ON RINGS WHOSE PRIME IDEALS ARE MAXIMAL

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ABSTRACT. We investigate in this paper the maximality of prime ideals in rings whose simple singular left $R$-modules are $p$-injective.

1. Introduction

Throughout this paper $R$ denotes an associative ring with identity, and all modules are unitary left $R$-modules. All prime ideals of a ring $R$ are assumed to be proper. We use $P(R)$ and $N(R)$ to represent the prime radical and the set of all nilpotent elements of $R$, respectively.

A left $R$-module $M$ is called to be principally left $p$-injective (briefly $p$-injective) if for any $a \in R$ and any left $R$-homomorphism of $Ra$ into $M$ extends to one of $R$ into $M$. A von Neumann regular ring is left $p$-injective as a left $R$-module. However, in general, the converse does not hold. Moreover, any strongly $\pi$-regular ring and left weakly $\pi$-regular ring which are the generalizations of von Neumann regular rings are not left $p$-injective as a left $R$-module, for example an upper triangular matrix ring over a field.

Various generalizations of von Neumann regularity of rings whose simple (singular) left $R$-modules are $p$-injective were studied in [11], [12] and [13]. In particular, Yuchimong [12] proved that if every simple left $R$-module is $p$-injective, then $R$ is left weakly regular. However,
there exists a ring whose simple singular left $R$-modules are $p$-injective which is not left weakly regular, for example, an upper triangular matrix ring over a field.

On the other hand, the connection between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal has been investigated by many authors [2, 3, 4, 6 and 8].

We investigate in this paper the maximality of prime ideals in rings whose simple singular left $R$-modules are $p$-injective.

2. Main Results

A ring $R$ is called 2-primal if its prime radical $\mathbf{P}(R)$ coincides with the set $\mathbf{N}(R)$ of all nilpotent elements of $R$ [1]. Note that commutative rings and reduced rings (i.e., rings without nonzero nilpotent elements) is a 2-primal ring.

Some of the earliest results known to us about 2-primal rings (although not so called at the time) and prime ideals were due to Shin [10]. Hirano [6] used the term $N$-ring for what we call a 2-primal ring. The name 2-primal rings originally came from the context of left near rings by Birkenmeier, Heatherly and Lee [1].

Following [10] and [3], for a prime ideal $P$ of a ring $R$, we put

$$O_P = \{a \in R \mid ab = 0 \text{ for some } b \in R \setminus P\} \quad \text{and}$$

$$\overline{O}_P = \{a \in R \mid a^n \in O_P \text{ for some positive integer } n\}.$$ In general, $\overline{O}_P$ is not a subset of a prime ideal $P$ and $O_P \subseteq \overline{O}_P$.

**Example 2.1.** Let $R$ be a ring of $2 \times 2$ matrices over a field $F$. Then $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a prime ideal of $R$. Let $a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$.

Then $ab = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in R \setminus P$. Thus $a \in O_P \subseteq \overline{O}_P$, but $a \notin P$.

Recall that a two-sided ideal $P$ of $R$ is completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. 

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In [10], Shin showed that a ring $R$ is 2-primal if and only if every minimal prime ideal of $R$ is completely prime. Recently, Kim and Kwak [9] showed that $R$ is a 2-primal ring if and only if $\overline{O}_P \subseteq P$ for each prime ideal $P$ of $R$, and so $O_P \subseteq P$ for each prime ideal $P$ of a 2-primal ring $R$. Note that if $P$ is a completely prime ideal of $R$, then $\overline{O}_P \subseteq P$.

**Lemma 2.2.** Let $M$ be a maximal left ideal of a ring $R$ which properly contains a prime ideal $P$ of $R$. If $O_P \subseteq P$, then $M$ is an essential left ideal of $R$.

**Proof.** Let $M$ be a maximal left ideal of a ring $R$ which properly contains a prime ideal $P$ of $R$. Then there exists $a \in M$ such that $a \in R \setminus P$. Assume to the contrary that $M = l(e)$ for some $0 \neq e^2 = e \in R$, where $l(e) = \{r \in R \mid re = 0\}$. Then $ae = 0$. Since $e \notin P$, $a \in O_P$ and so $a \in P$, which is a contradiction. \qed

**Corollary 2.3.** Let $R$ be a 2-primal ring. If $M$ is a maximal left ideal of $R$ which properly contains a prime ideal $P$, then $M$ is an essential left ideal of $R$.

The condition "$M$ is a maximal left ideal of $R$ which properly contains a prime ideal $P$" in Lemma 2.2 and Corollary 2.3 is not superfluous. The conditions "$O_P \subseteq P$" in Lemma 2.2 and "$R$ is a 2-primal ring" in Corollary 2.3 are also not superfluous, respectively.

**Example 2.4.** (1) Let $F$ be a field. We consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Then $R$ is a 2-primal ring. However, the maximal left ideal $M = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ of $R$ is not essential. Note that $M$ does not properly contain any prime ideal of $R$.

(2) In Example 2.1, $O_P \notin P$ (and so $R$ is not 2-primal) and $M = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in F \right\}$ is a maximal left ideal of $R$ which properly contains a prime ideal $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. However $M$ is not an essential left ideal of $R$. 

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THEOREM 2.5. Let $P$ be a prime ideal of a ring $R$ with $O_P \subseteq P$. If every simple singular left $R$-module is $p$-injective, then $P$ is maximal.

Proof. We claim that $RaR + P = R$ for $a \in R \setminus P$. If not, then there exists a maximal ideal $M$ of $R$ containing $RaR + P$. Moreover, $M$ is a maximal left ideal of $R$. Suppose not. Then there exists a maximal left ideal $K$ of $R$ such that $M \subseteq K$, then $K$ is essential by Lemma 2.2. Thus, by hypothesis, $R/K$ is $p$-injective, so any $R$-homomorphism of $Ra$ into $R/K$ extends to one of $R$ into $R/K$. Let $f : Ra \rightarrow R/K$ be defined by $f(ra) = r + K$. If $ra = sa$ for $r, s \in R$, then $(r - s)a = 0$ and so $r - s \in O_P \subseteq P \subseteq K$. So $r + K = s + K$. Hence $f$ is a well-defined left $R$-homomorphism. Since $R/K$ is $p$-injective, there exists $b \in R$ such that $1 + K = f(a) = ab + K$. So $1 - ab \in K$, whence $1 \in K$, which is also a contradiction. Therefore $M$ is a maximal essential left ideal of $R$, and so $R/M$ is $p$-injective. By the same method in the above proof, $P$ is a maximal ideal of $R$. \qed

The following example shows that the condition “every simple singular left $R$-module is $p$-injective” in Theorem 2.5 is not superfluous.

EXAMPLE 2.6. Let $R = \mathbb{Z}$ be the ring of integers and $P$ a prime ideal of $R$. Then $P = p\mathbb{Z}$, where $p$ is a zero or positive prime number. Thus $O_P = \{a \in \mathbb{Z} \mid ab = 0 \text{ for some } b \in \mathbb{Z} \setminus p\mathbb{Z}\} = \{0\}$ and so $O_P \subseteq P$ for all prime ideal $P$ of $R$. But, a prime ideal $\{0\}$ is not maximal. Now, we consider a simple singular $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$. Then it is not $p$-injective. In fact, $f : 4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by $f(4n) = n + 2\mathbb{Z}$ cannot be extended to one of $\mathbb{Z}$ into $\mathbb{Z}/2\mathbb{Z}$, by the similar method of the proof of Theorem 2.5.

PROPOSITION 2.7. Let $R$ be a ring whose simple singular left $R$-modules are $p$-injective. Then every completely prime ideal of $R$ is maximal.

Proof. Let $P$ be a completely prime ideal of $R$. Suppose that $P$ is not a maximal ideal of $R$. Then there exists a maximal ideal $M$ of $R$ such that $RaR + P \subseteq M$ for $a \in R \setminus P$. In fact, $M$ is a maximal left ideal of $R$. If not, then there exists a maximal left ideal $K$ of $R$ such that $M \subseteq K$, and so $K$ is an essential left ideal by Lemma 2.2. Since $R/K$
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is $p$-injective, so any $R$-homomorphism of $Ra$ into $R/K$ extends to one of $R$ into $R/K$. Let $f : Ra \to R/K$ be defined by $f(ra) = r + K$. Then $f$ is well-defined. For, if $ra = sa$ for $r, s \in R$, then $(r - s)a = 0 \in P$ and so $r - s \in P \subseteq K$. Hence $r + K = s + K$. Since $R/K$ is $p$-injective, there exists $c \in R$ such that $1 + K = f(a) = ac + K$. Thus $1 - ac \in K$, whence $1 \in K$; which is a contradiction. So $M$ is a maximal essential left ideal of $R$. Hence $R/M$ is also $p$-injective. Applying the same method in the above proof, $P$ is a maximal ideal of $R$. □

Recall that a ring $R$ is 2-primal if and only if every minimal prime ideal of $R$ is completely prime.

**Corollary 2.8.** Let $R$ be a 2-primal ring. If every simple singular left $R$-module is $p$-injective, then every prime ideal of $R$ is maximal. In particular, every prime factor ring of $R$ is a simple domain.

**Proof.** Let $R$ be a 2-primal ring. By Proposition 2.7 and the fact that every minimal prime ideal of $R$ is completely prime, every prime ideal of $R$ is maximal. Thus every prime ideal of $R$ is completely prime and so every prime factor ring of $R$ is a simple domain. □

As a parallel result to Theorem 2.5 and Proposition 2.7, we obtain the following.

**Proposition 2.9.** Let $R$ be a ring whose maximal essential left ideals are two-sided. Then we have the following.

1. Assume that $O_P \subseteq P$ for each prime ideal $P$ of $R$. If every simple singular left $R$-module is $p$-injective, then every prime ideal of $R$ is a maximal left ideal.
2. If every simple singular left $R$-module is $p$-injective, then every completely prime ideal of $R$ is a maximal left ideal.

**Proof.** These proofs can be easily showed by adapting the method of the proofs of Theorem 2.5 and Proposition 2.7, respectively. □

The condition “every simple singular left $R$-module is $p$-injective” in Proposition 2.7, Corollary 2.8 and Proposition 2.9 is not superfluous.
Example 2.10. In Example 2.6, the ring $R$ is commutative and so it is 2-primal, but a completely prime ideal $\{0\}$ is not maximal. Note that a simple singular $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is also not $p$-injective.

The following example shows that there exists a ring whose maximal essential left ideals are two-sided and prime ideals are (two-sided) maximal, but every simple singular left $R$-module need not to be $p$-injective. This is compared with the converse of Proposition 2.9(2).

Example 2.11. Let $T = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. Then $T$ is a commutative subring of a ring of $2 \times 2$ matrices over $\mathbb{Z}_2$, Mat$_2(\mathbb{Z}_2)$, where $\mathbb{Z}_2$ denotes the ring of all integers modulo 2. Let $R$ be the set of all sequences of Mat$_2(\mathbb{Z}_2)$, which are eventually in $T$, i.e.,

$$R = \{ (a_n) \mid a_n \in \text{Mat}_2(\mathbb{Z}_2) \text{ and } a_n \text{ is eventually in } T \}.$$

Then $R$ is a ring under $(a_n) + (b_n) = (a_n + b_n), (a_n)/(b_n) = (a_n b_n)$.

Moreover, $R$ is a semiprime PI-ring whose maximal essential left ideals are two-sided, but it is not von Neumann regular [7]. Since $R$ is a strongly $\pi$-regular ring (i.e., for every $a \in R$, there exists a positive integer $n = n(a)$, depending on $a$, such that $a^n \in a^{n+1} R$), every prime ideal is maximal by [4, Theorem 2.3]. However, there exists a simple singular left $R$-module which is not $p$-injective by the following: If $R$ is a PI-ring, then $R$ is von Neumann regular if and only if $R$ is a semiprime ring whose simple singular left modules are $p$-injective [13, Corollary 7].

The condition “$O_P \subseteq P$ for each prime ideal $P$ of $R$” in Proposition 2.9(1) is not superfluous.

Example 2.12. Let $R_1 = \{ (a_n) \mid a_n \in \text{Mat}_2(\mathbb{Z}_2) \text{ and } a_n \text{ is eventually in } T_1 \}$, where $T_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{Z}_2 \right\}$. Then $R_1$ is a subring of the ring $R$ in Example 2.11. Note that $R_1$ is a semiprime PI, von Neumann regular ring. Thus every simple singular left module is $p$-injective. However $P = \{ (a_n) \in R_1 \mid a_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \}$ is a prime ideal of $R_1$ and it can be easily checked that $P$ is not a maximal
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left ideal of \( R_1 \). Now, for \( x = \left\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \cdots \right\rangle, y = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \cdots \right\rangle \in R_1 \setminus P, xy = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \cdots \right\rangle \in P \) and so \( x \in O_P \). Therefore \( O_P \not\subseteq P \).

Recall that a ring \( R \) is said to be left (right) weakly \( \pi \)-regular if for every \( a \in R \), there exists a positive integer \( n = n(a) \), depending on \( a \), such that \( a^n \in Ra^nRa^n \) (\( a^nRa^nR \)). \( R \) is weakly \( \pi \)-regular if it is both right and left weakly \( \pi \)-regular [5].

**Theorem 2.13.** Let \( P \) be a completely prime ideal of a ring \( R \). If \( R/P(R) \) is a left (right) weakly \( \pi \)-regular ring, then \( P \) is a maximal ideal of \( R \).

**Proof.** Let \( P \) be a completely prime ideal of \( R \) with \( a \in R \setminus P \). First suppose that \( R/P(R) \) is a left weakly \( \pi \)-regular ring. Then \( P + RaR = R \). If not, then there exists a maximal two-sided ideal \( M \) of \( R \) such that \( P + RaR \subseteq M \). Since \( \overline{R} = R/P(R) \) is left weakly \( \pi \)-regular, \( \overline{Ra} = \overline{Ra}^n \overline{Ra}^n \) for some positive integer \( n \). So \( \overline{Ra}^n = \overline{M} \overline{a}^n \). Hence \( \overline{a}^n = \overline{b} \overline{a}^n \) for some \( \overline{b} \in \overline{M} \), and so \( \overline{1} - \overline{b} \overline{a}^n = \overline{0} \). Then \( (1 - b)a^n \in P \). Since \( P \) is completely prime and \( a \notin P \), \( 1 - b \in P \subseteq M \), which is a contradiction. Therefore \( P \) is a maximal ideal of \( R \). Similarly, for a right weakly \( \pi \)-regular ring \( R/P(R) \), we obtain that \( P \) is a maximal ideal of \( R \).

**Corollary 2.14.** Let \( R \) be a 2-primal ring. Then the following statements are equivalent:

1. \( R/P(R) \) is a left (right) weakly \( \pi \)-regular ring.
2. Every prime ideal of \( R \) is maximal.
3. \( R/J(R) \) is a left (right) weakly \( \pi \)-regular ring and \( J(R) \) is nil, where \( J(R) \) denotes the Jacobson radical of \( R \).

**Proof.** (1) \( \Rightarrow \) (2): Suppose that \( R/P(R) \) is right (left) weakly \( \pi \)-regular and \( P \) is a minimal prime ideal of \( R \). Since \( R \) is 2-primal, \( P \) is completely prime. By Theorem 2.13, \( P \) is a maximal ideal of \( R \). (2) implies (3) by [8, Proposition 5]. Since 2-primal and \( J(R) \) is nil, we have \( P(R) = J(R) \). So (3) implies (1).

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The condition "$R$ is a 2-primal ring" in Corollary 2.14 is not superfluous by [8, Example 10].

The following example shows that there exists a 2-primal ring $R$ such that every prime ideal of $R$ is maximal, but $R$ is neither right nor left weakly $\pi$-regular, even though $R/P(R)$ is left and right weakly $\pi$-regular.

**Example 2.15.** Let $W_1[F]$ be the first Weyl algebra over a field $F$ of characteristic zero. Then $W_1[F]$ is a simple domain which is not a division ring. Now, let

$$R = \begin{bmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{bmatrix}.$$  

Then $R$ is a 2-primal ring whose prime ideals are maximal. But $R$ is neither right nor left weakly $\pi$-regular by [2, Example 12].

**References**


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