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A PRODUCT FORMULA FOR LOCALIZATION OPERATORS

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ABSTRACT. The product of two localization operators with symbols F and G in some subspace of $L^2(\mathbb{C}^n)$ is shown to be a localization operator with symbol in $L^2(\mathbb{C}^n)$ and a formula for the symbol of the product in terms of F and G is given.

1. Weyl Transforms and Localization Operators

Let $\sigma \in L^2(\mathbb{R}^{2n})$. Then the Weyl transform W_σ associated to σ is the bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ given by

$$\langle W_\sigma f, g\rangle = (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi) W(f,g)(x,\xi)\,dx d\xi$$

for all functions f and g in $L^2(\mathbb{R}^n)$, where $\langle\, ,\, \rangle$ is the inner product in $L^2(\mathbb{R}^n)$ and $W(f,g)$ is the Wigner transform of f and g given by

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}\,dp$$

for all x and ξ in $\mathbb{R}^n$.

In order to give an account of a formula, in the paper [4] by Grossmann, Loupias and Stein, for the product of two Weyl transforms with symbols in $L^2(\mathbb{R}^{2n})$, we need the notion of a twisted convolution. To this end, we identify any point (q,p) in $\mathbb{R}^{2n}$ with the point $z = q + ip$ in $\mathbb{C}^n$, and define the symplectic form $[\, ,\,]$ on $\mathbb{C}^n$ by

$$[z,w] = 2\mathrm{Im}(z\cdot\overline{w}), \quad z,\, w \in \mathbb{C}^n,$$

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where

$$z = (z_1, z_2, \cdots, z_n),$$

$$w = (w_1, w_2, \cdots, w_n),$$

and

$$z \cdot \bar{w} = \sum_{j=1}^{n} z_j \bar{w}_j.$$

Now, for any fixed real number λ, we define the twisted convolution $f *_\lambda g$ of two measurable functions f and g on $\mathbb{C}^n$ by

$$(f *_\lambda g)(z) = \int_{\mathbb{C}^n} f(z-w) g(w) e^{i\lambda[z,w]} dw, \quad z \in \mathbb{C}^n,$$

where dw is the Lebesgue measure on $\mathbb{C}^n$, provided that the integral exists. The following theorem can be found in the paper [4] by Grossmann, Loupias and Stein.

THEOREM 1.1. *Let σ and τ be functions in $L^2(\mathbb{C}^n)$. Then $W_\sigma W_\tau = W_\omega$, where*

$$\hat{\omega} = (2\pi)^{-n} (\hat{\sigma} *_{\frac{1}{4}} \hat{\tau}).$$

REMARK 1.2. It should be pointed out immediately that the Fourier transform $\hat{f}$ of a function f in $L^2(\mathbb{C}^n)$ is defined by

$$\hat{f}(\zeta) = (2\pi)^{-n} \lim_{R \to \infty} \int_{|z| \le R} e^{-iz \cdot \zeta} f(z) dz, \quad \zeta \in \mathbb{C}^n,$$

where the limit is understood to be the limit in $L^2(\mathbb{C}^n)$ as $R \to \infty$.

Let φ be the function on $\mathbb{R}^n$ defined by

$$\varphi(x) = \pi^{-\frac{n}{4}} e^{-\frac{|x|^2}{2}}, \quad x \in \mathbb{R}^n.$$

For $z = q + ip$ in $\mathbb{C}^n$, we define the function φ_z on $\mathbb{R}^n$ by

$$\varphi_z(x) = e^{ip \cdot x} \varphi(x - q), \quad x \in \mathbb{R}^n.$$

Then, as an abridged version of Theorem 15.4 in the book [5] by Wong, we have the following theorem.

THEOREM 1.3. *Let $F \in L^2(\mathbb{C}^n)$. Then there exists a unique bounded linear operator $L_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $\langle L_F f, g\rangle$, for all f and g in $L^2(\mathbb{R}^n)$, is given by*

$$\langle L_F f, g\rangle = (2\pi)^{-n} \int_{\mathbb{C}^n} F(z)\langle f, \varphi_z\rangle\langle \varphi_z, g\rangle dz$$

for all simple functions F on $\mathbb{C}^n$ for which the Lebesgue measure of the set $\{z \in \mathbb{C}^n : F(z) \neq 0\}$ is finite.

For any F in $L^2(\mathbb{C}^n)$, we call the bounded linear operator $L_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the localization operator associated to the symbol F. The significance of localization operators in the study of signal analysis can be found in the papers [1, 2] and Section 2.7 of the book [3] by Daubechies.

The connection between Weyl transforms and localization operators is illuminated by the following theorem, i.e., Theorem 17.1 in the book [5] by Wong.

THEOREM 1.4. *Let Λ be the function on $\mathbb{C}^n$ defined by*

$$\Lambda(z) = \pi^{-n} e^{-|z|^2}, \quad z \in \mathbb{C}^n.$$

Then, for all F in $L^2(\mathbb{C}^n)$,

$$L_F = W_{F*\Lambda},$$

where $F * \Lambda$ is the convolution of F and Λ given by

$$(F * \Lambda)(z) = \int_{\mathbb{C}^n} F(z - w)\Lambda(w) dw, \quad z \in \mathbb{C}^n.$$

The aim of this paper is to study the product of two localization operators with symbols in $L^2(\mathbb{C}^n)$. In Section 2, we show that the product of two localization operators with symbols in $L^2(\mathbb{C}^n)$ is, in general, not a localization operator with symbol in $L^2(\mathbb{C}^n)$. In Section 3, we prove that the product of two localization operators with symbols in some subspace of $L^2(\mathbb{C}^n)$ is indeed a localization operator with symbol in $L^2(\mathbb{C}^n)$, and we give a formula for the symbol of the product in terms of a new convolution defined in Section 2.

2. A Necessary Condition

For any fixed real number λ, we define the λ-convolution $f *^{\lambda} g$ of two measurable functions f and g on $\mathbb{C}^n$ by

$$(f *^{\lambda} g)(z) = \int_{\mathbb{C}^n} f(z-\omega) g(\omega) e^{\lambda(z\cdot\bar{\omega} - |\omega|^2)} d\omega, \quad z \in \mathbb{C}^n,$$

provided that the integral exists. Then we have the following result.

THEOREM 2.1. *Let L_F and L_G be localization operators with symbols F and G, respectively, in $L^2(\mathbb{C}^n)$. If there exists a function H in $L^2(\mathbb{C}^n)$ such that*

$$L_F L_G = L_H,$$

then $\hat{H} = (2\pi)^{-n}(\hat{F} *^{\frac{1}{2}} \hat{G})$.

Proof. By Theorem 1.4,

$$W_{H*\Lambda} = W_{F*\Lambda} W_{G*\Lambda}\,. \tag{2.1}$$

Since

$$\hat{\Lambda}(\zeta) = (2\pi)^{-n} e^{-\frac{|\zeta|^2}{4}}, \quad \zeta \in \mathbb{C}^n, \tag{2.2}$$

it follows from Theorem 1.1, (2.1), (2.2), the definition of a twisted convolution and the fact that $(F * \Lambda)^\wedge = (2\pi)^n \hat{F}\hat{\Lambda}$ for all F in $L^2(\mathbb{C}^n)$ that for all ζ in $\mathbb{C}^n$,

$$\begin{aligned}
\hat{H}(\zeta) e^{-\frac{|\zeta|^2}{4}} &= (2\pi)^{-n}((\widehat{F*\Lambda}) *_{\frac{1}{4}} (\widehat{G*\Lambda}))(\zeta) \\
&= (2\pi)^{n}\{(\hat{F}\hat{\Lambda}) *_{\frac{1}{4}} (\hat{G}\hat{\Lambda})\}(\zeta) \\
&= (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{F}(\zeta-\omega) e^{-\frac{1}{4}|\zeta-\omega|^2} \hat{G}(\omega) e^{-\frac{1}{4}|\omega|^2} e^{i\frac{1}{4}[\zeta,\omega]} d\omega \\
&= (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{F}(\zeta-\omega)\hat{G}(\omega) e^{\frac{1}{4}\{-|\zeta-\omega|^2 - |\omega|^2 + i[\zeta,\omega]\}} d\omega. \qquad (2.3)
\end{aligned}$$

So, by (2.3),

(2.4)

$$\hat{H}(\zeta) = (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{F}(\zeta-\omega)\hat{G}(\omega) e^{\frac{1}{4}\{|\zeta|^2 - |\zeta-\omega|^2 - |\omega|^2 + i[\zeta,\omega]\}} d\omega, \quad \zeta \in \mathbb{C}^n.$$

Now, for all ζ and ω in $\mathbb{C}^n$,

$$\begin{aligned} & |\zeta|^2 - |\zeta-\omega|^2 - |\omega|^2 + i[\zeta,\omega] \\ = \; & |\zeta|^2 - |\zeta|^2 + 2\mathrm{Re}(\zeta\cdot\bar{\omega}) - |\omega|^2 - |\omega|^2 + i2\mathrm{Im}(\zeta\cdot\bar{\omega}) \\ = \; & 2(\zeta\cdot\bar{\omega}) - 2|\omega|^2. \end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we get, for all ζ in $\mathbb{C}^n$,

$$\begin{aligned} \hat{H}(\zeta) &= (2\pi)^{-n}\int_{\mathbb{C}^n}\hat{F}(\zeta-\omega)\hat{G}(\omega)e^{\frac{1}{2}(\zeta\cdot\bar{\omega}-|\omega|^2)}d\omega \\ &= (2\pi)^{-n}(\hat{F}*^{\frac{1}{2}}\hat{G})(\zeta), \end{aligned}$$

and the proof is complete. □

REMARK 2.2. In general, for F and G in $L^2(\mathbb{C}^n)$, it is not true that $\hat{F}*^{\frac{1}{2}}\hat{G}\in L^2(\mathbb{C}^n)$. So, the product of two localization operators with symbols in $L^2(\mathbb{C}^n)$ need not be a localization operator with symbol in $L^2(\mathbb{C}^n)$. This can best be seen from the following example.

EXAMPLE 2.3. Let W be the subset of $\mathbb{R}\times\mathbb{R}$ defined by

$$W = \{(q,p)\in\mathbb{R}\times\mathbb{R} : 0\le q,\, p\le 1\}. \tag{2.6}$$

We identify points ω and ζ in $\mathbb{C}$ with points (q,p) and (x,ξ) in $\mathbb{R}\times\mathbb{R}$ respectively. Let $F\in L^2(\mathbb{C})$ be defined by

$$\hat{F}(q,p) = e^{-\frac{1}{4}|q|}\chi(p), \quad q,\, p\in\mathbb{R}, \tag{2.7}$$

where χ is the characteristic function on $[-1,1]$, and let $G\in L^2(\mathbb{C})$ be defined by

$$\hat{G}(\omega) = \begin{cases} e^{\frac{1}{2}|\omega|^2}, & \omega\in W, \\ 0, & \omega\notin W. \end{cases} \tag{2.8}$$

Then, by (2.6)–(2.8),

$$\begin{aligned} & (\hat{F}*^{\frac{1}{2}}\hat{G})(\zeta) \\ = \; & \int_W \hat{F}(\zeta-\omega)\hat{G}(\omega)e^{-\frac{1}{2}|\omega|^2}e^{\frac{1}{2}\zeta\bar{\omega}}d\omega \\ = \; & \int_0^1\int_0^1 e^{-\frac{1}{4}|x-q|}\chi(\xi-p)e^{\frac{1}{2}(qx+p\xi)}e^{\frac{1}{2}i(q\xi-px)}dqdp \\ = \; & \left(\int_0^1 e^{-\frac{1}{4}|x-q|}e^{\frac{1}{2}qx+\frac{1}{2}iq\xi}dq\right)\left(\int_0^1\chi(\xi-p)e^{\frac{1}{2}p\xi-\frac{1}{2}ipx}dp\right) \end{aligned} \tag{2.9}$$

for all ζ in $\mathbb{C}$. But for $x > 1$ and $0 < \xi < 1$, we get from (2.9)

$$\begin{aligned}(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta) &= \left(\int_0^1 e^{-\frac{1}{4}x} e^{\frac{1}{4}q(1+2\zeta)} dq\right) \left(\int_0^1 e^{-\frac{1}{2}ip\zeta} dp\right) \\ &= \frac{4e^{-\frac{1}{4}x}}{1+2\zeta} \left(e^{\frac{1}{4}(1+2\zeta)} - 1\right) \frac{2i}{\zeta} \left(e^{-\frac{1}{2}i\zeta} - 1\right) \\ &= \frac{4e^{\frac{1}{4}x}}{1+2\zeta} \left(e^{\frac{1}{4}+\frac{1}{2}i\xi} - e^{-\frac{1}{2}x}\right) \frac{2i}{\zeta} \left(e^{\frac{1}{2}\xi} e^{-\frac{1}{2}ix} - 1\right)\end{aligned}$$

and hence $\hat{F} *^{\frac{1}{2}} \hat{G} \notin L^2(\mathbb{C})$.

From the proof of Theorem 2.1, we get the following corollary.

COROLLARY 2.4. *Let F and G be functions in $L^2(\mathbb{C}^n)$ such that $\hat{F} *^{\frac{1}{2}} \hat{G} \in L^2(\mathbb{C}^n)$. Then there exists a function H in $L^2(\mathbb{C}^n)$ such that $\hat{H} = (2\pi)^{-n}(\hat{F} *^{\frac{1}{2}} \hat{G})$ and*

$$L_F L_G = L_H.$$

In view of Remark 2.2 and Example 2.3, it is a natural problem to seek some subspace of $L^2(\mathbb{C}^n)$ such that the product of two localization operators with symbols in the subspace is indeed a localization operator with symbol in $L^2(\mathbb{C}^n)$.

3. A Product Formula

For any nonnegative real number c, we denote by $\mathcal{S}_c$ the set of all measurable functions F on $\mathbb{C}^n$ such that

$$|\hat{F}(\zeta)| \le e^{-c|\zeta|^2} |f(\zeta)|, \quad \zeta \in \mathbb{C}^n,$$

for some function f in $L^2(\mathbb{C}^n)$. It is clear that $\mathcal{S}_c$ is a subspace of $L^2(\mathbb{C}^n)$ for all $c \ge 0$. It is also clear that if $c \le d$, then $\mathcal{S}_d \subseteq \mathcal{S}_c$.

We can now give a formula for the product of two localization operators with symbols in $\mathcal{S}_c$, where $c > \frac{1+\sqrt{5}}{8}$.

THEOREM 3.1. *Let F and G be functions in $\mathcal{S}_c$, where $c > \frac{1+\sqrt{5}}{8}$. Then $L_F L_G = L_H$, where $H \in \bigcap_{0<d<c'} \mathcal{S}_d$, $c' = c - \frac{1}{4} - \frac{4c^2}{8c+1} > 0$, and*

$$\hat{H} = (2\pi)^{-n}(\hat{F} *^{\frac{1}{2}} \hat{G})$$

Proof. Let f and g be functions in $L^2(\mathbb{C}^n)$ such that

$$|\hat{F}(\zeta)| \leq e^{-c|\zeta|^2}|f(\zeta)| \tag{3.1}$$

and

$$|\hat{G}(\zeta)| \leq e^{-c|\zeta|^2}|g(\zeta)| \tag{3.2}$$

for all ζ in $\mathbb{C}^n$. Then, by (3.1), (3.2) and the definition of the $\frac{1}{2}$-convolution, we get, for all ζ in $\mathbb{C}^n$,

$$\begin{aligned} & |(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \\ = & \left| \int_{\mathbb{C}^n} \hat{F}(\zeta - \omega)\hat{G}(\omega)\, e^{\frac{1}{2}(\zeta\cdot\bar{\omega} - |\omega|^2)} d\omega \right| \\ \leq & \int_{\mathbb{C}^n} |\hat{F}(\zeta-\omega)||\hat{G}(\omega)|\, e^{\frac{1}{2}|\zeta||\omega|}\, e^{-\frac{1}{2}|\omega|^2} d\omega \\ \leq & \int_{\mathbb{C}^n} e^{-c|\zeta-\omega|^2}|f(\zeta-\omega)|\, e^{-c|\omega|^2}|g(\omega)|\, e^{\frac{1}{4}(|\zeta|^2+|\omega|^2)}\, e^{-\frac{1}{2}|\omega|^2} d\omega \\ \leq & \; e^{-(c-\frac{1}{4})|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta-\omega)||g(\omega)|\, e^{2c\mathrm{Re}(\zeta\cdot\bar{\omega})}\, e^{-(2c+\frac{1}{4})|\omega|^2} d\omega. \end{aligned} \tag{3.3}$$

But, for any positive number ε, we have

$$\begin{aligned} 2c\mathrm{Re}(\zeta\cdot\bar{\omega}) & \leq 2c|\zeta\cdot\bar{\omega}| \leq 2c|\zeta||\omega| \\ & = 2c\sqrt{\varepsilon}\,|\zeta|\frac{|\omega|}{\sqrt{\varepsilon}} \\ & \leq c\left(\varepsilon|\zeta|^2 + \frac{1}{\varepsilon}|\omega|^2\right) \end{aligned} \tag{3.4}$$

for all ζ and ω in $\mathbb{C}^n$. So, by (3.3) and (3.4), we have, for all ζ in $\mathbb{C}^n$,

$$|(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \leq e^{-(c-\frac{1}{4}-c\varepsilon)|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta-\omega)||g(\omega)|\, e^{-(2c+\frac{1}{4}-\frac{c}{\varepsilon})|\omega|^2} d\omega. \tag{3.5}$$

Since $c > \frac{1+\sqrt{5}}{8}$, it follows from (3.5) that for any positive number ε such that

$$\frac{c}{2c+\frac{1}{4}} < \varepsilon < 1 - \frac{1}{4c}, \tag{3.6}$$

there exists a positive constant d_ε such that

$$(3.7) \quad |(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \leq e^{-c_\varepsilon|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)|\, e^{-d_\varepsilon|\omega|^2} d\omega, \quad \zeta \in \mathbb{C}^n,$$

where

$$(3.8) \qquad c_\varepsilon = c - \frac{1}{4} - c\varepsilon.$$

Since, for any ε satisfying (3.6), the function $|g|e^{-d_\varepsilon|\cdot|^2}$ is in $L^1(\mathbb{C}^n)$, it follows from Young's inequality that the function h_ε on $\mathbb{C}^n$ defined by

$$(3.9) \qquad h_\varepsilon(\zeta) = \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)|\, e^{-d_\varepsilon|\omega|^2} d\omega, \quad \zeta \in \mathbb{C}^n,$$

is in $L^2(\mathbb{C}^n)$. Thus, by (3.7) and (3.9),

$$(3.10) \qquad |(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \leq e^{-c_\varepsilon|\zeta|^2} h_\varepsilon(\zeta), \quad \zeta \in \mathbb{C}^n,$$

for any ε satisfying (3.6). Now, by Plancherel's theorem, let $H \in L^2(\mathbb{C}^n)$ be such that

$$(3.11) \qquad \hat{H} = (2\pi)^{-n}(\hat{F} *^{\frac{1}{2}} \hat{G}).$$

Then, by (3.10) and (3.11), $H \in \mathcal{S}_{c_\varepsilon}$, and hence, by (3.6) and (3.8), $H \in \bigcap_{0<d<c'} \mathcal{S}_d$. That $L_F L_G = L_H$ is then of course a consequence of (3.11) and Corollary 2.4. □

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