ON $p$-ADIC $q$-BERNOULLI NUMBERS

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ABSTRACT. We give a proof of the distribution relation for $q$-Bernoulli polynomials $B_k(x : q)$ by using $q$-integral and evaluate the values of $p$-adic $q$-L-function.

1. Introduction

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$.

Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|q - 1|_p < p^{-\frac{1}{e-1}}$ so that $q^z = \exp(x \log_p q)$ for $|x|_p \leq 1$.

In the $p$-adic case, Carlitz’s $q$-Bernoulli number $\beta_k = \beta_k(q)$ are represented by a $q$-analogue form of Witt’s formula and investigated some properties (see [4], [5]). In [6], Koblitz constructed a $q$-analogue of the $p$-adic $L$-functions which interpolated Carlitz’s $q$-Bernoulli numbers $\beta_k(q)$. In the complex case (see [7]), Tsumura considered a $q$-analogue of the Dirichlet $L$-series which interpolated $q$-Bernoulli numbers $B_k(q)$.

Let $d$ be a fixed integer and let $p$ be a fixed prime number. We set

$$X = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{0 < a < dp, (a,p) = 1} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \}.$$
where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

For any positive integer $N$,

$$\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N : q]}$$

is known to be a distribution on $X$ (see [4], [5]).

This distribution yields an integral for each non-negative integer $m$:

$$\int_X [a]^m d\mu_q(a) = \beta_m(q) = I_q([a]^m),$$

where $\beta_m(q)$ are Carlitz's $q$-Bernoulli numbers (see [4], [5]).

In this paper, we show that $q$-Bernoulli numbers $B_k(q)$ can be represented as an integral by $q$-analogue $\mu_q$ of ordinary $p$-adic invariant measure and investigate some properties.

As an application, we give a proof of the distribution relation for $p$-adic $q$-Bernoulli polynomials $B_k(x; q)$ and construct $p$-adic $q$-Bernoulli measures to define $p$-adic $q$-$L$-series which interpolate $q$-Bernoulli numbers $B_k(q)$.

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2. $q$-Analogue of $p$-adic Bernoulli measures

In complex case [1], the Carlitz's numbers $\eta_k = \eta_k(q)$ are determined by

$$\eta_0 = 0, \quad (q\eta + 1)^k - \eta_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

These numbers $\eta_k$ induce Carlitz's $q$-Bernoulli numbers $\beta_k(q) = \beta_k$ as

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

In [7], H. Tsumura modified the above numbers $\eta_k$, that is,

$$B_0(q) = \frac{q - 1}{\log q}, \quad (qB(q) + 1)^k - B_k(q) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$
with the usual convention of replacing $B_i(q)$ by $B^i(q)$.

We use the notation

$$ [x] = [x : q] = \frac{1 - q^x}{1 - q}. $$

Note that $\lim_{q \to 1} [x : q] = x$. The $q$-Bernoulli numbers $B_k(q)$ satisfy the following relation

$$ B_m(q) = \frac{1}{(q - 1)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \frac{i}{[i]}. $$

This can be proved by the same method as [1: eq. 4.11].

In this paper, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. It is known (see [4], [5]) that

$$ \int_{\mathbb{Z}_p} q^{-x}[x]^m d\mu_q(x) = \frac{q - 1}{\log q} \int_{\mathbb{Z}_p} [x]^m d\mu_0(x), $$

where $\mu_0(x + p^N\mathbb{Z}_p) = \frac{1}{p^N}$. In the $p$-adic case, the $q$-Bernoulli numbers $B_k(q)$ can be represented by

$$ B_m(q) = \int_{\mathbb{Z}_p} q^{-x}[x]^m d\mu_q(x). $$

This is easily proved as in [4], [5].

Now, we define $q$-Bernoulli polynomials by

$$ B_m(x : q) = \int_{\mathbb{Z}_p} q^{-t}[x + t]^m d\mu_q(t). $$

Then these can be rewritten as

$$ (q^x B(q) + [x])^m = B_m(x : q), $$

for $m \geq 0$. 
Indeed we have
\[
\int_{\mathbb{Z}_p} q^{-t} [x + t]^n d\mu_q(t) = \int_{\mathbb{Z}_p} q^{-t}([x] + q^t [t])^n d\mu_q(t)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} [x]^{n-k} q^{kx} \int_{\mathbb{Z}_p} q^{-t}[t]^k d\mu_q(t)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} [x]^{n-k} q^{kx} B_k(q) = (q^x B(q) + [x])^n.
\]

As \(q \to 1\), we have \(B_k(q) \to B_k\) and \(B_k(x : q) \to B_k(x)\).

**Lemma 1.** For \(n \geq 0\), we have
\[
\int_{\mathbb{Z}_p} \chi(x) q^{-x}[x]^n d\mu_q(x) = \int_X \chi(x) q^{-x}[x]^n d\mu_q(x).
\]

This can be easily proved as in [4], [5].

The following lemma is used to construct the \(p\)-adic \(q\)-Bernoulli measures. A simple proof in the \(p\)-adic case can be given by using \(I_q\)-integration of \(q\)-Bernoulli numbers.

**Lemma 2.** For any positive integer \(d, k\) we have
\[
[d]^{k-1} \sum_{i=0}^{d-1} B_k \left( \frac{x + i}{d} : q^d \right) = B_k(x : q).
\]

**Proof.** From the definition of \(B_k(x : q)\), we can write
\[
B_k(x : q) = \int_X q^{-t}[x + t]^k d\mu_q(t) = \lim_{\rho \to \infty} \frac{1}{[dp^\rho]} \sum_{n=0}^{dp^\rho-1} [x + n]^k
\]
\[
= \lim_{\rho \to \infty} \frac{1}{[d]} \frac{1}{[p^\rho : q^d]} \sum_{i=0}^{d-1} \sum_{n=0}^{p^\rho-1} [x + i + dn]^k
\]
\[
= \lim_{\rho \to \infty} \frac{1}{[d]} \frac{1}{[p^\rho : q^d]} \sum_{n=0}^{d-1} \sum_{n=0}^{p^\rho-1} \left( \left[ \frac{x + i}{d} + n : q^d \right] [d] \right)^k
\]
\[
= \frac{1}{[d]} \sum_{i=0}^{d-1} [d]^k \int_{\mathbb{Z}_p} \left[ \frac{x + i}{d} + t : q^d \right]^k d\mu_q(t)
= [d]^{k-1} \sum_{i=0}^{d-1} B_k \left( \frac{x + i}{d} : q^d \right). \quad \square
\]

**Theorem 1.** For any positive integer \( N, k \) and \( d \), let \( \mu^*_{k} = \mu^*_{k, q} \) be defined by

\[
\mu^*_{k}(a + dp^N) = [dp^N : q]^{k-1} B_k \left( \frac{a}{dp^N} : q^{dp^N} \right).
\]

Then \( \mu^*_{k} \) is measure on \( X \).

**Proof.** It is suffices to check that

\[
\sum_{i=0}^{p-1} \mu^*_{k}(a + idp^N + dp^{N+1} \mathbb{Z}_p) = \mu^*_{k}(a + dp^N \mathbb{Z}_p).
\]

By definition of \( \mu^*_{k} \), we have

\[
\sum_{i=0}^{p-1} \mu^*_{k}(a + idp^N + dp^{N+1}) = [dp^{N+1} : q]^{k-1} \sum_{i=0}^{p-1} B_k \left( \frac{a + idp^N}{dp^{N+1}} : q^{dp^{N+1}} \right) = [dp^N : q]^{k-1} [p : q^{dp^N}]^{k-1} \sum_{i=0}^{p-1} B_k \left( \frac{\frac{a}{dp^N} + i}{p} : (q^{dp^N})^p \right) = [dp^N : q]^{k-1} B_k \left( \frac{a}{dp^N} : q^{dp^N} \right) = \mu^*_{k}(a + dp^N \mathbb{Z}_p).
\]

Thus we proved the above Theorem 1. \( \square \)

Let \( \chi \) be a Dirichlet character with conductor \( f \).
Now we define the generalized $q$-Bernoulli numbers as
\[ B_{k,x}(q) = \int_{\mathbb{Z}_p} \chi(x) q^{-x}[x]^k d\mu_q(x). \]

From the above definition, we have
\[ B_{k,x}(q) = [f]^{k-1} \sum_{a=0}^{f-1} \chi(a) B_k \left( \frac{a}{f} : q^f \right). \]

We can express a generalized $q$-Bernoulli number as an integral on $X$, by using the measure $\mu_x^*$.

**Theorem 2.** For any positive integer $k$, we have
\[ \int_X \chi(x) d\mu_x^*(x) = B_{k,x}(q). \]

**Proof.** From the definition of $\mu_x^*$, we see that
\[
\int_X \chi(x) d\mu_x^*
= \lim_{n \to \infty} \sum_{a=0}^{fp^n-1} \chi(a) \mu_x^*(a + fp^n\mathbb{Z}_p)
= \lim_{n \to \infty} \sum_{a=0}^{fp^n-1} \chi(a) [fp^n : q]^{k-1} B_k \left( \frac{a}{fp^n} : q^f p^n \right)
= \lim_{n \to \infty} [f]^{k-1} [p^n : q^f]^{k-1} \sum_{a=0}^{fp^n-1} \sum_{b=0}^{p^n-1} \chi(a + fb) B_k \left( \frac{a + fb}{fp^n} : q^f p^n \right)
= [f]^{k-1} \sum_{a=0}^{f-1} \chi(a) \lim_{n \to \infty} [p^n : q^f]^{k-1} \sum_{b=0}^{p^n-1} B_k \left( \frac{a + b}{p^n} : (q^f p^n) \right).
\]

By Lemma 2, we obtain that
\[ \int_X \chi(x) d\mu_x^*(x) = [f]^{k-1} \sum_{a=0}^{f-1} \chi(a) B_k \left( \frac{a}{f} : q^f \right) = B_{k,x}(q). \]

Therefore the above Theorem 2 is proved. \qed

Next we give a relation between $\mu_x^*$ and $\mu_q$. 
THEOREM 3. For any positive integer k, we have

\[ q^{-\frac{x}{k}} x^k d\mu_q(x) = d\mu_{q^k}(x). \]

Let \( p \) be a prime number and let \( \chi \) be a Dirichlet character with values contained in the algebraic closure of \( \mathbb{Q}_p \). We set \( p^* = p \) for \( p > 2 \), and \( p^* = 4 \) for \( p = 2 \), and denote by \( f = (f, p^*) \) the least common multiple of conductor \( f \) of \( \chi \) and \( p^* \).

Let \( B_{n,\chi}(q) \) denote the \( n \)-th generalized \( q \)-Bernoulli number belonging to the character \( \chi \).

Then we have \( q \)-analogue form of Witt’s formula in the \( p \)-adic cyclotomic field \( \mathbb{Q}_p(\chi) \) as follows:

\[ B_{n,\chi}(q) = \lim_{\rho \to \infty} \frac{1}{[fp^\rho]} \sum_{x=1}^{fp^\rho} \chi(x)[x]^n \]

for all \( n \geq 0 \).

Herein as usual we set \( \chi(x) = 0 \) if \( x \) is not prime to the conductor \( f \). In this section we shall give a few simple formulas of congruences for the generalized \( q \)-Bernoulli numbers. From the above formula for \( B_{n,\chi}(q) \) we have

\[ B_{n,\chi}(q) = \lim_{\rho \to \infty} \frac{1}{[fp^\rho]} \sum_{1 \leq x \leq fp^\rho}^* \chi(x)[x]^n \]

\[ + \lim_{\rho \to \infty} \frac{1}{[fp^{p-1}] [q^p] [p]} \sum_{y=1}^{fp^{p-1}} \chi(p) \chi(x)[p]^n [y : q^p]^n \]

where \( * \) means to takes sum over the rational integers prime to \( p \) in the given range.

Thus we have

\[ B_{n,\chi}(q) = \lim_{\rho \to \infty} \frac{1}{[fp^\rho]} \sum_{1 \leq x \leq fp^\rho}^* \chi(x)[x]^n \]

\[ + [p]^{n-1} \chi(p) B_{n,\chi}(q^p), \]
that is,

\[ B_{n,\chi}(q) - [p]^{n-1}\chi(p)B_{n,\chi}(q^p) = \lim_{\rho \to \infty} \frac{1}{[\tilde{f}^p\rho]} \sum_{1 \leq x \leq \tilde{f}^p\rho}^* \chi(x)[x]^n. \]

We choose a rational number \( c \in \mathbb{Z} \) such that \((c, \tilde{f}) = 1, c \neq \pm 1\).

Let \( x \) run over the range \( 1 \leq x \leq \tilde{f}^p\rho, (x, p) = 1 \), \( x_{\rho} \) run over the range \( 1 \leq x_{\rho} \leq \tilde{f}^p\rho, (x_{\rho}, p) = 1 \), and determine a number \( r_{\rho}(x) \in \mathbb{Z} \) by \( x_{\rho} = cx + r_{\rho}(x)\tilde{f}^p\rho \).

Taking the \( n \)-th power of this equality and making sum with the character \(\chi\) we obtain

\[
\frac{1}{[\tilde{f}^p\rho]} \sum_{1 \leq x_{\rho} \leq \tilde{f}^p\rho}^* \chi(x_{\rho})[x_{\rho}]^n
\]

\[
= \frac{1}{[\tilde{f}^p\rho]} \sum_{1 \leq x \leq \tilde{f}^p\rho}^* \chi(cx)[cx]^n + n \sum_{1 \leq x \leq \tilde{f}^p\rho}^* \chi(cx)[cx]^{n-1}[r_{\rho}(x) : q\tilde{f}^p\rho] \mod [\tilde{f}^p\rho]).
\]

If \( \rho \to \infty \), we get

\[
B_{n,\chi}(q) - [p]^{n-1}\chi(p)B_{n,\chi}(q^p)
\]

\[
= \chi(c)[c]^n(B_{n,\chi}(q^c) - [p : q^c]^{n-1}\chi(p)B_{n,\chi}(q^{pc})
\]

\[
+ n \lim_{\rho \to \infty} \sum_{1 \leq x \leq \tilde{f}^p\rho}^* \chi(cx)[cx]^{n-1}[r_{\rho}(x) : q\tilde{f}^p\rho].
\]

Thus we have

\[
\lim_{\rho \to \infty} \sum_{1 \leq x \leq \tilde{f}^p\rho}^* \chi(cx)[cx]^{n-1}[r_{\rho}(x) : q\tilde{f}^p\rho]
\]

\[
= \frac{1}{n} (B_{n,\chi}(q) - [p]^{n-1}\chi(p)B_{n,\chi}(q^p))
\]

\[
- \frac{1}{n} \chi(c)[c]^n(B_{n,\chi}(q^c) - [p : q^c]^{n-1}\chi(p)B_{n,\chi}(q^{pc}).
\]

Let we define the operator \( \chi^y = \chi^{y,k:q} \) on \( f(q) \) by

\[
\chi^y f(q) = [y]^{k-1}\chi(y)f(q^y),
\]

\[
\chi^x \chi^y = \chi^{x,k:q^y} \circ \chi^{y,k:q}.
\]

Then we see that \( \chi^x \chi^y = \chi^{xy} \).

Therefore we obtain the following


\textbf{PROPOSITION 1.} For \( n \geq 1 \), we have

\[
\lim_{\rho \to \infty} \sum_{1 \leq x \leq f \rho} \frac{\alpha^\rho}{\alpha^\rho} \frac{[\alpha x]^{n-1}}{[\alpha x]} = (1 - \alpha^p)(1 - [\alpha] \alpha^c) \frac{B_{n,\alpha}(\alpha)}{n}.
\]

Now we define

\[
\mu_k^\alpha = \frac{1}{\kappa} \left( \mu_{k,\alpha}(U) - [\alpha] \mu_{k,\alpha}(\frac{1}{\alpha} U) \right)
\]

where \( U \subset X \) is a compact open set. Note that \( \mu_k^\alpha \) is a measure on \( X \). Here, we define \( X^* = X - pX \). This measure yields an integral as follows.

\[
\int_{X^*} \chi(x) d\mu_k^\alpha(x) = (1 - \alpha^p)(1 - [\alpha] \alpha^c) \frac{B_k \chi(\alpha)}{k}
\]

Therefore we obtain the following.

\textbf{THEOREM 4.} For \( k \geq 1 \), we have

\[
\int_{X^*} \chi(x) d\mu_k^\alpha(x) = \lim_{\rho \to \infty} \sum_{1 \leq x \leq f \rho} \frac{\alpha^\rho}{\alpha^\rho} \frac{[\alpha x]^{k-1}}{[\alpha x]} \left[ \left[ -\frac{\alpha x}{f \rho} \right]_G : \chi(f \rho) \right],
\]

where \([\cdot]_G \) is Gauss's symbol.

Let \( \omega \) denote the Teichmüller character mod \( p^\rho \). For \( x \in X^* \), we set \( \langle x \rangle = \langle x : q \rangle = [x]/\omega(x) \). Note that \( \langle x \rangle^a \) is defined by \( \exp(s \log(x)) \), for \( |s|_p \leq 1 \).

For \( r \in \mathbb{Z} \), if we define

\[
\zeta_{p,q}(r) = \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{[p^k]} \sum_{m=0}^{p^k-1} \frac{1}{[m]^{r-1}},
\]

then we have

\[
\zeta_{p,q}(1-k) = -\frac{B_k(q)}{k}.
\]

Here, we can also define \( L_{p,q} \) as follows.

\[
L_{p,q}(r, \chi) = \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{[f \rho^k]} \sum_{1 \leq n \leq f \rho^k} \frac{\chi(n) \omega^{r-1}(n)}{[n]^{r-1}}
\]

\[
= \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{[f \rho^k]} \sum_{1 \leq n \leq f \rho^k} \chi(n) \langle n \rangle^{1-r}.
\]
Thus we find that
\[ L_{p,q}(1 - k, \chi \omega^k) = -\frac{1}{k} \lim_{m \to \infty} \frac{1}{[fp]^m} \sum_{1 \leq n \leq fp}^* \chi(n)[n]^k \]
\[ = -\frac{1}{k} \int_{X^*} \chi(x)[x]^k q^{-x} d\mu_q(x) \]
\[ = -\frac{1}{k} (B_{k,\chi}(q) - \chi(p)[p]^{k-1} B_{k,\chi}(q^p)). \]

Therefore we obtain the following

**Proposition 2. (q-analogue of \( L_p(1 - k, \chi \omega^{-k}) \))**

For \( k \geq 1 \), we have

\[ L_{p,q}(1 - k, \chi \omega^k) = -\frac{1}{k} (B_{k,\chi}(q) - \chi(p)[p]^{k-1} B_{k,\chi}(q^p)). \]

**References**


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