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ON RINGS WHOSE PRIME IDEALS ARE MAXIMAL

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Abstract. We investigate in this paper the maximality of prime ideals in rings whose simple singular left R-modules are p-injective.

1. Introduction

Throughout this paper R denotes an associative ring with identity, and all modules are unitary left R-modules. All prime ideals of a ring R are assumed to be proper. We use $P(R)$ and $N(R)$ to represent the prime radical and the set of all nilpotent elements of R, respectively.

A left R-module M is called to be *principally left p-injective* (briefly p-injective) if for any $a \in R$ and any left R-homomorphism of Ra into M extends to one of R into M. A von Neumann regular ring is left p-injective as a left R-module. However, in general, the converse does not hold. Moreover, any strongly π-regular ring and left weakly π-regular ring which are the generalizations of von Neumann regular rings are not left p-injective as a left R-module, for example an upper triangular matrix ring over a field.

Various generalizations of von Neumann regularity of rings whose simple (singular) left R-modules are p-injective were studied in [11], [12] and [13]. In particular, Yuechiming [12] proved that if every simple left R-module is p-injective, then R is left weakly regular. However,

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there exists a ring whose simple singular left R-modules are p-injective which is not left weakly regular, for example, an upper triangular matrix ring over a field.

On the other hand, the connection between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal has been investigated by many authors [2, 3, 4, 6 and 8].

We investigate in this paper the maximality of prime ideals in rings whose simple singular left R-modules are p-injective.

2. Main Results

A ring R is called *2-primal* if its prime radical $\mathbf{P}(R)$ coincides with the set $\mathbf{N}(R)$ of all nilpotent elements of R [1]. Note that commutative rings and reduced rings (i.e., rings without nonzero nilpotent elements) is a 2-primal ring.

Some of the earliest results known to us about 2-primal rings (although not so called at the time) and prime ideals were due to Shin [10]. Hirano [6] used the term *N-ring* for what we call a 2-primal ring. The name 2-primal rings originally came from the context of left near rings by Birkenmeier, Heatherly and Lee [1].

Following [10] and [3], for a prime ideal P of a ring R, we put

$$O_P = \{a \in R \mid ab = 0 \text{ for some } b \in R\backslash P\} \text{ and}$$

$$\overline{O}_P = \{a \in R \mid a^n \in O_P \text{ for some positive integer } n\}.$$

In general, $\overline{O}_P$ is not a subset of a prime ideal P and $O_P \subseteq \overline{O}_P$.

EXAMPLE 2.1. Let R be a ring of 2×2 matrices over a field F. Then $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a prime ideal of R. Let $a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$. Then $ab = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $b \in R\backslash P$. Thus $a \in O_P \subseteq \overline{O}_P$, but $a \notin P$.

Recall that a two-sided ideal P of R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$.

In [10], Shin showed that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime. Recently, Kim and Kwak [9] showed that R is a 2-primal ring if and only if $\overline{O}_P \subseteq P$ for each prime ideal P of R, and so $O_P \subseteq P$ for each prime ideal P of a 2-primal ring R. Note that if P is a completely prime ideal of R, then $\overline{O}_P \subseteq P$.

LEMMA 2.2. *Let M be a maximal left ideal of a ring R which properly contains a prime ideal P of R. If $O_P \subseteq P$, then M is an essential left ideal of R.*

Proof. Let M be a maximal left ideal of a ring R which properly contains a prime ideal P of R. Then there exists $a \in M$ such that $a \in R \backslash P$. Assume to the contrary that $M = l(e)$ for some $0 \neq e^2 = e \in R$, where $l(e) = \{r \in R \mid re = 0\}$. Then $ae = 0$. Since $e \notin P$, $a \in O_P$ and so $a \in P$, which is a contradiction. □

COROLLARY 2.3. *Let R be a 2-primal ring. If M is a maximal left ideal of R which properly contains a prime ideal P, then M is an essential left ideal of R.*

The condition "M is a maximal left ideal of R which properly contains a prime ideal P" in Lemma 2.2 and Corollary 2.3 is not superfluous. The conditions "$O_P \subseteq P$" in Lemma 2.2 and "R is a 2-primal ring" in Corollary 2.3 are also not superfluous, respectively.

EXAMPLE 2.4. (1) Let F be a field. We consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Then R is a 2-primal ring. However, the maximal left ideal $M = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ of R is not essential. Note that M does not properly contain any prime ideal of R.
(2) In Example 2.1, $O_P \nsubseteq P$ (and so R is not 2-primal) and

$$M = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in F \right\}$$

is a maximal left ideal of R which properly contains a prime ideal $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. However M is not an essential left ideal of R.

THEOREM 2.5. *Let P be a prime ideal of a ring R with $O_P \subseteq P$. If every simple singular left R-module is p-injective, then P is maximal.*

Proof. We claim that $RaR + P = R$ for $a \in R \backslash P$. If not, then there exists a maximal ideal M of R containing $RaR + P$. Moreover, M is a maximal left ideal of R. Suppose not. Then there exists a maximal left ideal K of R such that $M \subsetneq K$, then K is essential by Lemma 2.2. Thus, by hypothesis, R/K is p-injective, so any R-homomorphism of Ra into R/K extends to one of R into R/K. Let $f : Ra \to R/K$ be defined by $f(ra) = r + K$. If $ra = sa$ for $r, s \in R$, then $(r - s)a = 0$ and so $r - s \in O_P \subseteq P \subseteq K$. So $r + K = s + K$. Hence f is a well-defined left R-homomorphism. Since R/K is p-injective, there exists $b \in R$ such that $1 + K = f(a) = ab + K$. So $1 - ab \in K$, whence $1 \in K$, which is also a contradiction. Therefore M is a maximal essential left ideal of R, and so R/M is p-injective. By the same method in the above proof, P is a maximal ideal of R. □

The following example shows that the condition "every simple singular left R-module is p-injective" in Theorem 2.5 is not superfluous.

EXAMPLE 2.6. Let $R = \mathbb{Z}$ be the ring of integers and P a prime ideal of R. Then $P = p\mathbb{Z}$, where p is a zero or positive prime number. Thus $O_P = \{a \in \mathbb{Z} \mid ab = 0 \text{ for some } b \in \mathbb{Z}\backslash p\mathbb{Z}\} = \{0\}$ and so $O_P \subseteq P$ for all prime ideal P of R. But, a prime ideal $\{0\}$ is not maximal. Now, we consider a simple singular $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$. Then it is not p-injective. In fact, $f : 4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $f(4n) = n + 2\mathbb{Z}$ cannot be extended to one of $\mathbb{Z}$ into $\mathbb{Z}/2\mathbb{Z}$, by the similar method of the proof of Theorem 2.5.

PROPOSITION 2.7. *Let R be a ring whose simple singular left R-modules are p-injective. Then every completely prime ideal of R is maximal.*

Proof. Let P be a completely prime ideal of R. Suppose that P is not a maximal ideal of R. Then there exists a maximal ideal M of R such that $RaR + P \subseteq M$ for $a \in R \backslash P$. In fact, M is a maximal left ideal of R. If not, then there exists a maximal left ideal K of R such that $M \subsetneq K$, and so K is an essential left ideal by Lemma 2.2. Since R/K

is p-injective, so any R-homomorphism of Ra into R/K extends to one of R into R/K. Let $f : Ra \to R/K$ be defined by $f(ra) = r + K$. Then f is well-defined. For, if $ra = sa$ for $r, s \in R$, then $(r-s)a = 0 \in P$ and so $r - s \in P \subseteq K$. Hence $r + K = s + K$. Since R/K is p-injective, there exists $c \in R$ such that $1 + K = f(a) = ac + K$. Thus $1 - ac \in K$, whence $1 \in K$; which is a contradiction. So M is a maximal essential left ideal of R. Hence R/M is also p-injective. Applying the same method in the above proof, P is a maximal ideal of R. □

Recall that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime.

COROLLARY 2.8. *Let R be a 2-primal ring. If every simple singular left R-module is p-injective, then every prime ideal of R is maximal. In particular, every prime factor ring of R is a simple domain.*

Proof. Let R be a 2-primal ring. By Proposition 2.7 and the fact that every minimal prime ideal of R is completely prime, every prime ideal of R is maximal. Thus every prime ideal of R is completely prime and so every prime factor ring of R is a simple domain. □

As a parallel result to Theorem 2.5 and Proposition 2.7, we obtain the following.

PROPOSITION 2.9. *Let R be a ring whose maximal essential left ideals are two-sided. Then we have the following.*

(1) *Assume that $O_P \subseteq P$ for each prime ideal P of R. If every simple singular left R-module is p-injective, then every prime ideal of R is a maximal left ideal.*
(2) *If every simple singular left R-module is p-injective, then every completely prime ideal of R is a maximal left ideal.*

Proof. These proofs can be easily showed by adapting the method of the proofs of Theorem 2.5 and Proposition 2.7, respectively. □

The condition "every simple singular left R-module is p-injective" in Proposition 2.7, Corollary 2.8 and Proposition 2.9 is not superfluous.

EXAMPLE 2.10. In Example 2.6, the ring R is commutative and so it is 2-primal, but a completely prime ideal $\{0\}$ is not maximal. Note that a simple singular $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is also not p-injective.

The following example shows that there exists a ring whose maximal essential left ideals are two-sided and prime ideals are (two-sided) maximal, but every simple singular left R-module need not to be p-injective. This is compared with the converse of Proposition 2.9(2).

EXAMPLE 2.11. Let $T = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. Then T is a commutative subring of a ring of 2×2 matrices over $\mathbb{Z}_2$, $\mathrm{Mat}_2(\mathbb{Z}_2)$, where $\mathbb{Z}_2$ denotes the ring of all integers modulo 2. Let R be the set of all sequences of $\mathrm{Mat}_2(\mathbb{Z}_2)$, which are eventually in T, i.e.,

$$R = \{\langle a_n \rangle \mid a_n \in \ \mathrm{Mat}_2(\mathbb{Z}_2) \text{ and } \ a_n \text{ is eventually in } T\}.$$

Then R is a ring under $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle, \langle a_n \rangle \langle b_n \rangle = \langle a_n b_n \rangle$. Moreover, R is a semiprime PI-ring whose maximal essential left ideals are two-sided, but it is not von Neumann regular [7]. Since R is a strongly π-regular ring (i.e., for every $a \in R$, there exists a positive integer $n = n(a)$, depending on a, such that $a^n \in a^{n+1}R$), every prime ideal is maximal by [4, Theorem 2.3]. However, there exists a simple singular left R-module which is not p-injective by the following: If R is a PI-ring, then R is von Neumann regular if and only if R is a semiprime ring whose simple singular left modules are p-injective [13, Corollary 7].

The condition "$O_P \subseteq P$ for each prime ideal P of R" in Proposition 2.9(1) is not superfluous.

EXAMPLE 2.12. Let $R_1 = \{\langle a_n \rangle | \, a_n \in \ \mathrm{Mat}_2(\mathbb{Z}_2)$ and a_n is eventually in $T_1\}$, where $T_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. Then R_1 is a subring of the ring R in Example 2.11. Note that R_1 is a semiprime PI, von Neumann regular ring. Thus every simple singular left module is p-injective. However $P = \left\{ \langle a_n \rangle \in R_1 \mid a_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a prime ideal of R_1 and it can be easily checked that P is not a maximal left ideal of R_1. Now, for $x = \left\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \dots \right\rangle, y = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \dots \right\rangle \in R_1 \backslash P, xy = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \dots \right\rangle$ $\in P$ and so $x \in O_P$. Therefore $O_P \nsubseteq P$.

Recall that a ring R is said to be *left* (*right*) *weakly π-regular* if for every $a \in R$, there exists a positive integer $n = n(a)$, depending on a, such that $a^n \in Ra^nRa^n$ (a^nRa^nR). R is *weakly* π-regular if it is both right and left weakly π-regular [5].

THEOREM 2.13. *Let P be a completely prime ideal of a ring R. If $R/P(R)$ is a left (right) weakly π-regular ring, then P is a maximal ideal of R.*

Proof. Let P be a completely prime ideal of R with $a \in R \backslash P$. First suppose that $R/P(R)$ is a left weakly π-regular ring. Then $P + RaR = R$. If not, then there exists a maximal two-sided ideal M of R such that $P + RaR \subseteq M$. Since $\overline{R} = R/P(R)$ is left weakly π-regular, $\overline{R}\overline{a}^n = \overline{R}\overline{a}^n\overline{R}\overline{a}^n$ for some positive integer n. So $\overline{R}\overline{a}^n = \overline{M}\overline{a}^n$. Hence $\overline{a}^n = \overline{b}\overline{a}^n$ for some $\overline{b} \in \overline{M}$, and so $(\overline{1} - \overline{b})\overline{a}^n = \overline{0}$. Then $(1 - b)a^n \in P$. Since P is completely prime and $a \notin P$, $1 - b \in P \subseteq M$, which is a contradiction. Therefore P is a maximal ideal of R. Similarly, for a right weakly π-regular ring $R/P(R)$, we obtain that P is a maximal ideal of R. □

COROLLARY 2.14. *Let R be a 2-primal ring. Then the following statements are equivalent:*

(1) *$R/P(R)$ is a left (right) weakly π-regular ring.*
(2) *Every prime ideal of R is maximal.*
(3) *$R/J(R)$ is a left (right) weakly π-regular ring and $J(R)$ is nil, where $J(R)$ denotes the Jacobson radical of R.*

Proof. (1) $\Rightarrow$ (2): Suppose that $R/P(R)$ is right (left) weakly π-regular and P is a minimal prime ideal of R. Since R is 2-primal, P is completely prime. By Theorem 2.13, P is a maximal ideal of R. (2) implies (3) by [8, Proposition 5]. Since 2-primal and $J(R)$ is nil, we have $P(R) = J(R)$. So (3) implies (1). □

The condition "R is a 2-primal ring" in Corollary 2.14 is not superfluous by [8, Example 10].

The following example shows that there exists a 2-primal ring R such that every prime ideal of R is maximal, but R is neither right nor left weakly π-regular, even though $R/P(R)$ is left and right weakly π-regular.

EXAMPLE 2.15. Let $W_1[F]$ be the first Weyl algebra over a field F of characteristic zero. Then $W_1[F]$ is a simple domain which is not a division ring. Now, let

$$R = \begin{bmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{bmatrix}.$$

Then R is a 2-primal ring whose prime ideals are maximal. But R is neither right nor left weakly π-regular by [2, Example 12].

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