THE CATEGORY OF INTUITIONISTIC Fuzzy TOPOLOGICAL SPACES

SOK JONG LEE and EUN PYO LEE

Abstract. In this paper, we introduce the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods. We investigate the properties of continuous, open and closed maps in the intuitionistic fuzzy topological spaces, and show that the category of Chang's fuzzy topological spaces is a bireflective full subcategory of that of intuitionistic fuzzy topological spaces.

1. Introduction and Preliminaries

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues [3,4,5] introduced intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets.

In this paper, we introduce the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods. We investigate the properties of continuous, open and closed maps in the intuitionistic fuzzy topological spaces, and show that the category of Chang's fuzzy topological spaces [2] is a bireflective full subcategory of that of intuitionistic fuzzy topological spaces.

Let $X$ be a nonempty set. An *intuitionistic fuzzy set* (IFS for short) $A$ is an ordered pair

$$A = (\mu_A, \gamma_A)$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively, and $\mu_A + \gamma_A \leq 1$.

Received January 5, 1999.

1991 Mathematics Subject Classification: 54A40.

Key words and phrases: intuitionistic fuzzy point, intuitionistic fuzzy neighborhood, intuitionistic fuzzy topology.
Obviously every fuzzy set $\mu_A$ on $X$ is an IFS of the form $(\mu_A, 1-\mu_A)$.

**Definition 1.1** ([1]). Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs on $X$. Then

1. $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\gamma_A \geq \gamma_B$.
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
3. $A^c = (\gamma_A, \mu_A)$.
4. $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.
5. $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.
6. $0_\infty = (0, 1)$ and $1_\infty = (1, 0)$.

Let $f$ be a map from a set $X$ to a set $Y$. Let $A = (\mu_A, \gamma_A)$ be an IFS of $X$ and $B = (\mu_B, \gamma_B)$ an IFS of $Y$. Then $f^{-1}(B)$ is an IFS of $X$ defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$$

and $f(A)$ is an IFS of $Y$ defined by

$$f(A) = (f(\mu_A), 1 - f(1 - \gamma_A)).$$

**Definition 1.2** ([3]). An intuitionistic fuzzy topology (IFT) for short) on $X$ is a family $\mathcal{T}$ of IFSs in $X$ which satisfies the following properties:

1. $0_\infty, 1_\infty \in \mathcal{T}$.
2. If $A_1, A_2 \in \mathcal{T}$, then $A_1 \cap A_2 \in \mathcal{T}$.
3. If $A_i \in \mathcal{T}$ for each $i$, then $\bigcup A_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called an intuitionistic fuzzy topological space (IFTS for short).

Let $(X, \mathcal{T})$ be an IFTS. Then any member of $\mathcal{T}$ is called an intuitionistic fuzzy open set (IFOS for short) of $X$ and the complement an intuitionistic fuzzy closed set (IFCS for short).

**Definition 1.3** ([3]). Let $(X, \mathcal{T})$ be an IFTS and $A$ an IFS in $X$. Then the fuzzy closure is defined by

$$\text{cl}(A) = \bigcap \{F \mid A \subseteq F, F^c \in \mathcal{T}\}$$

and the fuzzy interior is defined by

$$\text{int}(A) = \bigcup \{G \mid A \supseteq G, G \in \mathcal{T}\}.$$
The category of intuitionistic fuzzy topological spaces

**Theorem 1.4 ([3]).** For an IFS $A$ of an IFTS $(X, T)$, we have:

1. $\text{int}(A)^c = \text{cl}(A^c)$.
2. $\text{cl}(A)^c = \text{int}(A^c)$.

**2. Intuitionistic Fuzzy Neighborhoods**

We are going to introduce the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods.

**Definition 2.1.** Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. An **intuitionistic fuzzy point** (IFP for short) $x_{(\alpha, \beta)}$ of $X$ is an IFS of $X$ defined by

$$x_{(\alpha, \beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if } y = x, \\ (0, 1) & \text{if } y \neq x. \end{cases}$$

In this case, $x$ is called the **support** of $x_{(\alpha, \beta)}$ and $\alpha$ and $\beta$ are called the **value** and the **nonvalue** of $x_{(\alpha, \beta)}$, respectively. An IFP $x_{(\alpha, \beta)}$ is said to **belong** to an IFS $A = (\mu_A, \gamma_A)$ of $X$, denoted by $x_{(\alpha, \beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Clearly an intuitionistic fuzzy point can be represented by an ordered pair of fuzzy points as follows:

$$x_{(\alpha, \beta)} = (x_{\alpha}, 1 - x_{1-\beta}).$$

**Theorem 2.2.** Let $A = (\mu_A, \gamma_A)$ be an IFS of $X$. Then $x_{(\alpha, \beta)} \in A$ if and only if $x_{\alpha} \in \mu_A$ and $x_{1-\beta} \in 1 - \gamma_A$.

**Proof.** Let $x_{(\alpha, \beta)} \in A \Leftrightarrow \alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x) \Leftrightarrow \alpha \leq \mu_A(x)$ and $1 - \beta \leq 1 - \gamma_A(x) \Leftrightarrow x_{\alpha} \in \mu_A$ and $x_{1-\beta} \in 1 - \gamma_A$. □

**Theorem 2.3.** Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs of $X$. Then $A \subseteq B$ if and only if $x_{(\alpha, \beta)} \in A$ implies $x_{(\alpha, \beta)} \in B$ for any IFP $x_{(\alpha, \beta)}$ in $X$.

**Proof.** Let $A \subseteq B$ and $x_{(\alpha, \beta)} \in A$. Then $\alpha \leq \mu_A(x) \leq \mu_B(x)$ and $\beta \geq \gamma_A(x) \geq \gamma_B(x)$. Thus $x_{(\alpha, \beta)} \in B$. Conversely, take any $x \in X$. Let $\alpha = \mu_A(x)$ and $\beta = \gamma_A(x)$. Then $x_{(\alpha, \beta)}$ is an IFP in $X$ and $x_{(\alpha, \beta)} \in A$. By the hypothesis, $x_{(\alpha, \beta)} \in B$. Thus $\mu_A(x) = \alpha \leq \mu_B(x)$ and $\gamma_A(x) = \beta \geq \gamma_B(x)$. Hence $A \subseteq B$. □

65
Theorem 2.4. Let \( A = (\mu_A, \gamma_A) \) be an IFS of \( X \). Then \( A = \bigcup \{ x(\alpha, \beta) \mid x(\alpha, \beta) \in A \} \).

Proof. Since \( x(\alpha, \beta) = (x_\alpha, 1 - x_\beta) \),
\[
\bigcup \{ x(\alpha, \beta) \mid x(\alpha, \beta) \in A \}
= (\bigvee \{ x_\alpha \mid x_\alpha \in \mu_A \}) \land \{ 1 - x_\beta \mid x_\beta \in 1 - \gamma_A \}.
\]
Clearly \( \bigvee \{ x_\alpha \mid x_\alpha \in \mu_A \} = \mu_A \). On the other hand \( \bigvee \{ x_\beta \mid x_\beta \in 1 - \gamma_A \} = 1 - \gamma_A \) and hence
\[
\gamma_A = 1 - \bigvee \{ x_\beta \mid x_\beta \in 1 - \gamma_A \} = \bigwedge \{ 1 - x_\beta \mid x_\beta \in 1 - \gamma_A \}.
\]
Hence \( \bigcup \{ x(\alpha, \beta) \mid x(\alpha, \beta) \in A \} = (\mu_A, \gamma_A) = A. \)

Definition 2.5. Let \( x(\alpha, \beta) \) be an IFP of an IFTS \((X, \mathcal{T})\). An IFS \( A \) of \( X \) is called an intuitionistic fuzzy neighborhood (IFN for short) of \( x(\alpha, \beta) \) if there is an IFOS \( B \) in \( X \) such that \( x(\alpha, \beta) \in B \subseteq A \).

Theorem 2.6. Let \((X, \mathcal{T})\) be an IFTS. Then an IFS \( A \) of \( X \) is an IFOS if and only if \( A \) is an IFN of \( x(\alpha, \beta) \) for every IFP \( x(\alpha, \beta) \in A \).

Proof. Let \( A \) be an IFOS of \( X \). Clearly \( A \) is an IFN of any \( x(\alpha, \beta) \in A \). Conversely, let \( x(\alpha, \beta) \in A \). Since \( A \) is an IFN of \( x(\alpha, \beta) \), there is an IFOS \( B_{x(\alpha, \beta)} \) in \( X \) such that \( x(\alpha, \beta) \in B_{x(\alpha, \beta)} \subseteq A \). So we have
\[
A = \bigcup \{ x(\alpha, \beta) \mid x(\alpha, \beta) \in A \} \subseteq \bigcup \{ B_{x(\alpha, \beta)} \mid x(\alpha, \beta) \in A \} \subseteq A
\]
and hence \( A = \bigcup \{ B_{x(\alpha, \beta)} \mid x(\alpha, \beta) \in A \} \). Since each \( B_{x(\alpha, \beta)} \) is an IFOS, \( A \) is an IFOS.

3. Continuous, Open and Closed Maps

Definition 3.1 ([3]). Let \((X, \mathcal{T})\) and \((Y, \mathcal{U})\) be IFSTs. Then a map \( f : X \to Y \) is said to be

1. continuous if \( f^{-1}(B) \) is an IFOS of \( X \) for each IFOS \( B \) of \( Y \), or equivalently, \( f^{-1}(B) \) is an IFCS of \( X \) for each IFCS \( B \) of \( Y \),
2. open if \( f(A) \) is an IFOS of \( Y \) for each IFOS \( A \) of \( X \),
3. closed if \( f(A) \) is an IFCS of \( Y \) for each IFCS \( A \) of \( X \),
4. a homeomorphism if \( f \) is bijective, continuous and open.
The category of intuitionistic fuzzy topological spaces

**Theorem 3.2.** Let \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) be a map. Then \( f \) is continuous if and only if for any IFP \( x_{(\alpha, \beta)} \) in \( X \) and any IFN \( B \) of \( f(x_{(\alpha, \beta)}) \), there is an IFN \( A \) of \( x_{(\alpha, \beta)} \) such that \( x_{(\alpha, \beta)} \in A \) and \( f(A) \subseteq B \).

**Proof.** Let \( x_{(\alpha, \beta)} \) be any IFP in \( X \) and \( B \) any IFN of \( f(x_{(\alpha, \beta)}) \). Then there is an IFOS \( C \) in \( Y \) such that \( f(x_{(\alpha, \beta)}) \in C \subseteq B \). Since \( f \) is continuous, \( f^{-1}(C) \) is an IFOS and

\[
x_{(\alpha, \beta)} \in f^{-1}(x_{(\alpha, \beta)}) \subseteq f^{-1}(C) \subseteq f^{-1}(B).
\]

Put \( A = f^{-1}(C) \). Then \( A \) is an IFN of \( x_{(\alpha, \beta)} \) and \( x_{(\alpha, \beta)} \in A \subseteq f^{-1}(B) \). Thus \( x_{(\alpha, \beta)} \in A \) and \( f(A) \subseteq ff^{-1}(B) \subseteq B \).

Conversely, let \( B \) be any IFOS of \( Y \). If \( f^{-1}(B) = 0_{\alpha, \beta} \), then it is obvious. Suppose \( x_{(\alpha, \beta)} \in f^{-1}(B) \). Then \( B \) is an IFN of \( f(x_{(\alpha, \beta)}) \). By hypothesis, there is an IFN \( A_{x_{(\alpha, \beta)}} \) of \( x_{(\alpha, \beta)} \) such that \( x_{(\alpha, \beta)} \in A_{x_{(\alpha, \beta)}} \) and \( f(A_{x_{(\alpha, \beta)}}) \subseteq B \). Since \( A_{x_{(\alpha, \beta)}} \) is an IFN of \( x_{(\alpha, \beta)} \), there is an IFOS \( C_{x_{(\alpha, \beta)}} \) in \( X \) such that

\[
x_{(\alpha, \beta)} \in C_{x_{(\alpha, \beta)}} \subseteq A_{x_{(\alpha, \beta)}} \subseteq f^{-1}f(A_{x_{(\alpha, \beta)}}) \subseteq f^{-1}(B).
\]

So we have

\[
f^{-1}(B) = \bigcup \{x_{(\alpha, \beta)} \mid x_{(\alpha, \beta)} \in f^{-1}(B)\}
\]
\[
\subseteq \bigcup \{C_{x_{(\alpha, \beta)}} \mid x_{(\alpha, \beta)} \in f^{-1}(B)\}
\]
\[
\subseteq f^{-1}(B)
\]

and hence \( f^{-1}(B) = \bigcup \{C_{x_{(\alpha, \beta)}} \mid x_{(\alpha, \beta)} \in f^{-1}(B)\} \). Thus \( f^{-1}(B) \) is an IFOS of \( X \). Therefore \( f \) is continuous. \( \square \)

**Theorem 3.3.** Let \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) be a map. Then the following statements are equivalent:

1. \( f \) is a continuous map.
2. \( f(\text{cl}(A)) \subseteq \text{cl}(f(A)) \) for each IFS \( A \) of \( X \).
3. \( \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for each IFS \( B \) of \( Y \).
4. \( f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B)) \) for each IFS \( B \) of \( Y \).
Proof. We already know that $(1) \iff (3) \iff (4)$ (See [3]). $(1) \implies (2)$

Let $f$ be a continuous map and $A$ any IFS of $X$. Since $\text{cl}(f(A))$ is an
IFS of $Y$, $f^{-1}(\text{cl}(f(A)))$ is an IFS of $X$. Thus

$$\text{cl}(A) \subseteq \text{cl}(f^{-1}(f(A))) \subseteq \text{cl}(f^{-1}(\text{cl}(f(A)))) = f^{-1}(\text{cl}(f(A))).$$

Hence

$$f(\text{cl}(A)) \subseteq ff^{-1}(\text{cl}(f(A))) \subseteq \text{cl}(f(A)).$$

$(2) \implies (3)$ Let $B$ be any IFS of $Y$. By $(2)$,

$$f(\text{cl}(f^{-1}(B))) \subseteq \text{cl}(f^{-1}(B)) \subseteq \text{cl}(B).$$

Thus

$$\text{cl}(f^{-1}(B)) \subseteq f^{-1}(f(\text{cl}(f^{-1}(B)))) \subseteq f^{-1}(\text{cl}(B)).$$

The conditions in Theorem 3.3 are not equivalent to the condition
that $\text{int}(f(A)) \subseteq f(\text{int}(A))$ for each IFS $A$ of $X$. This is shown by the
following two examples.

**Example 3.4.** Let $X = \{x, y\}$ and $A$ and $B$ be IFSs of $X$ defined as

$A(x) = (0.3, 0.4), \quad A(y) = (0.3, 0.4);$

and

$B(x) = (0.3, 0.4), \quad B(y) = (0.1).$

Define $T_1 = \{0_\sim, 1_\sim, A\}$ and $T_2 = \{0_\sim, 1_\sim, B\}$. Then clearly $T_1$ and
$T_2$ are IFTs on $X$. Consider the constant map $f : (X, T_1) \rightarrow (X, T_2)$
defined by $f(x) = x$ and $f(y) = x$. Then $f^{-1}(0_\sim) = 0_\sim, f^{-1}(1_\sim) = 1_\sim$
and $f^{-1}(B) = A$ are IFOSs of $(X, T_1)$ and hence $f$ is continuous. But
$\text{int}(f(B)) = \text{int}(B) = B$ in $(X, T_2)$ and $f(\text{int}(B)) = f(0_\sim) = 0_\sim$ in
$(X, T_1)$. Thus $\text{int}(f(B)) \nsubseteq f(\text{int}(B))$.

**Example 3.5.** Let $X = \{x, y\}$ and $A$ be an IFS of $X$ defined as

$A(x) = (0.4, 0.2), \quad A(y) = (1, 0).$

Define $T_1 = \{0_\sim, 1_\sim\}$ and $T_2 = \{0_\sim, 1_\sim, A\}$. Then clearly $T_1$ and $T_2$
are IFTs on $X$. Consider the constant map $f : (X, T_1) \rightarrow (X, T_2)$
defined by $f(x) = x$ and $f(y) = x$. Then for each IFS $B$ of $(X, T_1)$,
$f(B)(y) = (0, 1)$. Thus $\text{int}(f(B)) = 0_\sim \nsubseteq f(\text{int}(B))$. But $f^{-1}(A)$ is
not an IFOS in $(X, T_1)$ and hence $f$ is not continuous.

However, we obtain the following theorem for a bijection $f$. 

68
The category of intuitionistic fuzzy topological spaces

**Theorem 3.6.** Let \( f : (X, \mathcal{T}) \to (Y, \mathcal{U}) \) be a bijection. Then the following statements are equivalent:

1. \( f \) is a continuous map.
2. \( f(\text{cl}(A)) \subseteq \text{cl}(f(A)) \) for each IFS \( A \) of \( X \).
3. \( \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for each IFS \( B \) of \( Y \).
4. \( f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B)) \) for each IFS \( B \) of \( Y \).
5. \( \text{int}(f(A)) \subseteq f(\text{int}(A)) \) for each IFS \( A \) of \( X \).

**Proof.** By Theorem 3.3, it suffices to show that (4) is equivalent to (5). Let \( A \) be any IFS of \( X \). Then \( f(A) \) is an IFS of \( Y \). So \( f^{-1}(\text{int}(f(A))) \subseteq \text{int}(f^{-1}f(A)) \). Since \( f \) is one-to-one,

\[
f^{-1}(\text{int}(f(A))) \subseteq \text{int}(f^{-1}f(A)) = \text{int}(A).
\]

Thus \( f f^{-1}(\text{int}(f(A))) \subseteq f(\text{int}(A)) \). Since \( f \) is onto,

\[
\text{int}(f(A)) = ff^{-1}(\text{int}(f(A))) \subseteq f(\text{int}(A)).
\]

Conversely, let \( B \) be any IFS of \( Y \). Then \( f^{-1}(A) \) is an IFS of \( X \). Since \( f \) is onto,

\[
\text{int}(B) = \text{int}(ff^{-1}(B)) \subseteq f(\text{int}(f^{-1}(B))).
\]

Since \( f \) is one-to-one,

\[
f^{-1}(\text{int}(B)) \subseteq f^{-1}f(\text{int}(f^{-1}(B))) = \text{int}(f^{-1}(B)).
\]

Hence the theorem follows. \( \Box \)

**Theorem 3.7.** Let \( f : (X, \mathcal{T}) \to (Y, \mathcal{U}) \) be a map. Then the following statements are equivalent:

1. \( f \) is an open map.
2. \( f(\text{int}(A)) \subseteq \text{int}(f(A)) \) for each IFS \( A \) of \( X \).
3. \( \text{int}(f^{-1}(B)) \subseteq f^{-1}(\text{int}(B)) \) for each IFS \( B \) of \( Y \).
Proof. (1) \( \Rightarrow \) (2) Let \( A \) be any IFS of \( X \). Clearly, \( \text{int}(A) \) is an IFOS of \( X \). Since \( f \) is an open map, \( f(\text{int}(A)) \) is an IFOS in \( Y \). Thus
\[
f(\text{int}(A)) = \text{int}(f(\text{int}(A))) \subseteq \text{int}(f(A)).
\]

(2) \( \Rightarrow \) (3) Let \( B \) be any IFS of \( Y \). Then \( f^{-1}(B) \) is an IFS of \( X \). By (2),
\[
f(\text{int}(f^{-1}(B))) \subseteq \text{int}(ff^{-1}(B)) \subseteq \text{int}(B).
\]

Thus we have
\[
\text{int}(f^{-1}(B)) \subseteq f^{-1}(f(\text{int}(f^{-1}(B)))) \subseteq f^{-1}(\text{int}(B)).
\]

(3) \( \Rightarrow \) (1) Let \( A \) be any IFOS of \( X \). Then \( \text{int}(A) = A \) and \( f(A) \) is an IFS of \( Y \). By (3),
\[
A = \text{int}(A) \subseteq \text{int}(f^{-1}(f(A))) \subseteq f^{-1}(\text{int}(f(A))).
\]

Hence we have
\[
f(A) \subseteq ff^{-1}(\text{int}(f(A))) \subseteq \text{int}(f(A)) \subseteq f(A).
\]
Thus \( f(A) = \text{int}(f(A)) \) and hence \( f(A) \) is an IFOS in \( Y \). Therefore \( f \) is an open map.
\( \square \)

**Theorem 3.8.** Let \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) be a map. Then the following statements are equivalent:

1. \( f \) is a closed map.
2. \( \text{cl}(f(A)) \subseteq f(\text{cl}(A)) \) for each IFS \( A \) of \( X \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( A \) be any IFS of \( X \). Clearly, \( \text{cl}(A) \) is an IFCS in \( X \). Since \( f \) is a closed map, \( f(\text{cl}(A)) \) is an IFCS in \( Y \). Thus we have
\[
\text{cl}(f(A)) \subseteq \text{cl}(f(\text{cl}(A))) = f(\text{cl}(A)).
\]

(2) \( \Rightarrow \) (1) Let \( A \) be any IFCS of \( X \). Then \( \text{cl}(A) = A \). By (2),
\[
\text{cl}(f(A)) \subseteq f(\text{cl}(A)) = f(A) \subseteq \text{cl}(f(A)).
\]
Thus \( f(A) = \text{cl}(f(A)) \) and hence \( f(A) \) is an IFCS in \( Y \). Therefore \( f \) is a closed map.
\( \square \)

The conditions in Theorem 3.8 are not equivalent to the condition that \( f^{-1}(\text{cl}(B)) \subseteq \text{cl}(f^{-1}(B)) \) for each IFS \( B \) of \( Y \). This is shown by the following two examples.
The category of intuitionistic fuzzy topological spaces

**Example 3.9.** Let $X = \{x, y\}$ and $A, B$ and $C$ be IFSs of $X$ defined as

\[
A(x) = (0.3, 0.4), \quad A(y) = (0.3, 0.4);
\]
\[
B(x) = (0.3, 0.4), \quad B(y) = (0, 1);
\]
and
\[
C(x) = (1, 0), \quad C(y) = (0, 1).
\]

Define $\mathcal{T}_1 = \{0_\infty, 1_\infty, A^c\}$ and $\mathcal{T}_2 = \{0_\infty, 1_\infty, B^c, C^c\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are IFTs on $X$. Consider the constant map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = x$. Then $f(0_\infty) = 0_\infty$, $f(1_\infty) = C$ and $f(A) = B$ are IFCs of $(X, \mathcal{T}_2)$ and hence $f$ is a closed map. On the other hand, $f^{-1}(\text{cl}(A)) = f^{-1}(1_\infty) = 1_\infty$ in $(X, \mathcal{T}_2)$ and $\text{cl}(f^{-1}(A)) = \text{cl}(A) = A$ in $(X, \mathcal{T}_1)$. Thus $f^{-1}(\text{cl}(A)) \subset \text{cl}(f^{-1}(A))$.

**Example 3.10.** Let $X = \{x, y\}$ and $A$ be an IFS of $X$ defined as

\[
A(x) = (0, 1), \quad A(y) = (1, 0).
\]

Define $\mathcal{T}_1 = \{0_\infty, 1_\infty\}$ and $\mathcal{T}_2 = \{0_\infty, 1_\infty, A^c\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are IFTs on $X$. Consider the constant map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = x$. Let $B$ be any IFS of $X$. If $B(x) = (0, 1)$, then

\[
f^{-1}(\text{cl}(B)) = f^{-1}(A) = 0_\infty \subset \text{cl}(f^{-1}(B)).
\]

Suppose $B(x) \neq (0, 1)$. Then $f^{-1}(B) \neq 0_\infty$ and hence

\[
f^{-1}(\text{cl}(B)) \subset 1_\infty = \text{cl}(f^{-1}(B)).
\]

But $f(1_\infty) = A^c$ (note that $f$ is not onto) is not an IFC in $(X, \mathcal{T}_2)$ and hence $f$ is not a closed map.

However, in case $f$ is a bijection, we have the following theorem.

**Theorem 3.11.** Let $f : (X, \mathcal{T}) \to (Y, U)$ be a bijection. Then the following statements are equivalent:

1. $f$ is a closed map.
2. $\text{cl}(f(A)) \subset f(\text{cl}(A))$ for each IFS $A$ of $X$.
3. $f^{-1}(\text{cl}(B)) \subset \text{cl}(f^{-1}(B))$ for each IFS $B$ of $Y$. 

71
Proof. By Theorem 3.8, it suffices to show that (2) is equivalent to (3). Let $B$ be any IFS of $Y$. Then $f^{-1}(B)$ is an IFS of $X$. Since $f$ is onto,

$$\text{cl}(B) = \text{cl}(ff^{-1}(B)) \subseteq f(\text{cl}(f^{-1}(B))).$$

So $f^{-1}(\text{cl}(B)) \subseteq f^{-1}f(\text{cl}(f^{-1}(B)))$. Since $f$ is one-to-one,

$$f^{-1}(\text{cl}(B)) \subseteq f^{-1}f(\text{cl}(f^{-1}(B))) = \text{cl}(f^{-1}(B)).$$

Conversely, let $A$ be any IFS of $X$. Then $f(A)$ is an IFS of $Y$. Since $f$ is one-to-one,

$$f^{-1}(\text{cl}(f(A))) \subseteq \text{cl}(f^{-1}f(A)) = \text{cl}(A).$$

So $ff^{-1}(\text{cl}(f(A))) \subseteq f(\text{cl}(A))$. Since $f$ is onto,

$$\text{cl}(f(A)) = ff^{-1}(\text{cl}(f(A))) \subseteq f(\text{cl}(A)).$$

Hence the theorem follows. $\square$

From Theorem 3.6, Theorem 3.7 and Theorem 3.11 we have the following result.

**Theorem 3.12.** Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a bijection. Then the following statements are equivalent:

1. $f$ is a homeomorphism.
2. $f$ is continuous and closed.
3. $f(\text{cl}(A)) = \text{cl}(f(A))$ for each IFS $A$ of $X$.
4. $\text{cl}(f^{-1}(B)) = f^{-1}(\text{cl}(B))$ for each IFS $B$ of $Y$.
5. $f^{-1}(\text{int}(B)) = \text{int}(f^{-1}(B))$ for each IFS $B$ of $Y$.
6. $\text{int}(f(A)) = f(\text{int}(A))$ for each IFS $A$ of $X$.

4. The Category of Intuitionistic Fuzzy Topological Spaces

Let $\mathbf{CFT}$ be the category of all Chang's fuzzy topological spaces and fuzzy continuous maps and $\mathbf{IFT}$ the category of all IFTSs and continuous maps.
The category of intuitionistic fuzzy topological spaces

Recall ([3]) that there are functors \( G_1, G_2 : \text{IFT} \to \text{C Ft} \) defined by

\[
G_1(X, T) = (X, G_1(T)) \quad \text{and} \quad G_1(f) = f,
\]

where \( G_1(T) = \{ \mu_A \mid A = (\mu_A, \gamma_A) \in T \} \),

\[
G_2(X, T) = (X, G_2(T)) \quad \text{and} \quad G_2(f) = f,
\]

where \( G_2(T) = \{ 1 - \gamma_A \mid A = (\mu_A, \gamma_A) \in T \} \).

**Theorem 4.1.** Define \( F_1 : \text{C Ft} \to \text{IFT} \) by

\[
F_1(X, T) = (X, F_1(T)) \quad \text{and} \quad F_1(f) = f,
\]

where \( F_1(T) = \{ A = (\mu_A, \gamma_A) \mid \mu_A \in T, \mu_A + \gamma_A \leq 1 \} \). Then \( F_1 \) is a functor.

**Proof.** First, we show that \( F_1(T) \) is an IFT. Clearly \( 0_\sim, 1_\sim \in F_1(T) \). Let \( A_1 = (\mu_{A_1}, \gamma_{A_1}), A_2 = (\mu_{A_2}, \gamma_{A_2}) \) in \( F_1(T) \). Then \( \mu_{A_1}, \mu_{A_2} \in T \) and \( \mu_{A_1} + \gamma_{A_1} \leq 1, \mu_{A_2} + \gamma_{A_2} \leq 1 \). So \( \mu_{A_1} \leq 1 - \gamma_{A_1} \) and \( \mu_{A_2} \leq 1 - \gamma_{A_2} \). Thus \( \mu_{A_1} \wedge \mu_{A_2} \leq (1 - \gamma_{A_1}) \wedge (1 - \gamma_{A_2}) = 1 - (\gamma_{A_1} \vee \gamma_{A_2}) \). Hence \( \mu_{A_1} \wedge \mu_{A_2} \in T \) and \( (\mu_{A_1} \wedge \mu_{A_2}) + (\gamma_{A_1} \vee \gamma_{A_2}) \leq 1 \). Thus

\[
A_1 \cap A_2 = (\mu_{A_1} \wedge \mu_{A_2}, \gamma_{A_1} \vee \gamma_{A_2}) \in F_1(T).
\]

Let \( A_i = (\mu_{A_i}, \gamma_{A_i}) \in F_1(T) \) for all \( i \in \Gamma \). Then for each \( i \in \Gamma, \mu_{A_i} \in T \) and \( \mu_{A_i} + \gamma_{A_i} \leq 1 \). So \( \bigvee \mu_{A_i} \in T \) and \( (\bigvee \mu_{A_i}) + (\bigwedge \gamma_{A_i}) \leq 1 \). Hence

\[
\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \gamma_{A_i}) \in F_1(T).
\]

Therefore \( (X, F_1(T)) \) is an IFTS. Next, we show that if \( f : (X, T) \to (Y, U) \) is fuzzy continuous then \( f : (X, F_1(T)) \to (Y, F_1(U)) \) is continuous. Let \( B = (\mu_B, \gamma_B) \in F_1(U) \). Then \( \mu_B \in U \) and \( \mu_B + \gamma_B \leq 1 \). Since \( f : (X, T) \to (Y, U) \) is fuzzy continuous, \( f^{-1}(\mu_B) \in T \). Since \( \mu_B + \gamma_B \leq 1, f^{-1}(\mu_B)(x) + f^{-1}(\gamma_B)(x) = \mu_B(f(x)) + \gamma_B(f(x)) \leq 1 \) for any \( x \in X \). So \( f^{-1}(\mu_B) + f^{-1}(\gamma_B) \leq 1 \) and hence

\[
f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \in F_1(T).
\]

Therefore \( f : (X, F_1(T)) \to (Y, F_1(U)) \) is continuous. In all, \( F_1 \) is a functor. \( \square \)
THEOREM 4.2. The functor $F_1 : \text{CFt} \to \text{IFT}$ is a left adjoint of the functor $G_1 : \text{IFT} \to \text{CFt}$.

Proof. For any $(X, T)$ in $\text{CFt}$, $1_X : (X, T) \to G_1 F_1 (X, T) = (X, T)$ is a fuzzy continuous map. Consider $(Y, U) \in \text{IFT}$ and a fuzzy continuous map $f : (X, T) \to G_1 (Y, U)$. In order to show that $f : F_1 (X, T) = (X, F_1 (T)) \to (Y, U)$ is a continuous map, let $B = (\mu_B, \gamma_B) \in U$. Then $\mu_B \in G_1 (U)$. Since $f : (X, T) \to G_1 (Y, U) = (Y, G_1 (U))$ is fuzzy continuous, $f^{-1} (\mu_B) \in T$. Since $\mu_B + \gamma_B \leq 1$, $f^{-1} (\mu_B) + f^{-1} (\gamma_B) \leq 1$ and hence $f^{-1} (B) = (f^{-1} (\mu_B), f^{-1} (\gamma_B)) \in F_1 (T)$. Thus $f : F_1 (X, T) \to (Y, U)$ is continuous. Therefore $1_X$ is a $G_1$-universal map for $(X, T)$ in $\text{CFt}$. \hfill \Box

THEOREM 4.3. Define $F_2 : \text{CFt} \to \text{IFT}$ by

$$F_2 (X, T) = (X, F_2 (T)) \quad \text{and} \quad F_2 (f) = f,$$

where $F_2 (T) = \{ A = (\lambda_A, \theta_A) \mid 1 - \theta_A \in T, \lambda_A + \theta_A \leq 1 \}$. Then $F_2$ is a functor.

Proof. First, we show that $F_2 (T)$ is an IFT. Clearly $0, 1 \in F_2 (T)$. Let $A_1 = (\lambda_{A_1}, \theta_{A_1}), A_2 = (\lambda_{A_2}, \theta_{A_2})$ in $F_2 (T)$. Then $1 - \theta_{A_1}, 1 - \theta_{A_2} \in T$ and $\lambda_{A_1} + \theta_{A_1} \leq 1, \lambda_{A_2} + \theta_{A_2} \leq 1$. So $1 - (\theta_{A_1} \lor \theta_{A_2}) = (1 - \theta_{A_1}) \land (1 - \theta_{A_2}) \in T$ and $(\lambda_{A_1} \land \lambda_{A_2}) + (\theta_{A_1} \lor \theta_{A_2}) \leq 1$. Thus

$$A_1 \cap A_2 = (\lambda_{A_1} \land \lambda_{A_2}, \theta_{A_1} \lor \theta_{A_2}) \in F_2 (T).$$

Let $A_i = (\lambda_{A_i}, \theta_{A_i}) \in F_1 (T)$ for all $i \in \Gamma$. Then for each $i \in \Gamma$, $1 - \theta_{A_i} \in T$ and $\lambda_{A_i} + \theta_{A_i} \leq 1$. So $1 - \lor \theta_{A_i} = \lor (1 - \theta_{A_i}) \in T$ and $(\lor \lambda_{A_i}) + (\lor \theta_{A_i}) \leq 1$. Hence

$$\bigcup A_i = (\lor \lambda_{A_i}, \lor \theta_{A_i}) \in F_2 (T).$$

Therefore $(X, F_2 (T))$ is an IFTS. Next, we show that if $f : (X, T) \to (Y, U)$ is fuzzy continuous then $f : (X, F_2 (T)) \to (Y, F_2 (U))$ is continuous. Let $B = (\lambda_B, \theta_B) \in F_2 (U)$. Then $1 - \theta_B \in U$ and $\lambda_B + \theta_B \leq 1$. Since $f : (X, T) \to (Y, U)$ is fuzzy continuous, $1 - f^{-1} (\theta_B) = f^{-1} (1 - \theta_B) \in T$. Since $\lambda_B + \theta_B \leq 1$, $f^{-1} (\lambda_B) (x) + f^{-1} (\theta_B) (x) =$
The category of intuitionistic fuzzy topological spaces

\( \lambda_B(f(x)) + \theta_B(f(x)) \leq 1 \) for any \( x \in X \). So \( f^{-1}(\lambda_B) + f^{-1}(\theta_B) \leq 1 \)
and hence

\[ f^{-1}(B) = (f^{-1}(\lambda_B), f^{-1}(\theta_B)) \in F_2(T). \]

Therefore \( f : (X, F_2(T)) \to (Y, F_2(U)) \) is continuous. In all, \( F_2 \) is a functor. \( \square \)

**Theorem 4.4.** The functor \( F_2 : CFt \to IFt \) is a left adjoint of the functor \( G_2 : IFt \to CFt \).

**Proof.** For any \( (X, T) \) in \( CFt \), \( 1_X : (X, T) \to G_2F_2(X, T) = (X, T) \)
is a fuzzy continuous map. Consider \( (Y, U) \in IFt \) and a fuzzy continuous map \( f : (X, T) \to G_2(Y, U) \). We will show that \( f : F_2(X, T) = (X, F_2(T)) \to (Y, U) \) is a continuous map. Let \( B = (\mu_B, \gamma_B) \in U \). Then \( 1 - \gamma_B \in G_2(U) \). Since \( f : (X, T) \to G_2(Y, U) = (Y, G_2(U)) \) is fuzzy continuous, \( 1 - f^{-1}(\gamma_B) = f^{-1}(1 - \gamma_B) \in T \). Since \( \mu_B + \gamma_B \leq 1 \),
\( f^{-1}(\mu_B) + f^{-1}(\gamma_B) \leq 1 \) and hence \( f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \in F_2(T) \). Thus \( f : F_2(X, T) \to (Y, U) \) is continuous. Therefore \( 1_X \) is a \( G_2 \)-universal map for \( (X, T) \) in \( CFt \). \( \square \)

Let \( (X, T) \) be a Chang’s fuzzy topological space. Then \( T_c = \{ A = (\mu_A, 1 - \mu_A) \mid \mu_A \in T \} \) is an IFT. In this case, we call \( T_c \) a c-intuitionistic fuzzy topology and \( (X, T_c) \) a c-intuitionistic fuzzy topological space. Also \( c-IFt \) denote the category of all c-intuitionistic fuzzy topological spaces and continuous maps.

**Theorem 4.5.** Two categories \( CFt \) and \( c-IFt \) are isomorphic.

**Proof.** Defined \( F : CFt \to c-IFt \) by

\[ F(X, T) = (X, T_c) \quad \text{and} \quad F(f) = f. \]

Consider the restriction \( G_1 : c-IFt \to CFt \) of the functor \( G_1 \) in the beginning of this section. Then \( F \) and \( G_1 \) are functors. Clearly \( G_1F(X, T) = G_1(X, T_c) = (X, G_1(T_c)) = (X, T) \). In order to show that \( FG_1(X, T) = (X, T) \), let \( (X, T) \in c-IFt \) and \( A = (\mu_A, \gamma_A) \in T \).
Then \( \gamma_A = 1 - \mu_A \). So \( \mu_A \in G_1(T) \) and hence \( A = (\mu_A, \gamma_A) = (\mu_A, 1 - \mu_A) \in (G_1(T))_c \). Thus \( FG_1(X, T) = F(X, G_1(T)) = (X, (G_1(T))_c) = (X, T) \). \( \square \)

75
THEOREM 4.6. The category c-IFt is a bireflective full subcategory of IFt.

Proof. Clearly c-IFt is a full subcategory of IFt. Take any \((X, T)\) in IFt. Let \(T^* = \{A \in T \mid A = (\mu_A, 1 - \mu_A)\}\). Then \((X, T^*) \in c\text{-IFt}\) and \(1_X : (X, T) \to (X, T^*)\) is a continuous map. Consider \((Y, U) \in c\text{-IFt}\) and a continuous map \(f : (X, T) \to (Y, U)\). We need only to check that \(f : (X, T^*) \to (Y, U)\) is a continuous map. Let \(B \in U\). Since \((Y, U)\) is a c-intuitionistic fuzzy topological space, \(B = (\mu_B, 1 - \mu_B)\). Since \(f : (X, T) \to (Y, U)\) is continuous, \(f^{-1}(B) \in T\). Also,

\[
f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(1 - \mu_B)) = (f^{-1}(\mu_B), 1 - f^{-1}(\mu_B)).
\]

Thus \(f^{-1}(B) \in T^*\). Hence \(f : (X, T^*) \to (Y, U)\) is a continuous map. \(\Box\)

COROLLARY 4.7. The category CFt is a bireflective full subcategory of IFt.

References


Seok Jong Lee, Department of Mathematics, Chungbuk National University, Chongju 361-763, Korea
E-mail: sjlee@cucc.chungbuk.ac.kr

Eun Pyo Lee, Department of Mathematics, Seonam University, Namwon 590-170, Korea
E-mail: eplee@tiger.seonam.ac.kr