COINCIDENCE AND SADDLE POINT THEOREMS
ON GENERALIZED CONVEX SPACES

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Abstract. We give a new coincidence theorem for multimaps on
generalized convex spaces and apply it to deduce \( \varepsilon \)-saddle point and
saddle point theorems.

1. Introduction and Preliminaries

In [8], some \( \varepsilon \)-saddle point and saddle point theorems for convex sets
in topological vector spaces were obtained. These new results generalize
the corresponding ones of Komiya [2].

Now it is well-known that convex subsets of topological vector spaces
are generalized to convex spaces due to Lassonde [3], which are further
extended to the generalized convex spaces or \( G \)-convex spaces due to
Park [4,5,6,7]. This new class of spaces contains many known spaces
having certain abstract convexity without linear structure; see [5].

In the present paper, we deduce a new coincidence theorem for mul-
timaps on \( G \)-convex spaces, and use it to deduce new \( \varepsilon \)-saddle point
and saddle point theorems. Consequently, we show that main results
in [8] holds for much larger class of spaces.

A multimap \( T : X \rightrightarrows Y \) is a function from \( X \) into the power set
\( 2^Y \) of \( Y \) with fibers \( T^{-1}y = \{ x \in X : y \in Tx \} \) for \( y \in Y \). A function
\( f : X \rightarrow \mathbb{R} \) on a topological space \( X \) is said to be lower (resp. upper)
semicontinuous if the set \( \{ x \in X : f(x) > \alpha \} \) (resp. \( \{ x \in X : f(x) < \alpha \} \)) is open in \( X \) for every real number \( \alpha \).

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Given a set $A$, let $\langle A \rangle$ denote the collection of all nonempty finite subsets of $A$ and $|A|$ the cardinality of $A$. Let $\Delta_n$ be the standard $n$-simplex.

A generalized convex space or a G-convex space $(X, D; \Gamma)$ consists of a topological space $X$ and a nonempty set $D$ such that for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exist a subset $\Gamma(A)$ of $X$ and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $\phi_A(\Delta_J) \subset \Gamma(J)$ for every $J \in \langle A \rangle$, where $\Delta_J$ denotes the face of $\Delta_n$ corresponding to $J \in \langle A \rangle$; that is, if $\Delta_n = \text{co}\{e_0, e_1, \ldots, e_n\}, A = \{a_0, a_1, \ldots, a_n\}$, and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Examples of G-convex spaces [6] are convex spaces [3], C-spaces [1], and many others; see [5]. Given a G-convex space $(X, D; \Gamma)$ with $D \subset X$, a subset $K$ of $X$ is said to be $\Gamma$-convex if for each $A \in \langle D \rangle$, $A \subset K$ implies $\Gamma(A) \subset K$. For a nonempty subset $K$ of $X$ we define the $\Gamma$-convex hull of $K$

$$\Gamma\text{-co } K := \bigcap\{B \subset X : B \text{ is } \Gamma\text{-convex and } K \subset B\}.$$  

Then the $\Gamma$-convex hull of $K$ is the smallest $\Gamma$-convex set containing $K$.

If $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X, \Gamma)$. Let $\text{Int}_K A$ denote the interior of $A$ in $K$.

Given $\varepsilon > 0$, a function $f : X \times Y \to \mathbb{R}$ has an $\varepsilon$-saddle point $(x^*_\varepsilon, y^*_\varepsilon)$ if

$$f(x, y^*_\varepsilon) - \varepsilon < f(x^*_\varepsilon, y^*_\varepsilon) < f(x^*_\varepsilon, y) + \varepsilon$$

for all $x \in X$ and $y \in Y$; and a point $(x^*, y^*)$ is a saddle point of $f$ if

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$$

for all $x \in X$ and $y \in Y$; see [8].

Let $X$ and $Y$ be topological spaces, $K$ a subset of $X$ and $L$ a subset of $Y$. A function $f : X \times Y \to \mathbb{R}$ is said to be $\alpha$-transfer lower (resp. upper) semicontinuous on $K$ relative to $L$ if for each $(x, y) \in K \times L$, $f(x, y) > \alpha$ (resp. $f(x, y) < \alpha$) implies that there exists an open neighborhood $N(x)$ of $x$ in $K$ and a point $y' \in L$ such that $f(z, y') > \alpha$ (resp. $f(z, y') < \alpha$) for all $z \in N(x)$; and transfer lower (resp. upper) semicontinuous on $K$ relative to $L$ if $f$ is $\alpha$-transfer lower (resp. upper) semicontinuous on $K$ relative to $L$.
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upper) semicontinuous on $K$ relative to $L$ for each $\alpha \in \mathbb{R}$; see Tian [9]. These concepts are proper generalizations of lower (resp. upper) semicontinuous real-valued functions.

2. The Coincidence Theorem

We begin with the following lemmas due to the first author [4].

**Lemma 1.** Let $X$ be a Hausdorff compact space and $(Y,D;\Gamma)$ a $G$-convex space. Let $T : X \to Y$ and $S : X \to D$ be multimaps such that the following conditions are satisfied:

1. For each $x \in X$, $A \in \langle Sx \rangle$ implies $\Gamma(A) \subset Tx$; and
2. $X = \bigcup \{ \text{Int}_X S^{-1}y : y \in D \}$.

Then $T$ has a continuous selection $f : X \to Y$ such that $f = g \circ h$, where $g : \Delta_n \to Y$ and $h : X \to \Delta_n$ are continuous functions.

**Lemma 2.** Let $(X,\Gamma)$ be a Hausdorff compact $G$-convex space and $T : X \to X$ a multimap such that $Tx$ is a $\Gamma$-convex set for each $x \in X$, and $X = \bigcup \{ \text{Int}_X T^{-1}y : y \in X \}$. Then $T$ has a fixed point.

The following theorem improves and extends a result in [10, Theorem 1] to the case of a $G$-convex space.

**Theorem 1.** Let $X$ be a Hausdorff topological space, $(Y,D;\Gamma_Y)$ a $G$-convex space, $M$ and $P$ subsets of $X \times Y$. Suppose that there exist a compact $G$-convex space $(K,\Gamma_K)$ with $K \subseteq X$ and a subset $N$ of $K \times D$ such that

1. For each $x \in K$, $\Gamma$-co \{ $y \in D : (x,y) \notin N$ \} \subset \{ $y \in Y : (x,y) \notin M$ \};
2. For each $x \in K$ with \{ $y \in D : (x,y) \notin N$ \} \neq \emptyset, there exists $y' \in D$ such that $x \in \text{Int}_K \{ x' \in K : (x',y') \notin N \}$;
3. For each $y \in Y$, \{ $x \in K : (x,y) \in P$ \} is a $\Gamma$-convex subset of $(K,\Gamma_K)$;
4. $Y = \bigcup \{ \text{Int}_Y \{ y \in Y : (x,y) \in P \} : x \in K \}$; and
5. For all $(x,y) \in K \times Y$, $(x,y) \in P$ implies $(x,y) \in M$.

Then there exists a point $x_0 \in K$ such that $\{ x_0 \} \times D \subseteq N$. 

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\textit{Proof.} Suppose that the conclusion does not hold; that is, for each \( x \in K \) there is a point \( y_0 \in D \) such that \((x,y_0) \notin N\). For each \( x \in K \), let

\[ Sx = \{ y \in D : (x,y) \notin N \}, \quad Tx = \{ y \in Y : (x,y) \notin M \}. \]

Then for each \( x \in K \), \( \Gamma\text{-co}Sx \subset Tx \) by (1); \( K = \bigcup \{ \text{Int}_K S^{-}y : y \in D \} \) by (2). Define a multimap \( \tilde{S} : K \rightarrow Y \) by \( \tilde{S}x := \Gamma\text{-co}Sx \) for \( x \in K \).

Since \( K = \bigcup \{ \text{Int}_K \tilde{S}^{-}y : y \in Y \} \), by Lemma 1, there is a continuous function \( f : K \rightarrow Y \) such that \( f(x) \in \tilde{S}x \subset Tx \) for all \( x \in K \). Hence, \((x,f(x)) \notin M \) for all \( x \in K \).

On the other hand, we define a multimap \( H : Y \rightarrow K \) by

\[ Hy := \{ x \in K : (x,y) \in P \} \quad \text{for} \ y \in Y. \]

By (3), \( Hy \) is \( \Gamma \)-convex for every \( y \in Y \), and \( Y = \bigcup \{ \text{Int}_Y \text{H}^{-}x : x \in K \} \) by (4). A multimap \( F : K \rightarrow K \) defined by \( Fx := h \circ f(x) \) for \( x \in K \) has \( \Gamma \)-convex values and \( K = \bigcup \{ \text{Int}_K F^{-}y : y \in K \} \). In fact, for every \( x \in K \), there is a \( y \in K \) such that \( f(x) \in \text{Int}_Y \text{H}^{-}y \) and so \( x \in f^{-} \{ \text{Int}_Y \text{H}^{-}y \} \subset \text{Int}_K f^{-} \{ \text{H}^{-}y \} = \text{Int}_K F^{-}y \) by the continuity of \( f \). Since \((K, \Gamma_K)\) is a Hausdorff compact \( G \)-convex space, by Lemma 2, there is a point \( x_0 \in K \) such that \( x_0 \in Fx_0 = H(f(x_0)) \); and hence by (5), \((x_0,f(x_0)) \in M \). This contradiction proves the theorem. \( \square \)

Note that, if \( X \) and \( Y \) are \( C \)-spaces, Theorem 1 reduces to [10, Theorem 1].

Now we give a Fan-Browder type coincidence theorem for \( G \)-convex spaces which generalizes [1, Corollary 4.2] and [10, Theorem 5] for \( C \)-spaces.

\textbf{Theorem 2.} Let \( X \) be a Hausdorff topological space, \((Y,D; \Gamma_Y)\) a \( G \)-convex space, and \( T : X \rightarrow Y \) and \( S : Y \rightarrow X \) multimap. Suppose that there exist a compact \( G \)-convex space \((K, \Gamma_K)\) with \( K \subset X \) and a multimap \( A : K \rightarrow D \) such that

1. for each \( x \in K \), \( Ax \subset Tx \), and \( Tx \) is \( \Gamma \)-convex;
2. \( K = \bigcup \{ \text{Int}_K A^{-}y : y \in D \} \);
3. for each \( y \in Y \), \( Sy \cap K \) is \( \Gamma \)-convex in \((K, \Gamma_K)\); and
4. \( Y = \bigcup \{ \text{Int}_Y S^{-}x : x \in K \} \).
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Then there exist points \( x_0 \in K \) and \( y_0 \in Y \) such that \( y_0 \in Tx_0 \) and \( x_0 \in Sy_0 \).

Proof. Let

\[
P = \bigcup_{x \in X} \{x\} \times S^\perp x, \quad M = \{(x, y) \in X \times Y : y \not\in Tx\} \quad \text{and} \quad N = \{(x, y) \in K \times D : y \not\in Ax\}.
\]

Suppose that \( Tx \cap S^\perp x = \emptyset \) for all \( x \in K \). Then for all \( (x, y) \in K \times Y \), \( (x, y) \in P \) implies \( (x, y) \in M \). Since \( \{y \in D : (x, y) \not\in N\} \subset \{y \in Y : y \in Tx\} = \{y \in Y : (x, y) \not\in M\} \), and \( Tx \) is \( \Gamma \)-convex for each \( x \in K \), condition (1) of Theorem 1 is satisfied. By (2) it is clear that condition (2) of Theorem 1 holds.

For each \( y \in Y \), since \( \{x \in K : (x, y) \in P\} = Sy \cap K \), by assumption (3), condition (3) of Theorem 1 is also satisfied. By (4),

\[
Y = \bigcup \{\text{Int}_Y \{y \in Y : (x, y) \in P\} : x \in K\},
\]

that is, condition (4) of Theorem 1 holds. By Theorem 1, there exists a point \( x_0 \in K \) such that \( \{x_0\} \times D \subset N \); that is, \( y \not\in Ax_0 \) for all \( y \in D \). Consequently, we have \( Ax_0 = \emptyset \), which contradicts assumption (2) (since \( y_0 \in Ax_0 \) for some \( y_0 \in D \)). This completes the proof.

Note that, even if \( X \) and \( Y \) are \( C \)-spaces, Theorem 2 improves [10, Theorem 5].

3. Main Results

Using our coincidence theorem, we obtain a new \( \varepsilon \)-saddle point theorem for \( G \)-convex spaces which generalizes [8, Theorem 1] for topological vector spaces.

**Theorem 3.** Let \( X \) be a Hausdorff topological space, \( (Y, \Gamma_Y) \) a \( G \)-convex space, \( f : X \times Y \to \mathbb{R} \) a real-valued function and \( \varepsilon > 0 \). Suppose that there exists a compact \( G \)-convex space \( (K, \Gamma_K) \) with \( K \subset X \) such that

1. for any \( (x, y) \in X \times Y \), \( \inf_{v \in Y} f(x, v) > -\infty \) and \( \sup_{u \in X} f(u, y) < +\infty \);
(2) the function \( (x, y) \mapsto f(x, y) - \inf_{v \in Y} f(x, v) \) is \( \varepsilon \)-transfer upper semicontinuous on \( K \) relative to \( Y \), and the set \( \{ x \in K : f(x, y) > t \} \) is a nonempty \( \Gamma \)-convex set for each \( y \in Y \) and each \( t \in \mathbb{R} \);

(3) the function \( (x, y) \mapsto f(x, y) - \sup_{u \in X} f(u, y) \) is \( (-\varepsilon) \)-transfer lower semicontinuous on \( Y \) relative to \( K \), and \( \{ y \in Y : f(x, y) < t \} \) is a nonempty \( \Gamma \)-convex set for each \( x \in K \) and each \( t \in \mathbb{R} \).

Then \( f \) has a point \( (x^*_x, y^*_y) \in K \times Y \) such that \( f(x^*_x, y^*_y) - \varepsilon < f(x^*_x, y^*_y') < f(x^*_x, y^*_y) + \varepsilon \) for all \( x \in X \) and \( y \in Y \).

**Proof.** Let \( \varepsilon > 0 \). Define multimaps \( A : K \rightharpoonup Y \), \( T : X \rightharpoonup Y \) and \( S : Y \rightharpoonup X \) by

\[
Ax = \{ y \in Y : f(x, y) - \inf_{v \in Y} f(x, v) < \varepsilon \}
\]

\[
Tx = \{ y \in Y : f(x, y) - \inf_{v \in Y} f(x, v) < \varepsilon \}
\]

\[
Sy = \{ x \in X : f(x, y) - \sup_{u \in X} f(u, y) > -\varepsilon \}.
\]

Then for each \( x \in K \), \( Ax = Tx \), and \( Tx \) is a nonempty \( \Gamma \)-convex set. For each \( x \in K \), there exists a \( y \in Y \) such that \( f(x, y) - \inf_{v \in Y} f(x, v) < \varepsilon \).

By (2), there exists an open neighborhood \( N(x) \) of \( x \) in \( K \) and a point \( y' \in Y \) such that \( f(z, y') - \inf_{v \in Y} f(z, v) < \varepsilon \) for all \( z \in N(x) \), that is, \( N(x) \subset A^{-}y' \); and hence \( x \in \text{Int}_{K} A^{-}y' \). Thus \( K = \bigcup \{ \text{Int}_{K} A^{-}y' : y \in Y \} \). Moreover, \( Sy \cap K \) is a nonempty \( \Gamma \)-convex set for each \( y \in Y \) by (2). A similar argument shows by (3) that \( Y = \bigcup \{ \text{Int}_{Y} S^{-}x : x \in K \} \).

By Theorem 2, there exists \( (x^*_x, y^*_y) \in K \times Y \) such that \( y^*_y \in Tx^*_x \) and \( x^*_x \in Sy^*_y \); that is, \( f(x, y^*_y) - \varepsilon < f(x^*_x, y^*_y) < f(x^*_x, y^*_y) + \varepsilon \) for all \( x \in X \) and \( y \in Y \). This completes the proof. \( \square \)

For the case when \( X \) and \( Y \) are convex spaces in the sense of Lassonde [3] and for mere upper (resp. lower) semicontinuous functions, Theorem 3 improves [8, Theorem 1].

From Theorem 3 we deduce the following new saddle point theorem for spaces without linear structure.

**Theorem 4.** Let \( X \) be a Hausdorff topological space, \( (Y, \Gamma_Y) \) a Hausdorff \( G \)-convex space and \( f : X \times Y \to \mathbb{R} \) a real-valued function.
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Suppose that there exists a compact $G$-convex space $(K, \Gamma_K)$ with $K \subset X$ such that

1. for any $(x, y) \in X \times Y$, $\inf_{v \in Y} f(x, v) > -\infty$ and $\sup_{u \in X} f(u, y) < +\infty$;
2. the function $(x, y) \mapsto f(x, y) - \inf_{v \in Y} f(x, v)$ is upper semicontinuous on $K$ relative to $Y$, the function $x \mapsto f(x, y)$ is upper semicontinuous on $K$ for each $y \in Y$; and the set \( \{ x \in K : f(x, y) > t \} \) is a nonempty $\Gamma$-convex set for each $y \in Y$ and $t \in \mathbb{R}$;
3. the function $(x, y) \mapsto f(x, y) - \sup_{u \in X} f(u, y)$ is transfer lower semicontinuous on $Y$ relative to $K$, and \( \{ y \in Y : f(x, y) < t \} \) is a nonempty $\Gamma$-convex set for each $x \in K$ and each $t \in \mathbb{R}$;
4. for every sequence \( \{(x_n, y_n)\}_{n \in \mathbb{N}} \) in $K \times Y$ such that $(x_n, y_n)$ is an $\varepsilon_n$-saddle point of $f$ and $\varepsilon_n \to 0^+$, there exist a subsequence \( \{y_{n_k}\}_{k \in \mathbb{N}} \) and a point $y^* \in Y$ such that

\[
\liminf_{k \to \infty} f(x, y_{n_k}) \geq f(x, y^*) \quad \text{for all } x \in X.
\]

Then $f$ has a point $(x^*, y^*) \in K \times Y$ such that $f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$ for all $x \in X$ and $y \in Y$.

Proof. For each $n \in \mathbb{N}$ with $\varepsilon_n \to 0^+$, by Theorem 3, there is a point $(x_n^*, y_n^*) \in K \times Y$ such that

\[
f(x, y_n^*) - \varepsilon_n < f(x_n^*, y_n^*) < f(x_n^*, y) + \varepsilon_n \quad \text{for all } (x, y) \in X \times Y.
\]

By (4), there exist a subsequence \( \{y_{n_k}^*\}_{k \in \mathbb{N}} \) and a point $y^* \in Y$ such that

\[
\liminf_{k \to \infty} f(x, y_{n_k}^*) \geq f(x, y^*) \quad \text{for each } x \in X.
\]

Since $K$ is compact, there is a subnet \( \{x_{n_\alpha}^*\} \) of \( \{x_n^*\} \) and $x^* \in K$ such that \( \{x_{n_\alpha}^*\} \) converges to $x^*$.

For each $x \in X$ and each $\alpha$, we have

\[
f(x^*, y^*) = f(x^*, y^*) - f(x_{n_\alpha}^*, y^*) + f(x_{n_\alpha}^*, y^*)
\]
\[
> f(x^*, y^*) - f(x_{n_\alpha}^*, y^*) + f(x, y_{n_\alpha}^*) - 2\varepsilon_\alpha
\]

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and hence by the uppersemicontinuity of \( f(\cdot, y^*) \) on \( K \)

\[
\begin{align*}
    f(x^*, y^*) &\geq f(x^*, y^*) - \limsup_{\alpha} f(x^*_\alpha, y^*) + \liminf_{\alpha} f(x, y^*) \\
    &\geq f(x, y^*).
\end{align*}
\]

Next, for each \( y \in Y \) and each \( \alpha \), we have

\[
\begin{align*}
    f(x^*, y^*) &= f(x^*, y^*) - f(x^*, y^*_\alpha) + f(x^*, y^*_\alpha) \\
    &< f(x^*, y^*) - f(x^*, y^*_\alpha) + f(x^*_\alpha, y) + 2\varepsilon_{\alpha}
\end{align*}
\]

and hence by the uppersemicontinuity of \( f(\cdot, y) \) on \( K \)

\[
\begin{align*}
    f(x^*, y^*) &\leq f(x^*, y^*) - \liminf_{\alpha} f(x^*, y^*_\alpha) + \limsup_{\alpha} f(x^*_\alpha, y) \\
    &\leq f(x^*, y).
\end{align*}
\]

Thus, \((x^*, y^*) \in K \times Y\) is a saddle point of \( f \). This completes the proof. \( \Box \)

Note that Theorem 4 is a far-reaching generalization of [8, Theorem 2] and [2, Theorem 3].

Similarly, many other results for convex spaces or \( C \)-spaces can be extended to the framework of \( G \)-convex spaces. In the first author's works on \( G \)-convex spaces, he tried to restrict to write down only essential things.

References

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