^* - \{(c(X - [A - \{ p \}]) \cup \{ p \})\} = i(A - \{ p \})
\]
from (2) of Corollary 1. \( \square \)

Corollary 3. For any \( A \subset X^* \), the followings are hold.

(1) If \( p \in A \) and \( \exists B \in \Omega \cap L \) such that \( A \cap (X - B) = \emptyset \), then \( i^*(A) = i(A - \{ p \}) \cup \{ p \} \).

(2) If \( p \not\in A \) or \( p \in A \) and \( A \cap (X - B) \neq \emptyset \) for any \( B \in \Omega \cap L \), then \( i^*(A) = i(A - \{ p \}) \).

Proposition 5. For any \( A \subset X^* \), the followings are hold.

(1) \( c^*(i^*(A)) = c(i(A)) \cup \{ p \} \) if \( p \not\in A \) and for any \( B \in \Omega \cap L \), \( i(A) \cap (X - B) \neq \emptyset \).

(2) \( c^*(i^*(A)) = c(i(A)) \) if \( p \not\in A \) and there is \( B \in \Omega \cap L \), \( i(A) \cap (X - B) = \emptyset \).

(3) \( c^*(i^*(A)) = c(i(A - \{ p \})) \cup \{ p \} \) if \( p \in A \) and there is \( B \in \Omega \cap L \), \( i(A - \{ p \}) \cap (X - B) = \emptyset \).

(4) \( c^*(i^*(A)) = c(i(A - \{ p \})) \) if \( p \in A \) and for any \( B \in \Omega \cap L \), \( i(A - \{ p \}) \cap (X - B) \neq \emptyset \).

Proof. (1) Since \( p \not\in A \), \( p \not\in i(A) \) and \( i^*(A) = i(A) \) from Proposition 3. So \( c^*(i^*(A)) = c^*(i(A)) = c(i(A)) \cup \{ p \} \) according to (2) of Corollary 2.

(3) Since \( p \in A \), \( i^*(A) = i(A - \{ p \}) \cup \{ p \} \) from (1) of Proposition 4. It is obvious that \( p \not\in i(A - \{ p \}) \). So \( c^*(i^*(A)) = c^*(i(A - \{ p \})) \cup \{ p \} = c(i(A - \{ p \})) \cup \{ p \} \) according to (1) of Corollary 2.

The proofs for (2) and (4) are similar with that for (1) and (3) respectively. \( \square \)

Proposition 6. If \( p \not\in A \) and \( A \in K^*(X^*) \), then \( A \in K(X) \).

Proof. According to (1) and (2) of Proposition 5, \( A = c(i(A)) \) since \( p \not\in A \) and \( A = c^*(i^*(A)) \) from \( A \in K^*(X^*) \). \( \square \)

Proposition 7. If \( p \in A^* \in K^*(X^*) \), then \( A^* - \{ p \} \in K(X) \).

Proof. From (3) and (4) of Proposition 5, \( A^* - \{ p \} = c(i(A^* - \{ p \})) \) so \( A^* - \{ p \} \in K(X) \). \( \square \)
Proposition 8. If \( A \in K(X) \), then \( A \in K^*(X^*) \) and \( A \cup \{p\} \in K^*(X^*) \) if and only if there is \( B \in \Omega \cap L \), such that \( i(A) \cap (X - B) = \emptyset \).

Proof. It is easy to see that \( c^*(i^*(A)) = A \) and \( c^*(i^*(A \cup \{p\})) = A \cup \{p\} \) from (2) and (3) of Proposition 5 respectively. \( \square \)

Proposition 9. \( \{p\} \in K^*(X^*) \) if and only if \( (X, \tau) \) is a Lindelöf space.

Proof. If \( \{p\} \in K^*(X^*) \), then \( c^*(i^*(\{p\})) = \{p\} \). It follows that \( i^*(\{p\}) = \{p\} \) so \( \{p\} \) is an open set in \( (X^*, \tau^*) \). Moreover, \( X = X^* - \{p\} \in \Omega \cap L \) so \( X \) is a Lindelöf space.

Conversely, if \( X \) is a Lindelöf space, then \( \{p\} \) is an open and closed set in \( (X^*, \tau^*) \) so \( c^*(i^*(\{p\})) = \{p\} \). That is, \( \{p\} \in K^*(X^*) \). \( \square \)

Proposition 10. \( X \in K^*(X^*) \) if and only if \( (X, \tau) \) is a Lindelöf space.

Proof. If \( X \in K^*(X^*) \), then there is \( B \in \Omega \cap L \) satisfying \( i(X) \cap (X - B) = \emptyset \) from Proposition 5 so \( X \subset B \). On the other hand, \( B \subset X \). Finally, \( X = B \) is a Lindelöf set in \( (X, \tau) \). Conversely, if \( (X, \tau) \) is a Lindelöf space, then \( X \in \Omega \cap L \) so \( X \) is a closed set in \( (X^*, \tau^*) \). Since \( X \in \tau \subset \tau^* \), \( X \) is open in \( (X^*, \tau^*) \). Hence \( X \in K^*(X^*) \). \( \square \)

Corollary 4. \( \{p\} \in K^*(X^*) \) if and only if \( X \in K^*(X^*) \).

According to Kuratowski and Mostowski [3, p. 39], \( K(X) \) is a Boolean algebra with a unit with respect to the operations \( \circ \) and \( \bigcirc \) as well as that \( K^*(X^*) \) a Boolean algebra with a unit with respect to the operations \( \circ^* \) and \( \bigcirc^* \). It is easy to see that \( K(X) \) is a sub-algebra of \( K^*(X^*) \) if and only if \( X \) is a Lindelöf space from Propositions 6, 7 and 10.

Proposition 11. If \( A \subset X \) and \( A^* = A \cup \{p\} \) then

1. \( (A^*)^*_\epsilon = A' \cup \{p\} \) provided that \( i(X - A) \cap (X - B) \neq \emptyset \) for any \( B \in \Omega \cap L \);
2. \( (A^*)^*_\epsilon = A' \) provided that there is \( B \in \Omega \cap L \) satisfying \( i(X - A) \cap (X - B) = \emptyset \);
3. \( A'_\epsilon = A' \cup \{p\} \) provided that there is \( B \in \Omega \cap L \) satisfying \( i(X - A) \cap (X - B) = \emptyset \);
4. \( A'_\epsilon = A' \) provided that \( i(X - A) \cap (X - B) \neq \emptyset \) for any \( B \in \Omega \cap L \).

Proof. (1) From (1) of Proposition 5, \( (A^*)^*_\epsilon = c^*(i^*(X^* - A^*)) = c(i(X - A)) \cup \{p\} = A' \cup \{p\} \).

(2) From (2) of Proposition 5, \( (A^*)'_\epsilon = c^*(i^*(X^* - A^*)) = c(i(X - A)) = A' \).
(3) From (3) of Proposition 5, \(A'_* = c^*(i^*(X^* - A^*)) = c(i(X - A)) \cup \{p\} = A' \cup \{p\}.
(4) From (4) of Proposition 5, \(A'_* = c^*(i^*(X^* - A)) = c(i(X - A)) = A'. \quad \square

3. The retraction regular closed function

Definition 2. The retraction regular closed function \(f : K^*(X^*) \rightarrow K(X)\) is defined for any \(A^* \in K^*(X^*)\) by \(f(A^*) = A^*\) if \(p \notin A^*\) or \(f(A^*) = A^* - \{p\}\) if \(p \in A^*\).

Proposition 12. If \(p \in A^* \in K^*(X^*)\), then \(f((A^*)'_*) = (f(A^*))'_*\).

Proof. Let \(A \subset X\) and \(A^* = A \cup \{p\}\). If for any \(B \in \Omega \cap L\), \(i(X - A) \cap (X - B) \neq \emptyset\), then \(f((A^*)'_*) = f(A' \cup \{p\}) = A' = (f(A^*))'_*\) from (1) of Proposition 11 and Definition 2. On the other hand, if there is \(B \in \Omega \cap L\) satisfying \(i(X - A) \cap (X - B) = \emptyset\), then \(f((A^*)'_*) = f(A') = A' = (f(A^*))'_*\) from (2) of Proposition 11 and Definition 2. \(\square\)

Proposition 13. If \(p \notin A \in K^*(X^*)\), then \(f(A'_*) = (f(A))'_*\).

Proof. From (3) of Proposition 11 and Definition 2, if there is \(B \in \Omega \cap L\) satisfying \(i(X - A) \cap (X - B) = \emptyset\), then \(f(A'_*) = f(A' \cup \{p\}) = A' = (f(A))'_*\). On the other hand, if for any \(B_1 \in \Omega \cap L\), \(i(X - A) \cap (X - B) \neq \emptyset\), then \(f(A'_*) = f(A') = A' = (f(A))'_*\) from (4) of Proposition 11 and Definition 2. \(\square\)

Corollary 5. For any \(A^* \in K^*(X^*)\), \(f((A^*)'_*) = (f(A^*))'_*\).

Proposition 14. For any \(A^*, B^* \in K^*(X^*)\),

1. If \(p \in A^* - B^*\) and for any \(C \in \Omega \cap L\), \(i(A^* \cap B^*) \cap (X - C) \neq \emptyset\), then \(A^* \circ B^* = ((A^* - \{p\}) \circ (B^*) \cup \{p\});\)
2. If \(p \in B^* - A^*\) and there is \(C \in \Omega \cap L\) satisfying \(i(A^* \cap B^*) \cap (X - C) = \emptyset\), then \(A^* \circ B^* = B^* \circ (B^* - \{p\});\)
3. If \(p \in A^* \cap B^*\) and for any \(C \in \Omega \cap L\), \(i((A^* \cap B^*) - \{p\}) \cap (X - C) \neq \emptyset\), then \(A^* \circ B^* = ((A^* - \{p\}) \circ (B^* - \{p\});\)
4. If \(p \in A^* \cap B^*\) and there is \(C \in \Omega \cap L\) satisfying \(i((A^* \cap B^*) - \{p\}) \cap (X - C) = \emptyset\), then \(A^* \circ B^* = ((A^* - \{p\}) \circ (B^* - \{p\}) \cup \{p\});\)
5. If \(p \notin A^* \cup B^*\) and for any \(C \in \Omega \cap L\), \(i((A^* \cap B^*)) \cap (X - C) \neq \emptyset\), then \(A^* \circ B^* = (A^* \circ B^*) \cup \{p\};\)
(6) If \( p \notin A^* \cup B^* \) and there is \( C \in \Omega \cap L \) satisfying \( i(A^* \cap B^*) \cap (X - C) = \emptyset \), then \( A^* \odot^* B^* = A^* \odot B^* \).

Proof. (1) From (1) of Proposition 5, \( A^* \odot^* B^* = c(i((A^* - \{p\}) \cap B^*)) \cup \{p\} \).

(2) From (2) of Proposition 5, \( A^* \odot^* B^* = c(i(A^* \cap (B^* - \{p\}))) \).

(3) From (4) of Proposition 5, \( A^* \odot^* B^* = c(i((A^* \cap B^*) - \{p\}) = c(i((A^* - \{p\}) \cap (B^* - \{p\}))) \).

(4) From (3) of Proposition 5, \( A^* \odot^* B^* = c(i((A^* \cap B^*) - \{p\}) \cup \{p\} = c(i((A^* - \{p\}) \cap (B^* - \{p\}))) \).

(5) From (1) of Proposition 5, since \( p \notin A^* \cap B^* \), \( A^* \odot^* B^* = c(i(A^* \cap B^*)) \cup \{p\} \).

(6) From (2) of Proposition 5, \( A^* \odot^* B^* = c(i(A^* \cap B^*)) \).

Proposition 15. For any \( A^*, B^* \in K^*(X^*) \), \( f(A^* \odot^* B^*) = f(A^*) \odot f(B^*) \).

Proof. In order to reduce the length of the proof, we call

\[
i(A^* \cap B^*) \cap (X - C) = \emptyset
\]

the equality (*) briefly. The situation of the proof can be divided into the following eight cases:

1. \( p \in A^* - B^* \), for any \( C \in \Omega \cap L \) the equality (*) does not hold.
2. \( p \in A^* - B^* \), there is \( C \in \Omega \cap L \) satisfying the equality (*).
3. \( p \in B^* - A^* \), for any \( C \in \Omega \cap L \) the equality (*) does not hold.
4. \( p \in B^* - A^* \), there is \( C \in \Omega \cap L \) satisfying the equality (*).
5. \( p \in A^* \cap B^* \), for any \( C \in \Omega \cap L \) the equality (*) does not hold.
6. \( p \in A^* \cap B^* \), there is \( C \in \Omega \cap L \) satisfying the equality (*).
7. \( p \notin A^* \cup B^* \), for any \( C \in \Omega \cap L \) the equality (*) does not hold.
8. \( p \notin A^* \cup B^* \), there is \( C \in \Omega \cap L \) satisfying the equality (*).

The proof for each case is simple from Proposition 14 so it is omitted. \( \square \)

Proposition 16. For any \( A^*, B^* \in K^*(X^*) \), \( f(A^* \cup B^*) = f(A^*) \cup f(B^*) \).

Proof. It is easy to check the equality from Definition 2. \( \square \)

Proposition 17. For any \( A^*, B^* \in K^*(X^*) \), \( f(A^* \circ^* B^*) = f(A^*) \circ f(B^*) \).

Proof. It is the result of calculating from Propositions 16 and 15 and Corollary 5. \( \square \)

From above we obtain the following theorem.
Theorem 1. The retraction regular closed function \( f : K^*(X^*) \to K(X) \) defined for any \( A^* \in K^*(X^*) \) by \( f(A^*) = A^* \) if \( p \in A^* \), \( f(A^*) = A^* - \{p\} \) if \( p \in A^* \) from the regular closed algebra \( K^*(X^*) \) of the space with one point Lindelöfication topology into that algebra \( K(X) \) of the space \((X, \tau)\) is a homomorphism.

Proposition 18. Retraction regular closed function \( f \) is a surjective homomorphism if and only if for any \( A \in K(X) \), there exists \( B \in \Omega \cap L \) satisfying \( i(A) \cap (X - B) = \emptyset \).

Proof. It is obvious from the theorem, Proposition 8 and Definition 2. \( \square \)

Definition 3. Let \( P = \{(x, y) : x, y \in R, y > 0\} \) be the open upper half plane with the Euclidean topology \( \sigma \) and \( R_1 = \{(x, 0) : x \in R\} \) the real axis. The topology \( \tau \) on \( X = P \cup R_1 \) is generated by adding to \( \sigma \) all sets of the form \( \{(x, 0)\} \cup (P \cap U) \) where \( U \) is a Euclidean neighborhood of \((x, 0)\) in the plane. The topological space \((X, \tau)\) is called the space with half-disc topology (see Steen and Seebach [3, p. 96-97]).

Definition 4. Let \((X, \tau)\) be the space with half-disc topology, \( q \) a point not in \( X \) and \( X^* = X \cup \{q\} \). Suppose that \( \tau^* = \tau \cup \tau_1 \) where \( \tau_1 = \{E : E \subset X^* \text{ and } X^* - E \in \Omega \cap L\} \). The topological space \((X^*, \tau^*)\) is called the space with one point Lindelöfication topology of the space with half-disc topology.

Example 1. The retraction regular closed function \( f \) may be not surjective homomorphism. Let \((X^*, \tau^*)\) be the space with one point Lindelöfication topology of space \((X, \tau)\) with half-disc topology. If

\[
A = \{(x, y) : x, y \in R, 0 \leq y \leq 1, x \in \bigcup_{n=1}^{\infty} \left[ n - \frac{1}{n}, n + \frac{1}{n} \right]\}
\]

then \( A \in K(X) \) from Definition 3 and for any \( B \in \Omega \cap L \) the equality \( i(A) \cap (X - B) = \emptyset \) does not hold. So there is no \( A^* \in K^*(X^*) \) satisfying \( f(A^*) = A \) from Proposition 8, where \( A \) is a closed and non-Lindelöf subset in \((X, \tau)\) and not a closed set in \((X^*, \tau^*)\). Hence the retraction regular closed function \( f \) is not a surjective homomorphism.

Proposition 19. If retraction regular closed function \( f \) is a surjective homomorphism, then \((X, \tau)\) is a Lindelöf space.

Proof. Since \( X \in K(X) \), if \( f \) is surjective, then there is \( B \in \Omega \cap L \) satisfying \( X \cap (X - B) = \emptyset \) from Proposition 18. So \( B = X \). That is, \( X \) is a Lindelöf space. \( \square \)
Example 2. The retraction regular closed function \( f \) may be not an injective homomorphism. Let \((X^*, \tau^*)\) be the space with one point Lindelöfication topology of space \((X, \tau)\) with half-disc topology. If

\[
A = \{(x, y) : x, y \in R, 1 \leq x \leq 2 \text{ and } 2 \leq y \leq 3\}
\]

and

\[
B = \{(x, y) : x, y \in R, 0 \leq x \leq 3 \text{ and } 1 \leq y \leq 4\}
\]

then \(A, B \in K(X)\) from the definition of the space with half-disc topology. Since \(B \in \Omega \cap L\) and \(i(A) \cap (X - B) = \emptyset\), \(A \in K^*(X^*)\) and \(A \cup \{q\} \in K^*(X^*)\) from Proposition 8. Finally, \(f(A) = f(A \cup \{q\}) = A\), that is, \(f\) is not injective.

Proposition 20. For any \(A \subset X, A \in K^*(X^*)\) if and only if \(A \cup \{p\} \in K^*(X^*)\).

Proof. If \(A \in K^*(X^*)\), then \(c^*(i^*(A)) = A\) so there is \(B \in \Omega \cap L\) satisfying \(i(A) \cap (X - B) = \emptyset\) from Proposition 5. So that \(c^*(i^*(A \cup \{p\})) = A \cup \{p\} \in K^*(X^*)\) from Proposition 5 again. Since the process of the proof can be inverted, we omit the proof for the converse. \(\square\)

From Proposition 20, we can assert that any retraction regular closed function \(f\) from the regular closed Boolean algebra \(K^*(X^*)\) of the space with one point Lindelöfication topology into that algebra \(K(X)\) of topological space \((X, \tau)\) must not be injective.

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References


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