A PRODUCT FORMULA FOR LOCALIZATION OPERATORS

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Abstract. The product of two localization operators with symbols \( F \) and \( G \) in some subspace of \( L^2(\mathbb{C}^n) \) is shown to be a localization operator with symbol in \( L^2(\mathbb{C}^n) \) and a formula for the symbol of the product in terms of \( F \) and \( G \) is given.

1. Weyl Transforms and Localization Operators

Let \( \sigma \in L^2(\mathbb{R}^{2n}) \). Then the Weyl transform \( W_\sigma \) associated to \( \sigma \) is the bounded linear operator from \( L^2(\mathbb{R}^n) \) into \( L^2(\mathbb{R}^n) \) given by

\[
\langle W_\sigma f, g \rangle = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi
\]

for all functions \( f \) and \( g \) in \( L^2(\mathbb{R}^n) \), where \( \langle , \rangle \) is the inner product in \( L^2(\mathbb{R}^n) \) and \( W(f, g) \) is the Wigner transform of \( f \) and \( g \) given by

\[
W(f, g)(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot p} f \left( x + \frac{p}{2} \right) \overline{g \left( x - \frac{p}{2} \right)} dp
\]

for all \( x \) and \( \xi \) in \( \mathbb{R}^n \).

In order to give an account of a formula, in the paper [4] by Grossmann, Loupia and Stein, for the product of two Weyl transforms with symbols in \( L^2(\mathbb{R}^{2n}) \), we need the notion of a twisted convolution. To this end, we identify any point \( (q, p) \) in \( \mathbb{R}^{2n} \) with the point \( z = q + ip \) in \( \mathbb{C}^n \), and define the symplectic form \( \{ , \} \) on \( \mathbb{C}^n \) by

\[
[z, w] = 2\text{Im}(z \cdot \overline{w}), \quad z, w \in \mathbb{C}^n,
\]
where

\[ z = (z_1, z_2, \cdots, z_n), \]

\[ w = (w_1, w_2, \cdots, w_n), \]

and

\[ z \cdot \bar{w} = \sum_{j=1}^{n} z_j \bar{w}_j. \]

Now, for any fixed real number \( \lambda \), we define the twisted convolution \( f \ast \lambda g \) of two measurable functions \( f \) and \( g \) on \( \mathbb{C}^n \) by

\[ (f \ast \lambda g)(z) = \int_{\mathbb{C}^n} f(z - w)g(w)e^{i\lambda \bar{w} \cdot w}dw, \quad z \in \mathbb{C}^n, \]

where \( dw \) is the Lebesgue measure on \( \mathbb{C}^n \), provided that the integral exists. The following theorem can be found in the paper [4] by Grossmann, Loupias and Stein.

**Theorem 1.1.** Let \( \sigma \) and \( \tau \) be functions in \( L^2(\mathbb{C}^n) \). Then \( W_\sigma W_\tau = W_{\hat{\omega}} \), where

\[ \hat{\omega} = (2\pi)^{-n}(\hat{\sigma} \ast \hat{\tau}). \]

**Remark 1.2.** It should be pointed out immediately that the Fourier transform \( \hat{f} \) of a function \( f \) in \( L^2(\mathbb{C}^n) \) is defined by

\[ \hat{f}(\zeta) = (2\pi)^{-n} \lim_{R \to \infty} \int_{|z| \leq R} e^{-iz \cdot \zeta} f(z)dz, \quad \zeta \in \mathbb{C}^n, \]

where the limit is understood to be the limit in \( L^2(\mathbb{C}^n) \) as \( R \to \infty \).

Let \( \varphi \) be the function on \( \mathbb{R}^n \) defined by

\[ \varphi(x) = \pi^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n. \]

For \( z = q + ip \) in \( \mathbb{C}^n \), we define the function \( \varphi_z \) on \( \mathbb{R}^n \) by

\[ \varphi_z(x) = e^{ipx} \varphi(x - q), \quad x \in \mathbb{R}^n. \]

Then, as an abridged version of Theorem 15.4 in the book [5] by Wong, we have the following theorem.
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**Theorem 1.3.** Let $F \in L^2(\mathbb{C}^n)$. Then there exists a unique bounded linear operator $L_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $\langle L_F f, g \rangle$, for all $f$ and $g$ in $L^2(\mathbb{R}^n)$, is given by

$$\langle L_F f, g \rangle = (2\pi)^{-n} \int_{\mathbb{C}^n} F(z) \langle f, \varphi_z \rangle \langle \varphi_z, g \rangle dz$$

for all simple functions $F$ on $\mathbb{C}^n$ for which the Lebesgue measure of the set $\{ z \in \mathbb{C}^n : F(z) \neq 0 \}$ is finite.

For any $F$ in $L^2(\mathbb{C}^n)$, we call the bounded linear operator $L_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the localization operator associated to the symbol $F$. The significance of localization operators in the study of signal analysis can be found in the papers [1, 2] and Section 2.7 of the book [3] by Daubechies.

The connection between Weyl transforms and localization operators is illuminated by the following theorem, i.e., Theorem 17.1 in the book [5] by Wong.

**Theorem 1.4.** Let $\Lambda$ be the function on $\mathbb{C}^n$ defined by

$$\Lambda(z) = \pi^{-n} e^{-|z|^2}, \quad z \in \mathbb{C}^n.$$ 

Then, for all $F$ in $L^2(\mathbb{C}^n)$,

$$L_F = W_{F * \Lambda},$$

where $F * \Lambda$ is the convolution of $F$ and $\Lambda$ given by

$$(F * \Lambda)(z) = \int_{\mathbb{C}^n} F(z - w) \Lambda(w) dw, \quad z \in \mathbb{C}^n.$$ 

The aim of this paper is to study the product of two localization operators with symbols in $L^2(\mathbb{C}^n)$. In Section 2, we show that the product of two localization operators with symbols in $L^2(\mathbb{C}^n)$ is, in general, not a localization operator with symbol in $L^2(\mathbb{C}^n)$. In Section 3, we prove that the product of two localization operators with symbols in some subspace of $L^2(\mathbb{C}^n)$ is indeed a localization operator with symbol in $L^2(\mathbb{C}^n)$, and we give a formula for the symbol of the product in terms of a new convolution defined in Section 2.
2. A Necessary Condition

For any fixed real number \( \lambda \), we define the \( \lambda \)-convolution \( f \ast^\lambda g \) of two measurable functions \( f \) and \( g \) on \( \mathbb{C}^n \) by

\[
(f \ast^\lambda g)(z) = \int_{\mathbb{C}^n} f(z - \omega)g(\omega)e^{i\lambda(z \cdot \omega - |\omega|^2)}d\omega, \quad z \in \mathbb{C}^n,
\]

provided that the integral exists. Then we have the following result.

\text{THEOREM 2.1.} Let \( L_F \) and \( L_G \) be localization operators with symbols \( F \) and \( G \), respectively, in \( L^2(\mathbb{C}^n) \). If there exists a function \( H \) in \( L^2(\mathbb{C}^n) \) such that

\[
L_F L_G = L_H,
\]

then \( \hat{H} = (2\pi)^{-n}(\hat{F} \ast^\frac{1}{2} \hat{G}) \).

\text{Proof.} By Theorem 1.4,

\[
(2.1) \quad W_{H \ast \Lambda} = W_{F \ast \Lambda}W_{G \ast \Lambda}.
\]

Since

\[
(2.2) \quad \hat{\Lambda}(\zeta) = (2\pi)^{-n}e^{-\frac{K|\zeta|^2}{4}}, \quad \zeta \in \mathbb{C}^n,
\]

it follows from Theorem 1.1, (2.1), (2.2), the definition of a twisted convolution and the fact that \( (F \ast \Lambda) = (2\pi)^{n}\hat{F} \hat{\Lambda} \) for all \( F \) in \( L^2(\mathbb{C}^n) \) that for all \( \zeta \) in \( \mathbb{C}^n \),

\[
(2.3) \quad \hat{H}(\zeta) e^{-\frac{|K\zeta|^2}{4}} = (2\pi)^{-n}\{(\hat{F} \ast \hat{\Lambda}) \ast^\frac{1}{2} (\hat{G} \ast \hat{\Lambda})\}(\zeta) = (2\pi)^{n}\{(\hat{F} \hat{\Lambda}) \ast^\frac{1}{2} (\hat{G} \hat{\Lambda})\}(\zeta) = (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{F}(\zeta - \omega)e^{-\frac{1}{2}K|\omega|^2}\hat{G}(\omega)e^{-\frac{1}{2}|\omega|^2}e^{i\langle \zeta, \omega \rangle}d\omega.
\]

So, by (2.3),

\[
(2.4) \quad \hat{H}(\zeta) = (2\pi)^{n}\int_{\mathbb{C}^n} \hat{F}(\zeta - \omega)\hat{G}(\omega)e^{i\langle |K|^{2} - K \cdot \omega - |\omega|^2 + i\langle \zeta, \omega \rangle \rangle}d\omega, \quad \zeta \in \mathbb{C}^n.
\]
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Now, for all $\zeta$ and $\omega$ in $\mathbb{C}^n$,

\[
|\zeta|^2 - |\zeta - \omega|^2 - |\omega|^2 + i[\zeta, \omega] = 2|\zeta \cdot \omega| - 2|\omega|^2.
\]

(2.5)

Therefore, by (2.4) and (2.5), we get, for all $\zeta$ in $\mathbb{C}^n$,

\[
\hat{H}(\zeta) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{F}(\zeta - \omega) \hat{G}(\omega) e^{\frac{i}{2}(\zeta \cdot \omega - |\omega|^2)} d\omega
\]

\[
= (2\pi)^{-n}(\hat{F} \ast \hat{G})(\zeta),
\]

and the proof is complete. \qed

Remark 2.2. In general, for $F$ and $G$ in $L^2(\mathbb{C}^n)$, it is not true that $\hat{F} \ast \hat{G} \in L^2(\mathbb{C}^n)$. So, the product of two localization operators with symbols in $L^2(\mathbb{C}^n)$ need not be a localization operator with symbol in $L^2(\mathbb{C}^n)$. This can best be seen from the following example.

Example 2.3. Let $W$ be the subset of $\mathbb{R} \times \mathbb{R}$ defined by

\[
W = \{(q, p) \in \mathbb{R} \times \mathbb{R} : 0 \leq q, p \leq 1\}.
\]

We identify points $\omega$ and $\zeta$ in $\mathbb{C}$ with points $(q, p)$ and $(x, \xi)$ in $\mathbb{R} \times \mathbb{R}$ respectively. Let $F \in L^2(\mathbb{C})$ be defined by

\[
\hat{F}(q, p) = e^{-\frac{1}{2}q^2} \chi(p), \quad q, p \in \mathbb{R},
\]

where $\chi$ is the characteristic function on $[-1, 1]$, and let $G \in L^2(\mathbb{C})$ be defined by

\[
\hat{G}(\omega) = \begin{cases} e^{\frac{1}{2}|\omega|^2}, & \omega \in W, \\ 0, & \omega \notin W. \end{cases}
\]

Then, by (2.6)–(2.8),

\[
(\hat{F} \ast \hat{G})(\zeta)
\]

\[
= \int_W \hat{F}(\zeta - \omega) \hat{G}(\omega) e^{-\frac{1}{2}|\omega|^2} e^{\frac{i}{2}\zeta \cdot \omega} d\omega
\]

\[
= \int_0^1 \int_0^1 e^{-\frac{1}{2}z^2} \chi(z - p) e^{\frac{i}{2}(z^2 + p\xi)} e^{\frac{i}{2}z^2} dq dp
\]

\[
= \left( \int_0^1 e^{-\frac{1}{2}z^2} e^{\frac{i}{2}z^2} dq \right) \left( \int_0^1 \chi(z - p) e^{\frac{i}{2}z^2} dp \right)
\]

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for all $\zeta$ in $\mathbb{C}$. But for $x > 1$ and $0 < \xi < 1$, we get from (2.9)
\[
(\hat{F} \ast_{\frac{1}{2}} \hat{G})(\zeta) = \left( \int_0^1 e^{-\frac{1}{2}x} e^{\frac{i}{2} x^{(1+2\zeta)} dq} \right) \left( \int_0^1 e^{-\frac{1}{2}ix} dp \right)
= \frac{4 e^{-\frac{1}{2}x}}{1 + 2\zeta} \left( e^{\frac{1}{2} x^{(1+2\zeta)}} - 1 \right) \frac{2i}{\zeta} \left( e^{-\frac{1}{2}ix} - 1 \right)
= \frac{4 e^{i x}}{1 + 2\zeta} \left( e^{\frac{1}{2} x^{(1+2\zeta)}} - e^{-\frac{1}{2}ix} \right) \frac{2i}{\zeta} \left( e^{\frac{1}{2}ix} e^{-\frac{1}{2}ix} - 1 \right)
\]
and hence $\hat{F} \ast_{\frac{1}{2}} \hat{G} \not\in L^2(\mathbb{C})$.

From the proof of Theorem 2.1, we get the following corollary.

**Corollary 2.4.** Let $F$ and $G$ be functions in $L^2(\mathbb{C}^n)$ such that $\hat{F} \ast_{\frac{1}{2}} \hat{G} \in L^2(\mathbb{C}^n)$. Then there exists a function $H$ in $L^2(\mathbb{C}^n)$ such that $\hat{H} = (2\pi)^{-n}(\hat{F} \ast_{\frac{1}{2}} \hat{G})$ and
\[
L_F L_G = L_H.
\]

In view of Remark 2.2 and Example 2.3, it is a natural problem to seek some subspace of $L^2(\mathbb{C}^n)$ such that the product of two localization operators with symbols in the subspace is indeed a localization operator with symbol in $L^2(\mathbb{C}^n)$.

3. A Product Formula

For any nonnegative real number $c$, we denote by $\mathcal{S}_c$ the set of all measurable functions $F$ on $\mathbb{C}^n$ such that
\[
|\hat{F}(\zeta)| \leq e^{-c|\zeta|^2}|f(\zeta)|, \quad \zeta \in \mathbb{C}^n,
\]
for some function $f$ in $L^2(\mathbb{C}^n)$. It is clear that $\mathcal{S}_c$ is a subspace of $L^2(\mathbb{C}^n)$ for all $c \geq 0$. It is also clear that if $c \leq d$, then $\mathcal{S}_d \subseteq \mathcal{S}_c$.

We can now give a formula for the product of two localization operators with symbols in $\mathcal{S}_c$, where $c > \frac{1+\sqrt{5}}{8}$.

**Theorem 3.1.** Let $F$ and $G$ be functions in $\mathcal{S}_c$, where $c > \frac{1+\sqrt{5}}{8}$. Then $L_F L_G = L_H$, where $H \in \bigcap_{0 < d < c'} \mathcal{S}_d$, $c' = c - \frac{1}{4} - \frac{4c^2}{8c+1} > 0$, and
\[
\hat{H} = (2\pi)^{-n}(\hat{F} \ast_{\frac{1}{2}} \hat{G})
\]
Proof. Let $f$ and $g$ be functions in $L^2(\mathbb{C}^n)$ such that

$$|\hat{f}(\zeta)| \leq e^{-c|\zeta|^2} |f(\zeta)|$$

(3.1) and

$$|\hat{g}(\zeta)| \leq e^{-c|\zeta|^2} |g(\zeta)|$$

(3.2) for all $\zeta$ in $\mathbb{C}^n$. Then, by (3.1), (3.2) and the definition of the $\frac{1}{2}$-convolution, we get, for all $\zeta$ in $\mathbb{C}^n$,

$$|((\hat{f} *_{\frac{1}{2}} \hat{g}))(\zeta)|$$

(3.3) $$= \left| \int_{\mathbb{C}^n} \hat{f}(\zeta - \omega) \hat{g}(\omega) e^{\frac{i}{2} |\zeta - \omega|^2} d\omega \right|$$

$$\leq \int_{\mathbb{C}^n} |\hat{f}(\zeta - \omega)||\hat{g}(\omega)| e^{\frac{i}{2} |\zeta - \omega|^2} e^{-\frac{1}{2} |\omega|^2} d\omega$$

$$\leq \int_{\mathbb{C}^n} e^{-c|\zeta - \omega|^2} |f(\zeta - \omega)||g(\omega)| e^{\frac{i}{2} |\zeta|^2 + |\omega|^2} e^{-\frac{1}{2} |\omega|^2} d\omega$$

$$\leq e^{-(c - \frac{1}{2})|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)| e^{2c \text{Re}(\zeta \cdot \bar{\omega})} e^{-(2c + \frac{1}{2})|\omega|^2} d\omega.$$}

But, for any positive number $\varepsilon$, we have

$$2c \text{Re}(\zeta \cdot \bar{\omega}) \leq 2c|\zeta \cdot \bar{\omega}| \leq 2c|\zeta||\omega|$$

(3.4) $$= 2c\sqrt{\varepsilon} |\zeta| \frac{|\omega|}{\sqrt{\varepsilon}}$$

$$\leq c \left( \varepsilon|\zeta|^2 + \frac{1}{\varepsilon} |\omega|^2 \right)$$

for all $\zeta$ and $\omega$ in $\mathbb{C}^n$. So, by (3.3) and (3.4), we have, for all $\zeta$ in $\mathbb{C}^n$,

$$|((\hat{f} *_{\frac{1}{2}} \hat{g}))(\zeta)| \leq e^{-(c - \frac{1}{2} - \varepsilon)|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)| e^{-(2c + \frac{1}{2} - \varepsilon)|\omega|^2} d\omega.$$}

(3.5) Since $c > \frac{1 + \sqrt{5}}{8}$, it follows from (3.5) that for any positive number $\varepsilon$ such that

$$\frac{c}{2c + \frac{1}{4}} < \varepsilon < 1 - \frac{1}{4c},$$

(3.6)
there exists a positive constant \( d_\varepsilon \) such that

\[
(3.7) \quad |(\hat{F} \ast_{\varepsilon} \hat{G})(\zeta)| \leq e^{-c_\varepsilon|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)|e^{-d_\varepsilon|\omega|^2}d\omega, \quad \zeta \in \mathbb{C}^n,
\]

where

\[
(3.8) \quad c_\varepsilon = c - \frac{1}{4} - \varepsilon c.
\]

Since, for any \( \varepsilon \) satisfying (3.6), the function \(|g|e^{-d_\varepsilon|\cdot|^2}\) is in \(L^1(\mathbb{C}^n)\), it follows from Young's inequality that the function \(h_\varepsilon\) on \(\mathbb{C}^n\) defined by

\[
(3.9) \quad h_\varepsilon(\zeta) = \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)|e^{-d_\varepsilon|\omega|^2}d\omega, \quad \zeta \in \mathbb{C}^n,
\]

is in \(L^2(\mathbb{C}^n)\). Thus, by (3.7) and (3.9),

\[
(3.10) \quad |(\hat{F} \ast_{\varepsilon} \hat{G})(\zeta)| \leq e^{-c_\varepsilon|\zeta|^2}h_\varepsilon(\zeta), \quad \zeta \in \mathbb{C}^n,
\]

for any \( \varepsilon \) satisfying (3.6). Now, by Plancherel's theorem, let \(H \in L^2(\mathbb{C}^n)\) be such that

\[
(3.11) \quad \hat{H} = (2\pi)^{-n}(\hat{F} \ast_{\varepsilon} \hat{G}).
\]

Then, by (3.10) and (3.11), \(H \in \mathcal{S}_c\), and hence, by (3.6) and (3.8), \(H \in \bigcap_{\varepsilon < d} \mathcal{S}_d\). That \(L_F L_G = L_H\) is then of course a consequence of (3.11) and Corollary 2.4. \(\Box\)

References


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